VICTOR OSTRIK: INTRODUCTION TO FUSION CATEGORIES, I

(Note: the attached handout is by Chelsea Walton.)

In the world of classical symmetries, i.e. those given by group actions, there is a particularly nice subclass: finite groups. If you know your symmetry group is finite, you can take advantage of many simplifying assumptions. Likewise, in the setting of quantum symmetries, given by, say, $\mathbb{C}$-linear tensor categories, fusion subcategories form a very nice subclass for which many simplifying assumptions hold. And indeed, if G is a finite group, its category of finite-dimensional representations is a fusion category.

Recall that a *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a distinguished object $\mathbf{1}\in\mathcal{C}$ called the *unit*, together with natural isomorphisms implementing associativity of $\otimes$, via $(X\otimes Y)\otimes Z\overset{\cong}{\to} X\otimes(Y\otimes Z)$; and unitality of $\mathbf{1}$, via $\mathbf{1}\otimes X\overset{\cong}{\to} X\overset{\cong}{\to} X\otimes\mathbf{1}$. These must satisfy some axioms which we won't discuss in detail here; the most important one is the *pentagon axiom* on the associator.

Today, we work over an algebraically closed field k, not necessarily closed. Recall that a *k-linear category* $\mathcal{C}$ is one for which for all objects $x,y\in\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is a k-vector space, such that composition is bilinear. A *k-linear monoidal category* is a monoidal category that is also a k-linear category — and we also impose the consistency condition that the tensor product is a k-linear functor. we will impose a few more niceness conditions before arriving at the definition of a fusion category — in fact, as many as we can such that we still have examples!

In particular, we will only consider k-linear monoidal categories $\mathcal{C}$ such that

- all Hom-spaces are finite-dimensional over k,
- $\mathcal{C}$ is semisimple,[1]
- $\mathcal{C}$ has only finitely many isomorphism classes of simple objects,
- $\mathbf{1}$ is indecomposable, and
- $\mathcal{C}$ is *rigid*, a condition on duals of objects.

A category satisfying all of these axioms is a *fusion category.*

There are three ways we can come to an understanding of these categories: through the definition, through realizations and examples, and through diagrammatics. We will also heavily use semisimplicity, through the principle that *k-linear functors out of $\mathcal{C}$ are determined by their values on simple objects, and all choices are allowed.*

Example 0.1. Our running example is $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, where n is a natural number and ω is a degree-3 cocycle for $\mathbb{Z}/n$, valued in $k^\times$.

The objects of $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ are the elements of $\mathbb{Z}/n$, with the tensor product $i\otimes j\coloneqq i+j$. If $\omega=1$, then we use the obvious associator, i.e. the isomorphism

$$(i\otimes j)\otimes k\overset{\cong}{\longrightarrow} i\otimes(j\otimes k) \tag{0.2}$$

which corresponds to the identity under the identifications with $i+j+k$.[2] But in general, we can do something different: choose the map (0.2) which is $\omega(i,j,k)$ times the standard one.

A priori you can use any function $\mathbb{Z}/n\times\mathbb{Z}/n\times\mathbb{Z}/n\to k^\times$, but the pentagon axiom on associativity imposes the condition that ω is a cocycle.

Exercise 0.3. If you have not seen this before, verify that the pentagon axiom forces $\partial\omega=1$.

The simplest nontrivial example[3] is for $n=2$ and

$$\omega(i,j,k)\coloneqq\begin{cases}1, & \text{if } i=0,\ j=0, \text{ or } k=0\\ -1, & \text{otherwise.}\end{cases} \tag{0.4}$$ ◂

[1] A k-linear category is *semisimple* if it's equivalent to the category of modules over $k\oplus\cdots\oplus k$, where there is a finite number of summands.

[2] These multiplication rules are really special, in that we were able to just write down an associator. This is generally not true; for general multiplication rules you're interested in, you'll have to work a little harder.

[3] This is nontrivial provided $\mathrm{char}(k)\neq 2$.

$\mathbb{Z}/n$ was not special here — given any finite group G and a cocycle $\omega \in Z^3(G;k^\times)$, we obtain a fusion category $\mathcal{V}ec_G^\omega$ in the same way.

With ω as in (0.4), $\mathcal{V}ec_{\mathbb{Z}/2}^\omega$ looks like a new example, not equivalent to $\mathcal{V}ec_G^0$ for any G — but in order to understand that precisely, we need to discuss when two tensor categories are equivalent.

Definition 0.5. A *tensor equivalence* of tensor categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, i.e. a functor together with data of natural isomorphisms $F(X \otimes Y) \xrightarrow{\cong} F(X) \otimes F(Y)$ satisfying some axioms.

