ZHENGHAN WANG: TOPOLOGICAL ORDERS, I

This talk will be a mathematics talk about topological order (in topological phases of matter), which is a subject close to physics. For a general reference for this talk, see [RW18]. We will focus on bosonic/spin intrinsic order, as opposed to fermionic order or SPT phases; if this doesn't mean anything to you, that's OK.

When we consider topological orders in dimension $2+1$, we mean two dimensions of space, and one of time, so we will usually think about surfaces, and sometimes 3-manifolds. A $(2+1)$-*dimensional topological order* is equivalent to a unitary topological modular functor, which is equivalent to a unitary modular tensor category (sometimes called an *anyon model* in physics). There are several reasons to care about these phases.

(1) At least in theory, the ability to build topological phases of matter would allow one to build a quantum computer, which has numerous applications to the real world, including making money. This is being pursued in industry, e.g. by Microsoft. However, there is still much to do, in both theory and engineering, before this can be a reality.
(2) From a theoretical physics perspective, topological phases are very interesting. These are in the subfield of condensed-matter physics, which historically understood phases of matter via Landau's group symmetry-breaking paradigm; for example, crystals are understood via discrete translation symmetries, but liquids have continuous symmetries. Phase transitions correspond to symmetry breaking. But topological phases do not follow these rules, leading to a paradigm shift in physicists' perspectives on phases, to the perspective of quantum symmetries.
(3) Finally, these are interesting objects in their own mathematical right.

Analogously to the relationship between Riemann sums and definite integrals, there are two perspectives on quantum field theory that shed mathematical insight into it: one can work in the continuum (akin to the integral) or on the lattice, which is more discrete, akin to a Riemann sum. Sometimes we use integrals to approximate Riemann sums, even though that wasn't the original way information flowed; likewise, these topological phases are QFTs on the lattice, but we can study them with continuum limits. This is a part of the general mathematical goal of understanding quantum field theory.

We will focus on two examples: the toric code and Haah's code. The toric code is very, very well-studied — almost any question you might ask about it has been answered. Haah's code is newer, and poorly understood: it's an example of a fracton model, and we think that a proper understanding of such models will lie beyond quantum field theory.

We will first study the toric code via its robust ground state degeneracy, which is a TQFT, and which is a mathematically satisfying perspective even if it's still not completely understood. In the next lecture, we'll study the elementary excitations, which lead to unitary modular tensor categories, a different perspective. But there is a mathematical theorem relating unitary TQFTs in dimension $2+1$ and unitary modular tensor categories.

You should not just accept these definitions as final — this field is still in the process of being mathematically codified. Some of these definitions are matters of convenience, so that we can actually get somewhere, even if we don't have the most correct definitions.

Definition 0.1. A *quantum theory* is a triple (L, B, H), where L is a finite-dimensional Hilbert space, B is a basis of L, and $H\colon L \to L$ is a Hermitian operator, which thanks to B we can think of as a matrix.

The basis is an unusual ingredient when one studies QFTs, but is important in the story of quantum information: it's how you represent classical information.

Many quantum theories satisfying Definition 0.1 aren't related to physics, and are therefore somewhat useless. We focus on the examples which come from physics; after all, Definition 0.1 is trying to (partially) axiomatize things physicists are interested in, right?

Definition 0.2. An *n-dimensional quantum schema* is a rule assigning to every n-dimensional manifold with a triangulation and a finite-dimensional Hilbert space, a quantum theory.

Example 0.3. The 1-dimensional Ising chain is a 1-dimensional quantum schema. Given a circle with a triangulation, the Hilbert space is the tensor product of copies of $\mathbb{C}^2$ indexed by the vertices. Assume there are N vertices, and orient the circle so we can identify them with $1, \ldots, N$ in order, and call the Hilbert space L_N. Inside $\mathbb{C}^2$, let $|0\rangle$ and $|1\rangle$ be the standard basis vectors (i.e. $(1,0)$ and $(0,1)$, respectively). The Hamiltonian has the form

$$H = -\sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z. \tag{0.4}$$

This is physicists' notation: let's explan what's going on. The *Pauli matrices* are the standard basis of $\mathfrak{su}_2$:

$$\sigma^x = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^y = \sigma_2 := i\sigma^x\sigma^z. \tag{0.5}$$

These are Hermitian matrices which square to the identity.

The notation σ_i^z means that σ^z acts on the $\mathbb{C}^2$ at vertex i, and acts by the identity on the remaining factors, i.e. by $\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \sigma^z \otimes \cdots \otimes \mathrm{id}$. ◂

Given a quantum theory, the eigenvalues of the Hamiltonian $\lambda_0 \le \lambda_1 \le \ldots$ are called *energies* of the theory, and the nonzero eigenvectors of λ_0 are called *ground states.* Nonzero eigenvectors for other eigenvalues are called *excited states.*

Here's the most important definition (albeit, again, not quite the real definition).

