# TERRY GANNON: CONFORMAL NETS, I

Why care about conformal nets? Well, conformal field theory (CFT) is implicitly tied to most of the subjects in this conference, e.g. to a few talks explictly about CFTs later this week, but also relationships with modular tensor categories. Conformal nets are our current best understanding of CFT, and as such are closely related to many other topics present in this conference.

In the last three centuries, physics has given back a great deal to mathematics, first via classical mechanics leading to the study of differential equations (ordinary and partial), and then symplectic geometry; then quantum mechanics and its ramifications in functional analysis; and recently, the still ongoing mathematical understanding of quantum field theory (QFT). We are barely scratching the surface, and the mathematical understanding of quantum field theory is promising to be a much deeper gift to mathematics than classical mechanics. Witten wrote around the turn of the century that understanding QFT will be a distinguished feature of $21^{\text{st}}$-century mathematics.

Quantum field theory is very general. We will study a very special, simple case: quantum field theories in dimension $1+1$ (i.e. one dimension each of space and time) which are conformally invariant. Conformal invariance is a strong condition to impose on a QFT, and we will be rewarded with nice properties and interesting examples.

The Wightman axioms lead to a focus on quantum fields, which when applied to $(1+1)$-dimensional CFT lead to a axiomatization of CFTs through vertex operator algebras. This is different, almost rival, to the perspective of conformal nets we will discuss today. Heisenberg argued that, since quantum fields aren't physically observable objects, we shouldn't focus on them, and instead we should axiomatize the observables, those things that one can actually (in principle) measure in a physical theory of the universe. This leads to the Haag-Kastler axioms for QFT, and when we implement this for CFT, we will see conformal nets.

In classical mechanics, the state of a system is a point in a phase space, which is a symplectic manifold. Observable data, such as the position, momentum, etc., of particles, are functions on phase space. Quantum mechanics is different. The state of a system is a ray in the phase space, which is a Hilbert space $H$. Observables are Hermitian operators on $H$, such as $(i/\hbar)\frac{\partial}{\partial x}$. Measurements amount to projecting down onto eigenspaces for different operators, and these projections tell you the different probabilities.

To discuss conformal field theories, let's first discuss conformal symmetries, which are symmetries which preserve angles infinitesimally, but might not preserve distances. For example, $z \mapsto z^{-1}$ is a conformal transformation on the Riemann sphere — one says there's a *conformal compactification* of $\mathbb{C}$, which is the Riemann sphere. The story is similar on $\mathbb{R}^{m,n}$, leading to a conformal symmetry group $\mathrm{SO}(m+1, n+1)/\{\pm 1\}$, provided that $m, n \geq 1$ and $m+n \geq 3$.

But we care about $m = n = 1$, in which things change drastically. The conformal compactification of $\mathbb{R}^{1,1}$ $S^1 \times S^1$; we add all possible light rays. The conformal transformations of $S^1 \times S^1$ are huge — this group is $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$: two copies of the orientation-preserving diffeomorphisms of the circle! $\mathrm{Diff}^+(S^1)$ is a Lie group in a suitable infinite-dimensional sense, and its Lie algebra is $\mathrm{Vect}(S^1)$, the Lie algebra of vector fields on the circle.

Quantum field theory is all about representation theory; this is how it relates to tensor categories. So we will be interested in repesentations of the groups and Lie algebras we've seen so far — but since states are rays in $H$, rather than points, the correct notion of representation in this setting is projective representations. The best way to handle these is to pass to a central extension and obtain a *bona fide* representation, and therefore we will see central extensions of $\mathrm{Diff}^+(S^1)$ and $\mathrm{Vect}(S^1)$. This might make contact with familiar mathematics: complexify the Lie algebra and centrally extend, and what you obtain is the *Virasoro algebra*. So the representation theory of the Virasoro algebra, and those representations which lift to representations of a central extension of $\mathrm{Diff}^+(S^1)$, are important in $(1+1)$-dimensional CFT.

Inside $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$, we could consider the somewhat special, finite-dimensional subgroup $\mathrm{SO}(2,2)/\{\pm 1\} \cong \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Another fairly obvious subgroup is the subgroup of rotations, the diagonals in $\mathrm{SO}(2,2)$.

The Virasoro algebra has a nice basis, which is the standard basis that people use when discussing it: there are elements $L_n$ for each $n \geq 0$, and a central element $k$. The Lie bracket is

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} ck, \tag{0.1}$$

where $c$ is some constant, in fact $(m^3 - m)/12$. Since $k$ is central, all other brackets of basis elements vanish.

There is a standard trick in conformal field theory: focus on the two factors of $\mathrm{Diff}^+(S^1)$ separately. This leads to a significant simplification — a *chiral conformal field theory* is a CFT restricted to each factor of $S^1$. This isn't the full story: we'd have to fit the two pieces together into one, in order to understand the full story, but there are reasonable scenarios in which this works well. It doesn't work for everything, but it will work for the examples we focus on.