Choose cocycles ω and ω' for $\mathbb{Z}/n$, and let's consider tensor functors $F\colon \mathcal{V}ec_{\mathbb{Z}/n}^\omega \to \mathcal{V}ec_{\mathbb{Z}/n}^{\omega'}$. Furthermore, let's assume F is the identity on objects, so the data of F is the natural isomorphism $F(X\otimes Y) \cong F(X)\otimes F(Y)$. This is a choice of an element of $k^\times$ for every pair of objects, subject to some additional conditions:

Proposition 0.6. *F is a tensor functor iff $\omega = \omega' \cdot \partial\psi$.*

Corollary 0.7. *$\mathcal{V}ec_{\mathbb{Z}/n}^\omega \simeq \mathcal{V}ec_{\mathbb{Z}/n}^{\omega'}$ if ω and ω' are cohomologous.*

Recall that $H^3(\mathbb{Z}/n;k^\times) \cong \mathbb{Z}/n$, so we have n possibilities, some of which might coincide. If F isn't the identity on objects, it's fairly easy to see that as a function on objects, identified with a function $\mathbb{Z}/n \to \mathbb{Z}/n$, we must get a group homomorphism; if F is to be an equivalence, this homomorphism must be an isomorphism. One can run a similar argument as above and obtain a nice classification result.

Proposition 0.8. *The tensor equivalence classes of tensor categories $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$ are in bijection with the orbits $H^3(\mathbb{Z}/n;k^\times)/\operatorname{Aut}(\mathbb{Z}/n)$, via the map sending ω to its class in cohomology.*

The action of $\operatorname{Aut}(\mathbb{Z}/n) = (\mathbb{Z}/n)^\times$ on $H^3(\mathbb{Z}/n;k^\times) \cong \mathbb{Z}/n$ is not the first action you might write down! Given $a \in (\mathbb{Z}/n)^\times$ and $s \in H^3(\mathbb{Z}/n;k^\times)$, the action is

$$a \cdot s = a^2 s. \tag{0.9}$$

This is a standard fact from group cohomology.

Now let's discuss some realizations of fusion categories. If H is a semisimple Hopf algebra, then $\mathcal{C} := \mathcal{R}ep_H^{fd}$ is a fusion category. Let $F\colon \mathcal{C} \to \mathcal{V}ec$ denote the forgetful functor to finite-dimensional vector spaces. It turns out that one can reconstruct $\mathcal{C}$ as a fusion category from F, and in fact any fusion category $\mathcal{C}$ with a tensor functor to $\mathcal{V}ec$ is equivalent to $\mathcal{R}ep_H^{fd}$ for some Hopf algebra H. The data of the tensor functor to $\mathcal{V}ec$ is crucial!

Example 0.10. For example, $\mathcal{V}ec_{\mathbb{Z}/n} \simeq \mathcal{R}ep_{\mathbb{Z}/n}^{fd}$; we saw in the previous lecture that representations of $\mathbb{Z}/n$ are equivalent to modules over the Hopf algebra $k[\mathbb{Z}/n] := k[x]/(x^n - 1)$, with comultiplication $\Delta(x) := x \otimes x$.

However, if ω is nontrivial, $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$ admits no tensor functor to $\mathcal{V}ec$, and therefore cannot be seen using Hopf algebras. One can try to generalize the reconstruction program, using quasi-Hopf algebras, weak Hopf algebras, etc. ◀

Bimodules provide another approach to realizations: we look for a ring R and a tensor functor $F\colon \mathcal{C} \to \mathcal{B}imod_R$. Applying this to $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$, we get (R,R)-bimodules $F(i)$ for each $i \in \mathbb{Z}/n$ and isomorphisms $F(i) \otimes_R F(j) \xrightarrow{\cong} F(i+j)$. In particular, each $F(i)$ is (tensor-)invertible.

Example 0.11. An *inner automorphism* of a ring R is conjugation by some $r \in R^\times$. Inner automorphisms form a normal subgroup of $\operatorname{Aut}(R)$, and the quotient is called the *outer automorphism group* of R and denoted $\operatorname{Out}(R)$. An *outer action* of a group G on a ring R is a group homomorphism $\varphi\colon G \to \operatorname{Out}(R)$.

Given an outer automorphism θ of R, one obtains an (R,R)-bimodule R_θ, whose left action is the R-action on R by left multiplication, and whose right action is $r \cdot x = r\theta(x)$. We need to choose an element in $\operatorname{Aut}(R)$ mapping to θ to make this definition, but different choices lead to isomorphic bimodules.

