CRIS NEGRON: FINITE TENSOR CATEGORIES AND HOPF ALGEBRAS: A SAMPLING

Today, we work over an algebraically closed field k.

Example 0.1 (Small quantum groups)**.** Small quantum groups are important examples of Hopf algebras. Let $k = \mathbb{C}$ and let $\mathfrak{g}$ be a simple Lie algebra. Choose Cartan data for $\mathfrak{g}$, so that we have a set Δ of positive roots, and choose $q \in \mathbb{C}^\times$ of order p. The *small quantum group* associated to this data is the algebra generated by E_α, F_α, K_α for $\alpha \in \Delta$ subject to the *q-Serre relations*

$$E_\alpha^p = F_\alpha^p = K_\alpha^p - 1 = 0 \tag{0.2a}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{\langle \alpha,\beta\rangle} E_\alpha \tag{0.2b}$$

$$K_\alpha F_\beta K_\alpha^{-1} = q^{-\langle \alpha,\beta\rangle} F_\alpha. \tag{0.2c}$$

This is a finite-dimensional, non-semisimple Hopf algebra.

Let $u_q(\mathfrak{b})$, called the *quantum Borel*, denote the subalgebra of $u_q(\mathfrak{g})$ generated by the K_α and E_α elements; this is also finite-dimensional and non-semisimple. Let $G \subseteq u_q(\mathfrak{b})$ be the sub*group* generated by the K_α elements. ◀

One might ask: how much information is lost when we move from a Hopf algebra to its tensor category of representations?

Recall that a tensor category is an abelian, k-linear, rigid monoidal category $\mathcal{C}$ whose objects all have finite length, whose Hom spaces are finite-dimensional over k, and whose unit is simple.

Definition 0.3. Call $\mathcal{C}$ *finite* if it has finitely many simple objects and enough projectives.

This implies $\mathcal{C}$ is tensor equivalent to a category of representations of a finite-dimensional algebra. For example, the representation categories of $u_q(\mathfrak{g})$ and $u_q(\mathfrak{b})$ are finite tensor categories.

Definition 0.4. If $\mathcal{C}$ is semisimple and has finitely many simple objects, call $\mathcal{C}$ *fusion.*

There's a sequence of nested inclusions

$$\begin{aligned}\{\text{representations of finite groups over } \mathbb{C}\} &\subseteq \{\text{fusion categories}\}\\ &\subseteq \{\text{finite tensor categories}\}\\ &\subseteq \{\text{tensor categories}\}.\end{aligned} \tag{0.5}$$

For example, $\mathcal{R}ep_{\mathrm{SL}_n}$ is a tensor category that is not finite.

For Hopf algebras, taking the category of representations lands in tensor categories, and we can study how tensor equivalences of representation categories can be thought of in the language of Hopf algebras. This is sort of asking, what happens as the boundary of this map fron Hopf algebras to tensor categories?

Definition 0.6. A *Drinfeld twist* of a Hopf algebra A is a unit $J \in A \otimes A$ satisfying

$$(\varepsilon \otimes 1)H = (1 \otimes \varepsilon)J = 1 \tag{0.7}$$

and the *cocycle condition*

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J). \tag{0.8}$$

Given a Drinfeld twist J, we can build some new things.

- First, a new Hopf algebra denoted A^J, which is the same as A except the comultiplication is modified to $\Delta^J := J\Delta(-)J$.
- We also get a new fiber functor $F_J\colon \mathcal{R}ep_A \to \mathcal{V}ect$, which is the usual forgetful functor on objects and morphisms, but whose monoidal structure is modified: when defining the map

$$F_J(V) \otimes F_J(W) \to F_J(V \otimes W), \tag{0.9}$$

  take the usual map, then apply J.

Theorem 0.10. *When A is a finite-dimensional Hopf algebra,* all *fiber functors $\mathcal{R}ep_A \to \mathcal{V}ect$ arise from Drinfeld twists in this way.*

Theorem 0.11 (Ng-Schauenberg)**.** *Let A and B be finite-dimensional Hopf algebras such that $\mathcal{R}ep_A \simeq \mathcal{R}ep_B$ as tensor categories. Then there is a Drinfeld twist J of A such that, as Hopf algebras, $B \cong A^J$.*

Example 0.12 (Negron [Neg18])**.** Specializing to $A = u_q(\mathfrak{b})$, an equivalence $\mathcal{R}ep_B \simeq \mathcal{R}ep_{u_q(\mathfrak{b})}$ leads to an alternating bicharacter $J \in \mathrm{Alt}(G^\vee) \subset T_w(u_q(\mathfrak{b}))$, such that $B \cong u_q(\mathfrak{b})^J$. Since $\mathrm{Alt}(G^\vee)$ is a finite set, this is particularly nice. ◀

As we heard in Rowell's talk, the notion of being the category of representations of a Hopf algebra is not invariant under tensor equivalence, and more generally, Hopf algebra representation categories are not closed under reasonable operations on the class of tensor categories.

