ANNA BELIAKOVA: QUANTUM INVARIANTS OF LINKS AND 3-MANIFOLDS, I

The title of this talk is inspired from Turaev's talk, but we have a different aim in mind: Turaev studies things from a very general perspective, but we're going to focus on specific examples in detail.

There is a procedure called surgery which associates to a framed link in S^3 a closed, oriented 3-manifold. A famous theorem of Lickorish-Wallace asserts that every closed, oriented 3-manifold can be realized in this way, and conversely, two framed links yield diffeomorphic 3-manifolds iff they differ by a series of known moves.

Given a ribbon Hopf algebra, one can build an invariant of framed links; for $\mathcal{U}_q(\mathfrak{sl}_2)$, for example, this is the colored Jones polynomial. Using a procedure called integration, we obtain 3-manifold invariants, in this case the Witten-Reshetikhin-Turaev invariants. And there's a way to build them directly from 3-manifolds, which uses finiteness.

There is another way to obtain framed link invariants from $\mathcal{U}_q(\mathfrak{sl}_2)$, yielding *Kashaev invariants*, which are of quantum dimension zero. These are sometimes also called *logarithmic invariants*. The corresponding 3-manifold invariants are called *Hennings CGP invariants*.

More recently, these colored link invariants have been unified into a more general invariant, called the *Habiro cyclotomic invariant*, yielding a unified Witten-Reshetikhin-Turaev invariant for 3-manifolds. We'll discuss this invariant in the second talk later this week.

Let $L\colon (S^1 \times I)^{\amalg k} \hookrightarrow S^3$ be a framed link, and let $\nu(L)$ denote its normal bundle, embedded in S^3 via the tubular neighborhood theorem. Given this data, *surgery on L* is the closed, oriented 3-manifold

$$S^3(K_f) \coloneqq S^3 \setminus \nu(L) \cup_f (D^2 \times S^1)^{\amalg k}, \tag{0.1}$$

where f is the identification of $\partial(S^3 \setminus \nu(L))$ and $(D^2 \times S^1)^{\amalg k}$ given by the framing. There are two moves K1 and K2 which change the framed link but don't change the diffeomorphism class of the 3-manifold obtained under surgery.

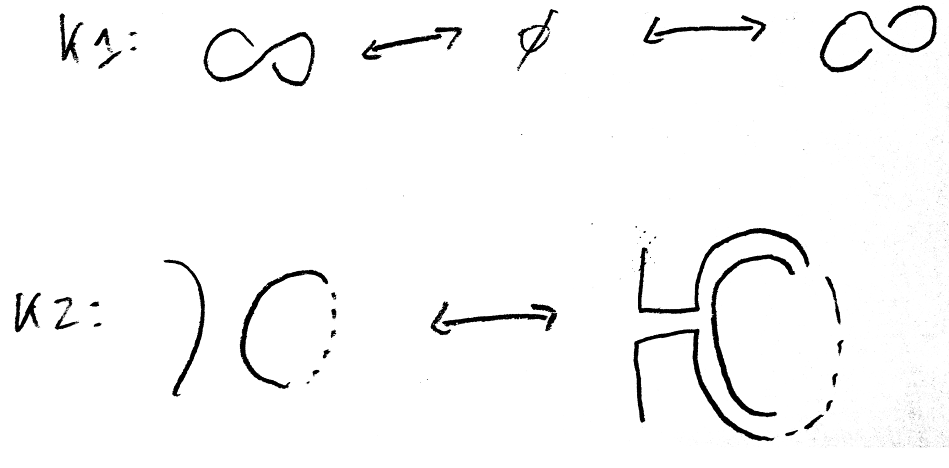

FIGURE 1. Moves on a framed link which do not change the diffeomorphism class of the 3-manifold obtained by surgery.

- The simpler move, denoted K1, exchanges a figure-8 with an empty set.
- K2 is a little more elaborate.