Anyways, given an outer action of $\mathbb{Z}/n$ on R, we obtain (R,R)-bimodules $R_{\varphi(i)}$ indexed by the objects $i \in \mathcal{V}ec_{\mathbb{Z}/n}$ and isomorphisms between $R_{\varphi(i)} \otimes R_{\varphi(j)} \xrightarrow{\cong} R_{\varphi(i+j)}$. This data stitches together into a tensor functor $\mathcal{V}ec_{\mathbb{Z}/n} \to \mathcal{B}imod_R$. ◀

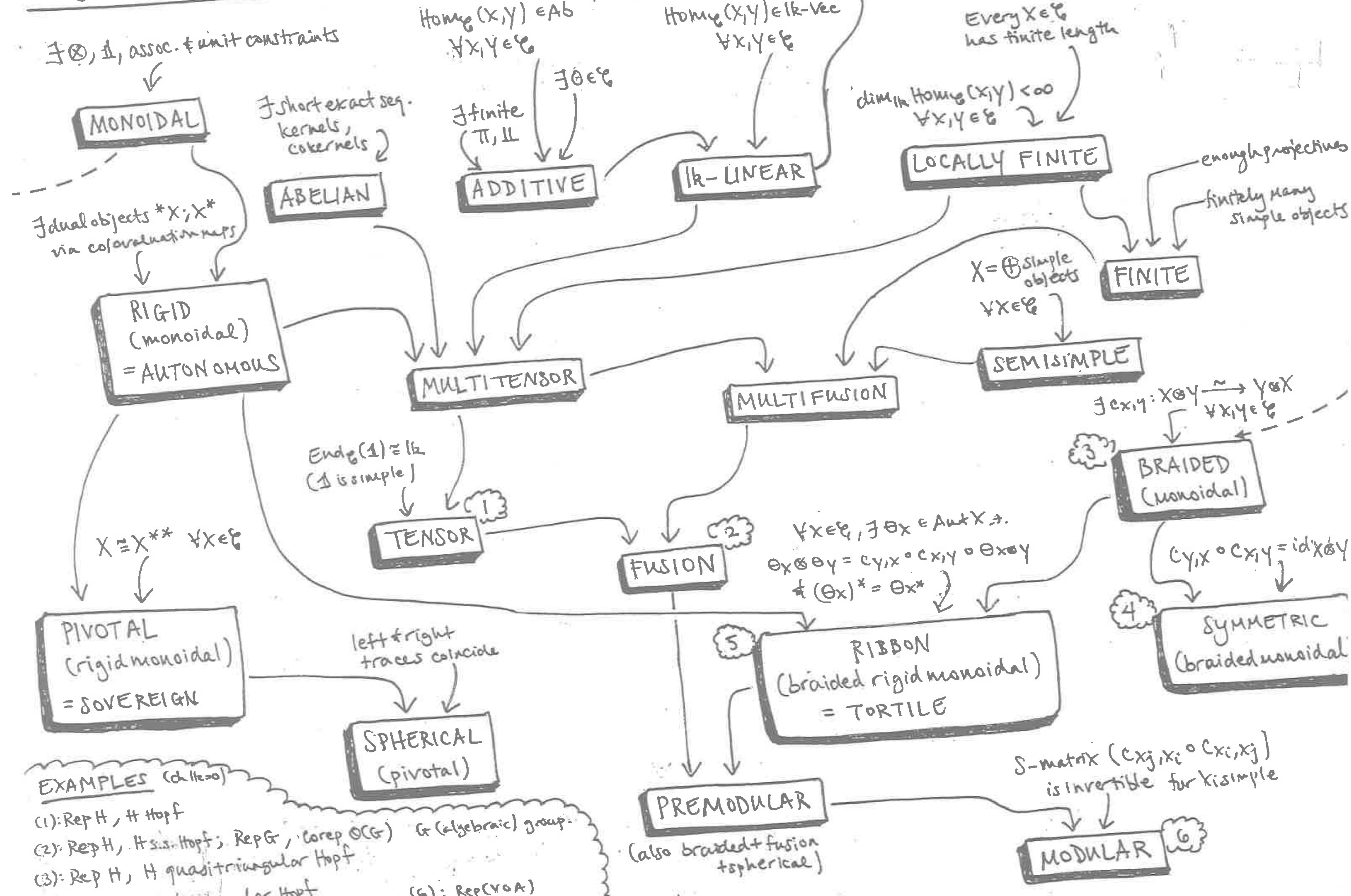
Diagrammatics represents the objects of a fusion category $\mathcal{C}$ as points, and morphisms as lines. One can then impose relations on certain morphisms, and therefore diagrammatics provide a generators-and-relations approach to the structure of a given fusion category. Next time, we'll see how to do this for $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$, and see more examples.

References

MONOIDAL CATEGORIES
– cheat sheet –
by. Chelsea Walton

KEY: on Tensor categories (2015)

$\Bbbk$ = field. $\mathcal{C}$ = category



EXAMPLES (ch $\Bbbk$ = 0)

(1): Rep H, H Hopf

(2): Rep H, H s.s. Hopf; Rep G, Corep $\mathcal{O}(G)$ G (algebraic) group.

(3): Rep H, H quasitriangular Hopf

(4): Rep H, H triangular Hopf

(5): Rep H, H ribbon Hopf

(6): Rep(VOA)

(1)–(6): Vec (fin-dim'l)