**Definition 0.2.** A *conformal net* is data of

- a Hilbert space $H$, called the *state space*; and
- for every interval[1] $I \subset S^1$, a von Neumann algebra $A(I)$ of bounded linear operators on $H$, called the *algebra of observables* on $I$,

such that the algebra generated by all $A(I)$s is $B(H)$, and satisfying a crucial axiom called *locality*: if $I_1$ and $I_2$ are disjoint intervals, with $O_1 \in A(I_1)$ and $O_2 \in A(I_2)$, then $[O_1, O_2] = 0$.

For a conformal field theory, we need a projective representation $U$ of $\mathrm{Diff}^+(S^1)$. This will enforce the condition of conformal invariance (well, really covariance): for every $\gamma \in \mathrm{Diff}^+(S^1)$, we get a unitary operator $U(\gamma)$, and we impose as part of the definition of a conformal net that

$$U(\gamma)A(I)U(\gamma)^* = A(\gamma(I)). \tag{0.3}$$

By differentiating, we obtain a representation of the Virasoro algebra. The *Hamiltonian* of the theory is $L_0$. We ask that in the Virasoro representation, $L_0$ is diagonalizable and has nonnegative eigenvalues. These are the possible energies in this theories, so we want these to be nonnegative. There's a final axiom, involving the vacuum.

The easiest way to get your hands on von Neumann algebras is: pick your favorite group $G$, which can be infinite, and a unitary representation $V$, maybe infinite-dimensional. Then you get lots of unitary operators; single out those which commute with the group action, the symmetries of the representation. These form a von Neumann algebra, and, up to isomorphism, all von Neumann algebras arise in this way. If you'd prefer, there's a list of axioms on a $*$-algebra giving the definition of a von Neumann algebra, but it does not get the idea across as effectively.

You might have guessed from the notation that these $A(I)$ form a net: whenever $I_1 \subset I_2$, $A(I_1) \subset A(I_2)$: if you can measure something inside a smaller space(time), you can measure it inside the bigger space(time).

Locality is asking that nothing can travel faster than the speed of light. Two regions which are separated from each other cannot influence each other infinitely fast; you can think of simultaneously performing two experiments in the different regions.

Plenty of thought went into the axioms of a conformal net, but it's clear that there's still a lot of work to do before we get to the level of mathematical comfort with this definition that we're at in, say, symplectic geometry.

**Example 0.4.** The silliest example involves $H = \mathbb{C}$. ◀

**Example 0.5.** A better example is to begin with a vertex operator algebra $A$. The quantum fields in this model of the CFT are the vertex operators, which are operator-valued distributions; hit them with some test function $f(\theta)$ which is supposed inside an interval $I$. After some difficult functional analysis, this gives operators which make up $A(I)$. This is beginning to be understood, thanks to work of Carpi, Kawahigashi, Longo, and Wiener [CKLW18]. ◀

**Example 0.6.** Let $L\mathrm{SU}(n)$ denote the *loop group* of $\mathrm{SU}(n)$, i.e. the infinite-dimensional Lie group of maps $S^1 \to \mathrm{SU}(n)$. If one chooses a good representation of the loop group (this is related to the conditions needed to obtain a modular tensor category of such representations). Then, in a similar way, one can build a conformal net, which was a difficult undertaking by Wasserman and others. ◀

[1] By an *interval* in $S^1$, we mean an open, connected subset.

The axioms of a conformal net are rich enough to produce some interesting phenomena. For example, if $I$ is an interval, the interior of $S^1 \setminus I$, called $I'$, is also an interval, and these two intervals don't overlap. *Haag duality* tells us that these two must commute, and we end up with $A(I) = (A(I'))'$. We also see that $A(I)$ is always a particular kind of irreducible von Neumann algebra (called a *factor*), type $\mathrm{III}_1$. This is a very special type of von Neumann algebra, and we will see some consequences of this next time.

Conformal nets exist so that we can study their representation theory, so let's discuss what a representation is. The definition might not be surprising: a representation $\pi$ of a conformal net $A$ is data of, for each interval $I \subset S^1$, an algebra map $\pi(I)\colon A(I) \to B(K)$, where $K$ is some Hilbert space not necessarily related to $H$. This is required to satisfy some axioms: notably, when $I_1 \subset I_2$, we require $\pi(I_2)|_{I_1} = \pi(I_1)$.

The tautological representation of $A$ acting on itself by the identity map is called the *vacuum representation*. Later, when we see that representations of a conformal net form a tensor category, the vacuum representation will be the tensor unit.

Any representation of a conformal net is automatically compatible with $\mathrm{Diff}^+(S^1)$ in the following sense: given $x \in A(I)$ and $\gamma \in \mathrm{Diff}^+(S^1)$,

$$\pi(\gamma(I))(U(\gamma)xU(\gamma)^*) = U_\pi(\gamma)\pi(I)(x)U_\pi(\gamma)^*. \tag{0.7}$$

Next, we'll discuss how to build a tensor category on the category of such representations; if the CFT has a finiteness condition called *rationality*, it will be a modular tensor category. So a conformal net is a very complicated way to obtain a modular tensor category, and we will discuss this and related questions.

## References


[CKLW18] Sebastiano Carpi, Yasuyuki Kawahigashi, Roberto Longo, and Mihály Weiner. From vertex operator algebras to conformal nets and back. *Mem. Amer. Math. Soc.*, 254(1213):vi+85, 2018. https://arxiv.org/abs/1503.01260. 2