COLLEEN DELANEY: MODULAR DATA AND BEYOND

This talk fits into the same line of ideas as Eric Rowell's, and to some extent also Anna Beliakova's, though here we work in the semisimple case, rather than the non-semisimple one. We always work over the ground field $k = \mathbb{C}$. We will discuss the modular data associated to a modular tensor category, how to think of them as quantum invariants, the role of the S- and T-matrices, and then invariants beyond modular data.

Let $\mathcal{C}$ be a braided fusion category; thus we have data of $\mathcal{C}$, the monoidal tensor product $\otimes$, the unit $\mathbf{1}$, and the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. We assume the braiding is *non-degenerate*, i.e. if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $Y \in \mathcal{C}$, then $X \cong \mathbf{1}$.

We can introduce more niceness into $\mathcal{C}$, in two ways.

- We can consider twists $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$, leading to the notion of a ribbon structure.
- We can consider a pivotal structure $\phi_x : x \overset{\cong}{\to} x^{**}$. This can satisfy a niceness condition called *sphericality*.

If we have both of these, and the nondegeneracy condition, we obtain a modular tensor category. Then the S-matrix is $S_{a,b} = \mathrm{tr}(c_{b,a} \circ c_{a,b})$, as (a, b) ranges over all pairs of isomorphism classes of irreducible objects. The T-matrix is diagonal, with the diagonal entries the twists of the irreducible objects.

Modular tensor categories were first written down by Moore and Seiberg [MS89], who were investigating two-dimensional conformal field theory. To Moore and Seiberg, the S- and T-matrices were part of the data of a modular tensor category, though this is not the only approach. In particular, they extend to a representation of the entire modular group, which is the reason for their name. In this sense, modular tensor categories are a gift from quantum field theory.

Subsequently, people studied many aspects of modular tensor categories; in the 2000s there was strong focus on the invariants they define, and more recently, Bartlett, Douglas, Schommer-Pries, and Vicary [BDSPV15] showed that modular tensor categories are equivalent data to once-extended $(2 + 1)$-dimensional TQFTs.

There was a longstanding conjecture that the modular data is a complete invariant of the modular tensor category. This conjecture was disproven in 2017, and this and subsequent developments will be a significant part of this talk.

As a tool of convenience, we will work with *skeletal modular tensor categories*, in which there is a single object in each isomorphism class. Let $\mathcal{L}$ denote the set of irreducible objects. This dehydrates modular tensor categories to equations: the structure coefficients N_{ab}^c, i.e. the number of copies of c inside $a \otimes b$, where $a, b, c \in \mathcal{L}$; the *F-symbols* $(F_a^{abc})_{ef}$ encoding the associator, and the *R-symbols* R_c^{ab} encoding the braiding. This loses some of the elegance and coherence, but can be useful for throwing mathematical or computer tools at the problem, and this has been a successful technique.

This leads to a graphical calculus for modular tensor calculus. We follow the conventions of Barkeshli-Bonderson-Cheng-Wang [BBCW19], which applies modular tensor categories to physics. Morphisms are represented by braided trivalent ribbon graphs.

The first Reidemeister move is not allowed in this calculus, but the second and third Reidemeister moves hold. Standard results in elementary knot theory implies this defines an invariant of framed links. Our pictures don't make the framing explicit, but we use the *blackboard framing*, which is the usual one you'd have if your links were mostly on the blackboard, with no surprising framing changes at crossings.

Remark 0.1. Since modular tensor categories also define anyon models, you can think of this graphical calculus as describingp processes (such as particle creation or collision) in these physical systems. In this case the invariant we've just described calculates some sort of amplitudes. ◂

Two basic elementary invariants of modular tensor categories are the rank and the *fusion ring* (i.e. the ring generated by the isomorphism classes of simple objects, with fusion as multiplication). These are "classical" (in the mathematical sense), in that the fusion ring is the decategorification of the modular tensor category, and these invariants don't see this categorical structure.

But we also have invariants that the fusion ring doesn't see, such as the *quantum dimension* d_a of an irreducible object a, namely $\mathrm{tr}(\mathrm{id}_a)$, and the *global dimension* D, the square root of the sum of the squares of the quantum dimensions of the irreducibles. We can also evaluate any link whose strands are colored by irreducible objects.

As a simple example, consider the figure-8 colored by an irreducible a, which the graphical calculus assigns an element of $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$, albeit maybe not a fast way to compute it:

$$\mathrm{tr}(c_{a,a}) = \sum_c \frac{d_c}{d_a} R_c^{aa} = \theta_a = T_{aa}. \tag{0.2}$$

So this tells you the T-matrix. In a similar but more complicated way, the graphical calculus tells you that if you label the strands of the Hopf link with a and b,

$$\frac{1}{D}\,\mathrm{tr}(c_{b,a} \circ c_{a,b}) = \frac{1}{D} \sum_c N_c^{ab} \frac{\theta_c}{\theta_a \theta_b} d_c = S_{ab}, \tag{0.3}$$

so you see the S-matrix.