We will define a universal $\mathfrak{sl}_2$ framed link invariant. Given $n \in \mathbb{N}$, let $\{n\} := \sigma^n - \sigma^{-n}$, where σ is a formal variable, and let $[n] = \{n\}/\{1\}$. Now, we define the quantum group

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle e, F^{(n)}, K \rangle, \tag{0.2}$$

where $F^{(n)} := F^n/[n]!$, $e = \{1\}E$, and $\sigma^H = K = \exp((h/2)H)$. Let

$$E = \sigma^{H\otimes H} 2 \sum_{n=0}^{\infty} \sigma^{\frac{n(n-1)}{2}} F^{(n)} \otimes e^n \subset \mathcal{U}_k \mathbin{\widehat{\otimes}} \mathcal{U}_k. \tag{0.3}$$

Then R is a simple tensor: write $R = \alpha \otimes \beta$. Now we label pieces of a framed link: if, traveling upwards, left travels over right, label the left with β and the right with α; if right travels over left, label the left with $\overline{\beta}$ and the right with $\overline{\alpha}$. Label a cup (coevaluation) with k and a cap (evaluation) with k^{-1}. Call the resulting element of the universal enveloping algebra J_L. Notice that this always lands in the center of $\mathcal{U}_k$, which is freely generated by the *Casimir element*

$$C := \{1\}FE + \sigma K + \sigma^{-1}K^{-1}. \tag{0.4}$$

Let V_n be the n-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. Then let $J_L(V_n)$ denote the action of J_L on V_n. For example, the Casimir acts on V_n by $\sigma^n + \sigma^{-n}$.

Theorem 0.5 (Habiro [Hab08])**.** *Let K_0 be a 0-framed knot. Then*

$$J_{K_0} = \sum_{m=0}^{\infty} a_m \sigma_m, \tag{0.6}$$

where $a_m \in \mathbb{Z}[q^{\pm 1}]$ and

$$\sigma_m = \prod_{i=1}^{m} \left(c^2 - (\sigma + \sigma^{-1})^2\right). \tag{0.7}$$

Example 0.8. For the knot 4_1, $J_{4_1} = \sum_{m=0}^{\infty} \sigma_m$. For the knot 3_1, we obtain

$$J_{3_1} = \sum_{m=0}^{\infty} (-1)^m q^{m(m-3)/2} \sigma_m. \tag{0.9}$$

◂

In general,

$$J_{K_0}(V_n) = \sum_{m=0}^{n-1} a_m \prod_{i=1}^{n} \left(q^n + q^{-n} - q^i - q^{-i}\right) = \sum a_m \prod_{i=1}^{m} \{n+i\}\{n-i\}. \tag{0.10}$$

This recovers the Witten-Reshetikhin-Turaev invariant as follows: let ξ be a p^{th} root of unity and plug in $q = \xi$. Then define

$$F_{K_a}(\xi) = \sum_{n=0}^{p-1} [n]^2 J_K(V_n)|_{q=\xi} = \sum_{n=0}^{p-1} [n]^2 q^{a(m^2-1)/4} J_{K_0}(V_n)|_{q=\xi}. \tag{0.11}$$

Then, the Witten-Reshetikhin-Turaev invariant of $S^3(K_a)$ at ξ is $F_{K_a}(\xi)/F_{\text{unknot}}(\xi)$, where the unknot has framing given by the sign of a.

Remark 0.12. There is another invariant of 3-manifolds given similar-looking data, called the *Turaev-Viro invariant*, computed by triangulating the 3-manifold and labeling tetrahedra by $6j$-symbols. Beliakova-Durhuus [BD96], Walker, and Turaev showed that the Turaev-Viro invariant of M is equal to the Reshetikhin-Turaev invariant of $M \mathbin{\#} (-M)$, i.e. the square of the Reshetikhin-Turaev invariant of M. ◂

Theorem 0.13 (Beliakova-Chen-Lê [BCL14])**.** *For all closed, oriented 3-manifolds M and all ξ, the Witten-Reshetikhin-Turaev invariant of M at ξ is in $\mathbb{Z}[\xi]$.*

That is, we can write the Witten-Reshetikhin-Turaev invariant of M as a polynomial in ξ of degree at most $p-1$, and this is telling us that the coefficients are integers.

