EMILY PETERS: SUBFACTORS AND PLANAR ALGEBRAS, I

In the subject of planar algebra, one can do a lot of math by drawing pictures and reasoning carefully about them. So these talks will have plenty of pictures.

References for today's talk:

- Jones, "Planar algebras I," [Jon99] the original reference.
- The speaker's thesis.
- Heunen and Vicary, "Categories for quantum theory."

Definition 0.1. A *Temperly-Lieb diagram* of size n is an embedding of n disjoint copies of $[0, 1]$ into $[0, n] \times [0, 1]$, such that the boundaries of the embedded intervals lie on integer-valued points.

That is, we take an $n \times 2$ rectangle of points, and draw lines pairing them, such that no two lines cross. We identify two Temperly-Lieb diagrams which are isotopic. Let TL_n denote the complex vector space spanned

FIGURE 1. A Temperly-Lieb diagram in TL_7.

by Temberly-Lieb diagrams of size n. Addition is formal. TL_n acquires an algebra structure by *stacking*: place one diagram on top of another, as illustrated in Figure 2.

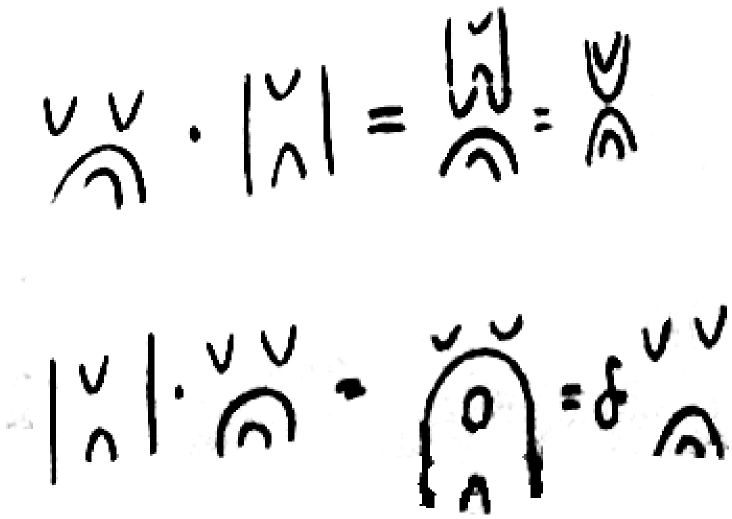

FIGURE 2. Composition of some Temperly-Lieb diagrams.

The identity operator for multiplication is the Temperly-Lieb diagram $||||\dots|$.

This algebra has some additional interesting structure.

- There's a trace $\mathrm{TL}_n \to \mathbb{C}$: given a Temperly-Lieb diagram, close up the embedded intervals in a process akin to a braid closure, then replace each circle with a factor of δ.
- A $*$-structure, by reversing the diagram horizontally.
- This defines a Hermitian form on TL_n, by $\langle x, y \rangle \coloneqq \mathrm{tr}(y^* x)$. This is an inner product if $\delta \geq 2$.

Since the trace depends on δ, we will write $\mathrm{TL}_n(\delta)$ for the Temperly-Lieb algebra with trace given by δ.

There is an embedding $\mathrm{TL}_n \hookrightarrow \mathrm{TL}_{n+1}$, given by adding a single vertical interval on the right-hand side of a diagram. Call the colimit $\mathrm{TL}(\delta)$.

Exercise 0.2. Check that this inclusion respects multiplication, the identity, and the trace, assuming we use the same value of δ in both cases.

This is the basic example of a *planar algebra*. In general, a planar algebra is a collection of vector spaces $V_0, V_1, V_2, \dots$, together with an action by something called the *planar operad*. Fortunately, you don't need to know what an operad is to understand the planar operad. This operad is given by *planar diagrams* (which are also called "spaghetti-and-meatballs diagrams"). These are diagrams of embeddings of compact 1-manifolds inside many-holed annuli, together with marked points on the boundaries of the annuli; the compact 1-manifolds are the spaghetti, and the holes in the annulus are the meatballs. These compose in a manner reminiscent of operator product expansion.

An action of the planar operad means, for each planar diagram, a multilinear map $\bigotimes V_i \to V_0$; we also ask for these maps to be compatible with compositions.