Definition 0.6. An n-dimensional Hamiltonian schema is *(sharply) gapped* if there is a constant $c > 0$ such that for all n-manifolds and triangulations, in the quantum system assigned by the schema, the eigenvalues of the Hamiltonian satisfy $\lambda_1 - \lambda_0 \ge c$.

Crucially, c does not depend on the triangulation. Sharply gapped schemas are almost topological.

Tentative definition 0.7. An n-dimensional Hamiltonian schema is *topological* if it's gapped, and if there exists a unitary topological modular functor (i.e. a unitary TQFT-like object in dimension $2+1$, which is once-extended, but isn't necessary finite enough to assign partition functions to all 3-manifolds) Z such that for any closed 2-manifold Σ, $Z(\Sigma)$ is isomorphic to the space of ground states of the Hamiltonian schema on Y for any triangulation.

We expect that a topological Hamiltonian schema represents mathematically a topological phase of matter.

Definition 0.8. A *topological phase of matter* is a path component of the space of topologically ordered Hamiltonians.

Unfortunately, we're not yet sure what the space of topologically ordered Hamiltonians is, but we want to say that two Hamiltonians are equivalent if there's a path deforming one into the other, through topological Hamiltonian schema — in particular, the path cannot close the gap: c must always be greater than some $\varepsilon > 0$. Understanding this carefully in general would require opening the can of worms called renormalization.

The toric code is the model organism in topological phases. If you want to understand just about anything about topological phases of matter, you should probably begin by thinking about it for this example.

Example 0.9 (Toric code)**.** The toric code, first studied by Kitaev [Kit03], realizes the topological order given by untwisted $\mathbb{Z}/2$-Dijkgraaf-Witten theory, corresponding to the modular tensor category $D(\mathbb{Z}/2)$, with four simple objects $\{1, e, m, \psi\}$ with $e \otimes e = m \otimes m = \psi \otimes \psi = 1$ and $e \otimes m = m \otimes e = \psi$ and twists $\theta_1 = \theta_e = \theta_m = 1$ and $\theta_\psi = -1$.[1]

The toric code schema begines with a closed surface Y and a triangulation (or more generally, a cellulation), which is often just taken to be the torus with a cellulation given by a square tiling of the plane. The Hilbert space L is a tensor product of $\mathbb{C}^2$ over all of the edges in the cellulation. Thus L is canonically identified with the group algebra for the group $(\mathbb{Z}/2)^{|E|}$, if E is the set of edges.

The Hamiltonian is

$$H = -\sum_{\text{all vertices } v} A_v - \sum_{\text{all faces } P} B_P, \tag{0.10}$$

[1] So the anyon ψ is a fermion, but this is still a bosonic phase, because we started with bosonic spins, or mathematically, vector spaces and not super vector spaces.

for some operators A_v and B_P we will define.[2] A_v is the tensor product of σ^z on all of the edges adjacent to v, and the identity on the remaining edges. B_P is the tensor product of σ^x on all edges in ∂P, and the identity on the remaining edges.

Here are three important properties of the toric code.

(1) All A_v and B_P operators commute with each other. This is clear for A_v and $A_{v'}$, and for B_P and $B_{P'}$, since we just have a bunch of σ^z or σ^x operators, or for A_v and B_P when $v \notin \overline{P}$, but the interesting bit is when v and P are adjacent; then σ^x and σ^z don't commute, but they anticommute, and there is an even number of edges affected by both A_v and B_P, so two minuses make a plus and $[A_v, B_P] = 0$.

(2) The space of ground states of this model on a closed surface Σ is canonically identified with the space of $\mathbb{C}$-valued functions on $H_1(\Sigma; \mathbb{Z}/2)$. This uses the fact that the Hamiltonian is *frustration-free*, which means that the ground states are precisely those stabilized by all A_v and B_P operators. Looking at B_P gives you cycles; then looking at A_v kills boundaries.

(3) The elementary excitations for the toric code form the unitary modular tensor category $D(\mathbb{Z}/2)$. ◀

Exercise 0.11. Modify the toric code to

$$H = \sum_v \varepsilon_1 A_v + \sum_P \varepsilon_2 B_P, \tag{0.12}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Which of these phases are topologically ordered, and which aren't? If you understand that, you probably understand this lecture very well.

References

[Kit03] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003. https://arxiv.org/abs/quant-ph/9707021. 2

[RW18] Eric C. Rowell and Zhenghan Wang. Mathematics of topological quantum computing. *Bull. Amer. Math. Soc. (N.S.)*, 55(2):183–238, 2018. https://arxiv.org/abs/1705.06206. 1

[2] Why "P"? Because in the physics literature, faces are often referred to as *plaquettes*.