TERRY GANNON: CONFORMAL NETS, II

Recall from last time we are interested in conformally invariant quantum field theory in one space and one time dimension, also known as 2d CFT. Our basic space is $\mathbb{R}^{1,1}$, which we compactify into $S^1 \times S^1$, leading to a splitting of the conformal group as $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$. If we only impose half of this, asking for $\mathrm{Diff}^+(S^1)$-covariance for a theory on S^1, we get what's called a *chiral* theory. This is sort of a cheat, and so we should expect/hope to recombine later. This is relatively well understood, at least for chiral theories that aren't too complicated, is well-understood, using modular categories. This is studied by Fuchs, Schweigert, and their collaborators. This trick works in certain simple settings (rational CFT), but otherwise is a bit too sneaky of a trick.

If you study VOAs or conformal nets, you must talk about their representations too — that's why they are. Recall that a conformal net is data of for each open interval $I \subset S^1$, a von Neumann algebra $A(I)$ that's a subalgebra of the algebra of bounded operators on some Hilbert space. This is a type III_1 factor, which is the interesting kind — other kinds are less exciting.

Recall that a representation of this conformal net is data of maps $\pi_I : A(I) \to B(K)$, where K is some other Hilbert space (and $B(K)$ denotes its algebra of bounded operators) which are compatible with the maps $A(I_1) \to A(I_2)$ induced from an inclusion $I_1 \subset I_2$. Moreover, each π_1 should be a continuous $*$-algebra homomorphism.

Even if you don't care about physics, good physics leads to great mathematics, and so there's a very good chance that the mathematics of the future will be reminiscent of conformal nets and, more generally, the mathematical approach to CFT.

There is a rival mathematical formalism for CFTs called vertex operator algebras (VOAs). In this formalism, a quantum field is an operator-valued distribution on spacetime. Given an operator φ and a test function $f(z)$ (a Laurent polynomial), the pairing $\int \varphi(z) f(z)\,\mathrm{d}z$, which is to be interpreted formally, produces an operator. For $f(z) = z^n$, we obtain the operator $\varphi_n \in \mathrm{End}(V)$, where V is the state space. This state space is graded by energy levels; the interesting cases are when each homogeneous component is finite-dimensional.

Often, φ is written in a Laurent-series-like manner, albeit in a formal sense:

$$\varphi = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n-1}, \tag{0.1}$$

and indeed everything in VOAs takes on a formal feel. As for conformal nets, there is a locality axiom: for all φ and ϕ, and z and w, there is some N for which

$$(z-w)^N[\varphi(z), \phi(w)] = 0. \tag{0.2}$$

This means $[\varphi(z), \phi(w)]$ is a finite linear combination of Dirac deltas and their derivatives at z and w. However, (0.2) is not how locality is usually stated in the VOA literature; usually, it's expressed in terms of operators $Y(n, z) = \varphi(z)$. In a VOA, these axioms automatically give you a representation of the Virasoro algebra, a central extension of the Lie algebra of vector fields on S^1, related to the $\mathrm{Diff}^+(S^1)$-action arising in CFT.

Conformal nets are more like Lie groups, and VOAs like Lie algebras — certainly, conformal nets involve some analysis, and VOAs are more algebraic. Conformal nets are forced to be unitary, and VOAs do not; one can also study them over characteristic p. And VOAs are friendlier in general.

Non-unitary or non-semisimple VOAs/CFTs are a new and intereting world, with rich examples, such as the triplet model, or symplectic fermions; even though we don't have many examples, we expect almost every CFT to be non-logarithmic. There is no analogue of this in conformal nets.

In a VOA, we have a lot of extra structure: a doubly infinite tower of mapping class group representations. This encodes the relationships to moonshine phenomena and more, and this is a vital component of the theory. This is much harder to see using conformal nets.

Associated to a nice VOA and its associated modular tensor category $\mathcal{C}$ there are characters

$$\chi_M(\tau) = \sum_{n=0}^{\infty} \dim M_n q^n, \tag{0.3}$$

and if you write $q = e^{2\pi i \tau}$, then these are modular functions.

Conjecture 0.4 (Atkin-Swinnerton-Dyer [ASD71])**.** Given a modular function $f(\tau)$ for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, if $f(\tau) = \sum q^n a_n$ and $a_n \in \mathbb{Z}$, then Γ contains some $\Gamma(N)$.