Now, let's write $F_{K_a}(\xi)$ using a Gauss sum:

$$F_{K_a}(\xi) = \sum_{m\geq 0} a_m \sum_{n=0}^{p-1} q^{a(m^2-1)/4}\{n+m\}\cdots[n]^2\cdots\{n-m\}. \tag{0.14}$$

This lives in $\mathbb{Z}[q^{\pm n}, q]$. Plugging in $a = \pm 1$, we see that

$$\sum_{n=0}^{p-1} q^{a(m^2-1)/4} q^{bn} = q^{-b^2/a}\gamma_a. \tag{0.15}$$

Let $L_a(q^{bn} := q^{-b^2/a}$ and

$$I_M = (?) \sum_{m\geq 0} a_m L_a(\{n+m\}\cdots[n]^2\cdots\{n-m\}). \tag{0.16}$$

Let $(q)_n := (1-q)\cdots(1-q^n) \in \mathbb{Z}[q]$ and $\check{I}_n \subset \mathbb{Z}[q]$ denote the ideal spanned by $(q)_n$. Then $\check{I}_n \subset \check{I}_{n+1} \subset \check{I}_{n+2} \subset \ldots$, and we can complete to

$$\widehat{\mathbb{Z}}[q] := \varprojlim_n \mathbb{Z}[q]/(q)_n, \tag{0.17}$$

which is the ring of analytic functions on the roots of unity, and is called the *Habiro ring*. An element of $\widehat{\mathbb{Z}}[q]$ can be represented as

$$f = \sum_{k=0}^{\infty} f_k (q)_k, \tag{0.18}$$

where $f_k \in \mathbb{Z}[q]$. This defines an embedding $\widehat{\mathbb{Z}}[q] \hookrightarrow \mathbb{Z}[[1-q]]$, and $f \in \widehat{\mathbb{Z}}[q]$ is determined uniquely by its values at roots of unity. The value $\omega_\xi f$ is well-defined.

Theorem 0.19 (Habiro [Hab08]). *If M is an integral homology sphere, there is a unique $I_M \in \widehat{\mathbb{Z}}[q]$ such that for any ξ, $\omega_\xi I_M$ is the Witten-Reshetikhin-Turaev invariant for ξ and M.*

For example, if M is the Poincaré homology sphere,

$$I_M = \frac{q}{1-q}\sum (-1)^k q^{k(k+1)/2}(q^{k+1})_{k+1}. \tag{0.20}$$

If M is a rational homology sphere with $b_1(M) > 0$ the theorem, proven by Beliakova-Bühler-Lê [BBL11], is not quite as simple. Recently, Habiro-Lê [HL16] have generalized Theorem 0.19 to the analogues of these invariants defined using an arbitrary simple Lie algebra.

Next time, we'll see how even non-semisimple invariants are determined by Witten-Reshetikhin-Turaev invariants.

References

[BBL11] Anna Beliakova, Irmgard Bühler, and Thang Lê. A unified quantum SO(3) invariant for rational homology 3-spheres. *Invent. Math.*, 185(1):121–174, 2011. https://arxiv.org/abs/0801.3893. 3

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[BD96] Anna Beliakova and Bergfinnur Durhuus. On the relation between two quantum group invariants of 3-cobordisms. *J. Geom. Phys.*, 20(4):305–317, 1996. https://arxiv.org/abs/q-alg/9502010. 2

[Hab08] Kazuo Habiro. A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres. *Invent. Math.*, 171(1):1–81, 2008. https://arxiv.org/abs/math/0605314. 2, 3

[HL16] Kazuo Habiro and Thang T. Q. Lê. Unified quantum invariants for integral homology spheres associated with simple Lie algebras. *Geom. Topol.*, 20(5):2687–2835, 2016. https://arxiv.org/abs/1503.03549. 3

Beliakova

MSRI <u>lecture 1:</u> — 1 — Quantum invariants of links and 3-mfd I

<u>Plan of the lectures</u>

$$\left\{\begin{matrix}\text{framed}\\ \text{links in } S^3\end{matrix}\right\}\Big/_{K1,K2} \xrightarrow[1:1]{\text{surgery}} \left\{\begin{matrix}\text{cl. oriented}\\ \text{3-mfd}\end{matrix}\right\}$$

ribbon Hopf algebra H $\downarrow$ $\qquad\qquad$ $\downarrow$ "finiteness"

Witten-Reshetikhin-T

$$\left\{\begin{matrix}\text{quantum}\\ \text{invar.}\end{matrix}\right\} \xrightarrow{\text{"integrate"}} \left\{\begin{matrix}\text{WRT}\\ \text{Hennings}\\ \text{CGP}\end{matrix}\right\}$$