For example, if a group G acts on a tensor category $\mathcal{C}$, then we can *equivariantize*, building a new tensor category $\mathcal{C}^G$, the category of objects $V \in \mathcal{C}$ with compatible structural isomorphisms $g \cdot V \xrightarrow{\cong} V$ for all $g \in G$. An embedding $\mathcal{V}ect \hookrightarrow \mathcal{C}$ induces an embedding $\mathcal{R}ep_G \hookrightarrow \mathcal{C}^G$.

Theorem 0.13 (Drinfeld-Gilyaki-Nikshych-Ostrik [DGNO10])**.** *Equivariantization defines a bijection between tensor equivalence classes of tensor categories with a G-action and tensor categories with a specified embedding of $\mathcal{R}ep_G$.*

This ultimately implies that even if $\mathcal{C}$ admits a fiber functor (as representation categories of Hopf algebras must), $\mathcal{C}^G$ might not, because there are categories containing $\mathcal{R}ep_G$ but not admitting a fiber functor.

Tensor categories have connections with 2d conformal field theory, hence vertex operator algebras.

- Given a *rational conformal field theory* (equivalently, a rational vertex operator algebra), Y. Huang shows how to extract a modular fusion category.
- Given a *logathmic conformal field theory*, a series of papers by Huang-Lepowski-Zhang construct a modular tensor category, maybe with some additional assumptions. See in particular [HLZ11].

The upshot is that given an *finite* logarithmic vertex operator algebra V, its category of representations is a finite, braided tensor category which is nondegenerate and pivotal (hence ribbon).[1]

Example 0.14. Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $p \in \mathbb{Z}_+$, one can construct a non-rational vertex operator algebra denoted $W_p(\mathfrak{g})$, which is cut out of a lattice model by an action of $u_q(\mathfrak{n})$ by something called short-screening operations. This was studied by Lentner and others.

These are understood in type A_1 and, at $p = 2$, type B_n: $W_p(\mathfrak{sl}_2)$ is the *triplet model* of Kausch (1991), and $W_2(\mathrm{B}_n)$ is the *symplectic fermion model* of Kausch [Kau00]. As established by Flandoli-Lentner [FL18], these have non-semisimple, modular representation theories. ◀

Conjecture 0.15. There is a modular equivalence $F_\otimes$ from the category of representations of $u_q(\mathfrak{g})$ to the category of representations of $W_p(\mathfrak{g})$, where $q := \exp(i\pi/p)$.

This is mostly done for $\mathfrak{g} = \mathfrak{sl}_2$, but is completely open in general.

Remark 0.16. You should be careful with what's precisely meant by $u_q(\mathfrak{g})$. See work of Negron and several others. ◀

To finish, let's talk a little bit about cohomology. Suppose $\mathcal{C}$ is a finite, but not semisimple, tensor category. Then let $\mathrm{Proj}\,\mathcal{C}$ denote the subcategory of projective objects in $\mathcal{C}$; this has finitely many indecomposables $P_1, \dots, P_n$, canonically labeled by the isomorphism classes of simple objects. $\mathrm{Proj}\,\mathcal{C}$ has a *strong* fusion rule, with

$$P_i \otimes P_j \cong \bigoplus_k P_k^{\oplus N_{ij}^k} \tag{0.17}$$

for some natural numbers N_{ij}^k. In fact, something stronger is true: $\mathrm{Proj}\,\mathcal{C}$ can be described by discrete/number-theoretic data. But what happens on the rest of $\mathcal{C}$?

The *stable category* of $\mathcal{C}$ is $\mathrm{Stab}\,\mathcal{C} := \mathcal{C}/\mathrm{Proj}\,\mathcal{C}$. This is not an abelian category, though it is triangulated — in particular, it has a shift functor $\Sigma\colon \mathrm{Stab}\,\mathcal{C} \to \mathrm{Stab}\,\mathcal{C}$. This data is regulated by geometry and continuous

[1] NOTE by the notetaker: I missed a few of the references starting at this point of the talk. Sorry about that.

invariants, called *support theory* or *tensor-triangulated geometry*, related to the Proj variety of $\mathrm{End}^*_{\mathrm{Stab}\,\mathcal{C}}(\mathbf{1})$. There's a lot more that could be said about this approach to the stable category, but that is a story for another day.

References

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