Example 0.3. The Temperly-Lieb algebra is a planar algebra, where the planar diagrams act by insertion. ◀

Example 0.4. The *graph planar algebra* on a simply laced graph Γ takes as its V_n the complex vector space spanned by the set of loops on Γ of length n.

There are a few different ways we can compose loops. Of course, we can concatenate loops with the same origin, as in algebraic topology; but there's another option. Assume both loops are of even length, and let p and p' be their respective halfway points; then, we can define their composition to be 0 if $p \neq p'$, and to be the first half of the first loop, then the second half of the second loop, if $p = p'$. These two composition laws correspond to two planar diagrams, and these should give you the general story.

There's also a trace, where again you close off all boundaries with circles. This procedure is slightly ambiguous, so we simply sum up over all possibilities. ◀

We've seen in a few different talks so far the idea of a monoidal category, along with many variations of their definition.

Definition 0.5. A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and a distinguished object $\mathbf{1} \in \mathcal{C}$ called the *unit*, together with data of an *associator*, a natural isomorphism $(-\otimes-)\otimes- \overset{\cong}{\Rightarrow} -\otimes(-\otimes-)$ and *left and right unitors*, natural isomorphisms $\mathbf{1}\otimes- \overset{\cong}{\Rightarrow} -$ and $-\otimes\mathbf{1} \overset{\cong}{\Rightarrow} -$; these are subject to some coherence conditions.

The point of recalling this definition is that we'll relate it to all the pictures in not just this lecture, but also the other ones this week. This is a point that is often unclear to people — if you already know why you can do diagrammatics for various kinds of categories, it might feel not worth reviewing, but if not, it's certainly confusing.

The idea is, we can draw objects, morphisms, and equations in a monoidal category as diagrams in 2d.

- A morphism $f\colon A \to B$ is a box from a strand labeled by B to a strand labeled by A.
- Composition is stacking vertically.
- The tensor product is stacking horizontally, both of objects and of morphisms.
- The monoidal unit is the empty diagram.

Two diagrams which are related under planar isotopy are considered equal.

Theorem 0.6. *A well-typed equation between morphisms in a monoidal category follows from the axioms of a monoidal category iff it holds true in the graphical language described above.*

As a simple example, how do vertical and horizontal composition (namely, composition of morphisms, resp. tensor product) interact? If you do vertical, then horizontal, or horizontal, then vertical, you get the same diagrams, and therefore they must be equal: given maps $f\colon A \to B$, $g\colon B \to C$, $h\colon D \to E$, and $k\colon E \to F$,

$$(f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k) \tag{0.7}$$

as maps $A \otimes D \to C \otimes F$.

A monoidal category is *rigid* if it has left and right duals for all objects. Evaluation and coevaluation correspond to cups and caps; thus we obtain an identity called the *Zorro diagram* or *snake diagram* allowing us to pull taut a single strand with a cup and a cap. This allows us to freely do planar isotopy.

Now let $\mathcal{C}$ be a rigid monoidal category and $X \in \mathcal{C}$; we will obtain a planar algebra by "zooming in" on this object X. Specifically, take $V_n \coloneqq \mathrm{End}(X^{\otimes n})$. The rest of the data comes from diagrammatics.

Why care? Well, the formalism of planar algebras is different enough from that of monoidal categories to lend different tools to the study of things in their intersections. For example, monoidal categories and planar algebras have different notions of smallness. For example, in a semisimple rigid monoidal category, you might measure the number of simple objects. In a planar algebra generated by X as above, smallness is more traditionally measured with the *Frobenius-Perron dimension.* This can be understood in general semisimple rigid monoidal categories $\mathcal{C}$; it is a map $K_0(\mathcal{C}) \to \mathbb{R}$ which is positive on simple eigenvalues. Specifically, suppose

$$X \otimes Y = \sum c_{XY}^Z Z, \tag{0.8}$$

where the sum is over isomorphism classs of simple objects Z of $\mathcal{C}$; this defines a matrix in the entries X and Y; its Frobenius-Perron eigenvalue is the Frobenius-Perron dimension of X.

References

[Jon99] Vaughan F. R. Jones. Planar algebras, I. 1999. https://arxiv.org/abs/math/9909027. 1