{logarithmic}

Costantino-Geer-Patureau

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$\left\{\begin{matrix}\text{Habiro}\\ \text{cyclo expansion}\end{matrix}\right\} \xrightarrow[\text{transform}]{\text{Laplace}} \left\{\begin{matrix}\text{unified}\\ \text{invariants}\end{matrix}\right\}$$

(1) framed link: $S^1\times I \sqcup \ldots \sqcup S^1\times I \hookrightarrow S^3$

$f^6 \mapsto 1$

(2) $M = S^3 \setminus N(K) \cup_f D^2\times S^1 = S^3(K_f)$

K1 $\to$  K2:

K2-move

Example:

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(1) ~~Universal link invariant~~

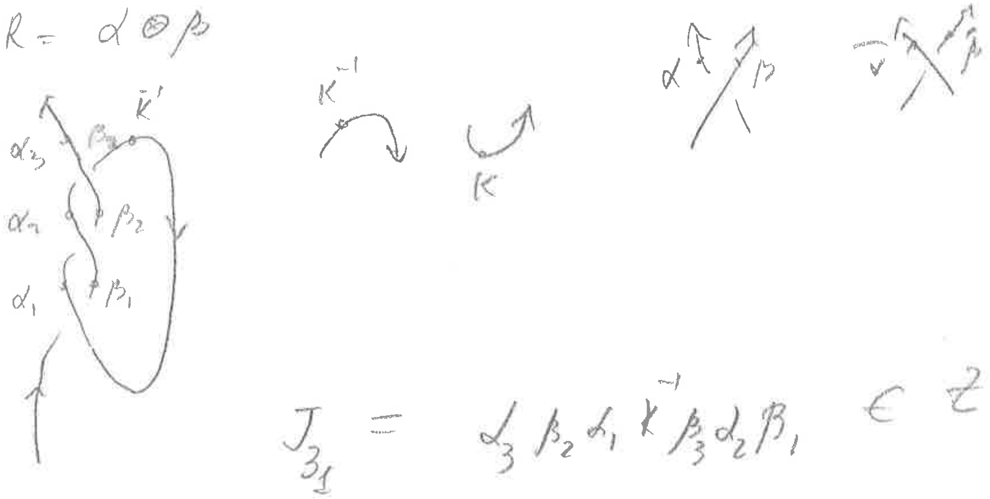
Universal R-matrix

$K = v^H = e^{\frac{h}{2}H}$

$$R = v^{\frac{H\otimes H}{2}} \sum v^{\frac{n(n-1)}{2}} F^{(n)} \otimes e^n \in U_h \otimes U_h$$

$$e = \{1\}E = (v - v^{-1})E \qquad F^{(n)} = \frac{F^n}{[n]!} \qquad [n] = \frac{v^n - v^{-n}}{v - v^{-1}} = \frac{\{n\}}{\{1\}}$$

$R = \alpha \otimes \beta$



$$J_{3_1} = \alpha_3 \beta_2 \alpha_1 K^{-1} \beta_3 \alpha_2 \beta_1 \in Z(U_h)$$

- $Z(U_h)$ is freely generated by

$$C = \{1\}^2 FE + vK + v^{-1}K^{-1}$$

- colored Jones $J_K \circlearrowright V_n$ n-dim irrep $U_q(sl_2)$

$J_K(V_n)$

$C \ni V_n = \sigma^n + \sigma^{-n}$

- Thm (Habiro's cyclotomic expansion)

let K_0 be a 0-framed knot

$$J_{K_0} = \sum_{m=0}^{\infty} a_m \sigma_m \qquad a_m \in \mathbb{Z}[q^{\pm 1}]$$

$$\sigma_m = \prod_{i=1}^{m} \left[C^2 - (\sigma^i + \sigma^{-i})^2 \right]$$

Remarks: ① Proof is by combination of topolog. argument $J_{K_0} \in Z(U^{ev}) \swarrow C^2$ generated

J_{ASL}

② $\exists$ a general. to links, a_m are known only for very few knots and links

Cor: $J_K(V_n) = \sum a_m \prod_{i=1}^{m} (q^n + q^{-n} - q^i - q^{-i}) =$

$$= \sum_{m=0}^{n-1} a_m \prod_{i=1}^{m} \{n+i\}\{n-i\}$$

Example: $J_{4_1} = \sum_{m=0}^{\infty} \sigma_m \qquad J_{3_1} = \sum_{m\geq 0} (-1)^m q^{\frac{m(m+3)}{2}} \sigma_m$