This conjecture is from the 1970s, and is still open, but the evidence has been leaning in favor of its truth. It suggests that there isn't much of a difference between modular tensor categories and conformal nets or VOAs.

We have relatively good control over the tensor products of conformal nets, which has something to do with the fact that the representation theory of type III_1 factors is nearly trivial — all left modules are isomorphic. The theory of bimodules is more interesting: the classification of (M, M)-bimodules, where M is a type III_1 factor, is equivalent to the classification of $*$-endomorphisms of M up to conjugation by unitary operators. The tensor product of bimodules passes to composition of morphisms, which is nice — but direct sum is trickier to translate.

One can try to see explicitly what happens in the finite-dimensional case, with matrices, but it turns out the answer is specific to the infinite-dimensional case. If $\lambda, \mu \colon M \rightrightarrows M$, then $(\lambda \oplus \mu)(x) = P\lambda(x)P^* + Q\mu(x)Q^*$, where P and Q are ways to split M into two copies: $P^*P = Q^*Q = I$, and $P^*Q = 0$.

You can use these to build exotic fusion categories with good control on their modular data: the magic trick is to realize simple objects in your fusion category as endomorphisms of an algebra, which makes tensor product easy and direct sum hard. And because tensor product comes from composition, this category is automatically strict, which is a nice but generally more artificial condition. This leads to many interesting exotic fusion categories, and it really seems like almost all fusion categories and modular tensor categories are exotic, even though currently we know of very few exotic examples. Maybe this is an unusual view from within the field.

Likewise, the representations of a conformal net can be realized as endomorphisms of said conformal net. This is a useful tool for getting a handle on braidings, etc., within the representation category. If π is a representation of the conformal net A and λ^π is the corresponding endomorphism $A(I) \to A(I)$ for each interval $I \subset S^1$, then composition passes to tensor product, but you can also see the braiding. This is just as in any quantum field theory, though in higher dimensions this braiding is symmetric.

The dimension of a representation is $d_\pi := \sqrt{[A(I) : \lambda_I^\pi(A(I))]} \in [1, \infty]$. From this data, one can get a subfactor: $A(I) \subset A(I')$, where I' is the interior of $S^1 \setminus I$. This is unfortunately a boring subfactor, in that $A(I) = A(I')$, so we instead need to choose two points inside I', and divides the circle into four intervals, I_1, I_2, I_3, and I_4, which we assume are in cyclic order around the circle. Then we have a subfactor

$$(A(I_1) \cup A(I_3)) \subset (A(I_3) \cup A(I_4)). \tag{0.5}$$

This is used in one of the most important definitions in the area.

Definition 0.6. A *rational conformal net* is one in which the index of the subfactor (0.5) is finite.

Rational conformal nets should model rational CFTs. In this case, the sum of the squares of the (Frobenius-Perron) dimensions of the *superselection sectors* (i.e. the models) is finite. Crucially, this means all of the representations are finite, and in fact the category of representations of a rational conformal net is a modular tensor category!

There is also a definition of rationality for VOAs, and the category of modules for a rational VOA is a modular tensor category, but this is harder to work with.

One useful thing you can do with a rational conformal net with a group action (e.g. the one associated to the E_8 lattice, with an involution) is *gauging* or the *orbifold construction*. It is a theorem that the result is always rational, but this is still a conjecture in the world of VOAs. In the E_8 example, the result of gauging, depending on the involution, can be the D_8 theory, whose modular tensor category is the Drinfeld double of $\mathbb{Z}/2$; or we can get $\mathrm{A}_1 \oplus \mathrm{E}_7$, whose modular tensor category is $D^\omega(\mathbb{Z}/2)$, the twisted Drinfeld double.

This tells us that what's actually acting is a finite group and a 3-cocycle, i.e. what's really acting on the conformal net is a fusion category! This suggests that exotic fusion categories should lead to exotic orbifolds and CFTs. There is related work by Marcel Bischoff.

Just as finite groups are finite collections of objects that act, these are the quantum analogue: finite collections of things that act, albeit in this more modern manner.

References

[ASD71] A. O. L. Atkin and H. P. F. Swinnerton-Dyer. Modular forms on noncongruence subgroups. In *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968)*, pages 1–25, 1971. 2