- WRT invariant $\quad M = S^3(K_a) \quad q = \xi \quad \xi^p = 1$

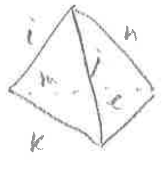
$$q\text{-}\dim V_n = tr^{V_n} K = q^{n-1} + \dots + q^{-n-1} = [n]$$

$$F_{K_a}(\xi) = \sum_{n=0}^{p-1} [n]^2 J_K(V_n)\Big|_{q=\xi} =$$

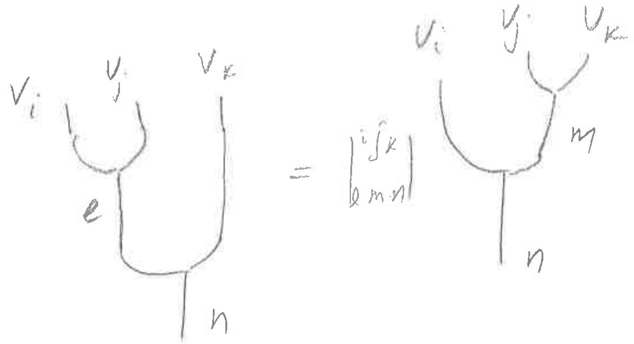
$$= \sum_{n=0}^{p-1} [n]^2 q^{a\frac{n^2-1}{4}} J_{K_0}(V_n)\Big|_{q=\xi}$$

$$WRT(M, \xi) := \frac{F_{K_a}(\xi)}{F_{U_{\text{signal}}}(\xi)}$$

Remark: $\exists$ another construction Turaev–Viro which uses simplicial decomposition, rather than surgery presentation



$$\to \left|\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right|$$



Thm ('94 B-Durhuus, Turaev, Walker)

$$TV(M) = WRT(M \# -M) = |WRT(M)|^2$$

<u>Thm</u>: (Masbaum for prime roots) $\forall M \quad \forall \xi \quad \mathfrak{g} = sl_2$
B-Chen-Le

$$WRT(M) \in \mathbb{Z}[\xi]$$

$$\Leftrightarrow \quad WRT(M) = \sum_{i=0}^{p-1} a_i \xi^i \qquad a_i \in \mathbb{Z}$$

change the root.

a_i are not stable of $\xi \to \zeta$

③ Unification

$$F_{K_a}(\xi) = \sum_{m=0} a_m \sum_{n=0}^{p-1} q^{a\frac{n^2-1}{4}} \left.\frac{\{n+m\} \ldots [n]^2 \ldots \{n-m\}}{\mathbb{Z}[q^{\pm 1}, q]}\right|_{q=\xi}$$

$$\sum_{n=0}^{p-1} q^{a\frac{n^2-1}{4}} q^{bn} = q^{-b^2/a} \gamma a$$

Laplace transform $\mathcal{L}_a(q^{bn}) = q^{-b^2/a}$

$$\sum_m a_m \mathcal{L}_a(\{n+m\} \ldots \{n-m\})$$ does not depend on the root of unity

Thm: (Habiro $a = \pm 1$) Given $M \in \mathbb{Z}HS$

$\exists!\ I_M \in \varprojlim_n \mathbb{Z}[q]/J_n = \widehat{\mathbb{Z}[q]}$

analytic functions on roots of unity

Manin ring of functions on the

$J_n = (1-q) \ldots (1-q^n)$ s.t. $ev_\xi I_M = WRT_\xi(M)$ [order of the elem]

τ

Remarks:

- $\widehat{\mathbb{Z}[q]} \hookrightarrow \mathbb{Z}[[1-q]]$

- $f \in \widehat{\mathbb{Z}[q]}$ is determined by its evaluations in ∞ many points

$\Longrightarrow \{WRT_\xi(M)\}$ is determined

by $\{WRT_\xi(M) \mid \mathrm{ord}_\xi = \text{prime}$
or $p^n, n \in \mathbb{N}\}$
for fixed prime p

- B-Bühler-Le we generalize $M \in \mathbb{Q}HS$
(linking matrix no zero eigenvalue is invertible)