Why so much fuss about modular data? Because all of the invariants we've described so far are determined by modular data, as in the following examples.

- The rank is the dimension of the S- and T-matrix.
- The fusion rules can be recovered from the formula

$$N_c^{ab} = \frac{1}{D} \sum_e \frac{S_{ea} S_{eb} S_{ec}^*}{d_e}. \tag{0.4}$$

  The mantra is "the S-matrix diagonalizes the fusion ring."
- The first column of the S-matrix is the quantum dimensions of the irreducibles, which also determine the global quantum dimension.
- The modular representation (which is a projective representation) is determined by S and T, viz. the generators s and t of $\mathrm{SL}_2(\mathbb{Z})$.
- The *central charge* c satisfies

$$e^{2\pi i c/8} = \sum_{a \in \mathcal{L}} d_a^2 \theta_a. \tag{0.5}$$

  Researchers in modular tensor categories also study higher Gauss sums, which are also determined by modular data.
- Frobenius-Schur indicators, one of the more subtle and bothersome aspects of modular tensor categories, in that they can get in the way of things you think are true but actually aren't. Anyways, they're all determined by modular data.

There are probably still yet other invariants we will apply to modular tensor categories, but which are determined by modular data.

Johnson-Freyd and Scheimbauer [JFS17] write down a 4-category of braided fusion categories. Not all of these are modular, but we can use modular data to understand this 4-category. Another thing we can do is enumerate the modular tensor categories of low rank; there are combinatorial constraints on modular data that shine a lot of light on the low-rank classification (and likewise for a related classification of "super modular tensor categories").

There are various ways to build new modular tensor categories from preexisting ones, such as the Deligne tensor product. In a Deligne tensor product, the S- and T-matrices are tensor products, so the modular data is a strong sign that your category factors as a tensor product. In other settings, it's not as clear how the modular data transforms, but this is something people are working on, and in practice partial information plus the usual combinatorial constraints are often sufficient. Some of these operations include taking the Drinfeld center, and de-equivariantization of a G-action on a modular tensor category.[1] Similarly, there's also G-equivariantization, and combining these is called *gauging*; Cui-Galindo-Plavnik-Wang [CGPW16] showed this sends modular tensor categories to modular tensor categories.

So if you can find additional interesting operations, and understand how the modular data behaves under those operations, that would be excellent! This was how a newer operation called "zesting" was discovered, borne out of attempts to disambiguate modular tensor categories with similar-looking modular data.

[1]This is not always a modular tensor category, but work of Nikshych shines light on when it is.

People are also interested in studying "exotic modular tensor categories," i.e. those not built out of group theory or Lie theory in the usual ways. For example, Grossman-Izumi [GI19] construct modular data from metric groups, and Bonderson-Rowell-Wang [BRW19] discuss realizations of exotic modular data.

So it's difficult to overstate how useful modular data is. But it's not a complete invariant, and this is a recent development! We'll finish by summarizing what we know now, and how life has changed in the aftermath.

Theorem 0.6 (Mignard-Schauenberg [MS17])**.** *g For $G \coloneqq \mathbb{Z}/q \rtimes \mathbb{Z}/p$, where p and q are primes, $p \mid q-1$, and $p > 2$, the modular categories $Z(\mathcal{V}ec_G^\omega)$ are not always determined explicitly by their modular data; an explicit example is $p = 5$ and $q = 11$.*

Lots of insight and brute force went into this discovery, but the five examples for $p = 5$ and $q = 11$ (since $H^3(G; \mathrm{U}_1) \cong \mathbb{Z}/5$) are fairly explicit. These categories are rank 49, which is huge; it's open whether there are smaller examples.

One generalization of the S-matrix is the "punctured S-matrix," corresponding to adding punctures to the torus. This determines the *W-matrix*, whose (a, b) entry is what the graphical calculus assigns to the *Whitehead link* colored by the two irreducible objects a and b.

Theorem 0.7 (Bonderson-Delaney-Galindo-Rowell-Tran-Wang [BDG+18])**.** *The T- and W-matrices can distinguish the modular tensor categories from Theorem 0.6.*

References

[BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. Symmetry fractionalization, defects, and gauging of topological phases. *Phys. Rev. B*, 100:115147, Sep 2019. https://arxiv.org/abs/1410.4540. 1

[BDG+18] Parsa Bonderson, Colleen Delaney, César Galindo, Eric C. Rowell, Alan Tran, and Zhenghan Wang. On invariants of modular categories beyond modular data. 2018. https://arxiv.org/abs/1805.05736. 3

[BDSPV15] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category. 2015. https://arxiv.org/abs/1509.06811. 1

[BRW19] Parsa Bonderson, Eric C. Rowell, and Zhenghan Wang. On realizing modular data. 2019. https://arxiv.org/abs/1910.07061. 3

[CGPW16] Shawn X. Cui, César Galindo, Julia Yael Plavnik, and Zhenghan Wang. On gauging symmetry of modular categories. *Comm. Math. Phys.*, 348(3):1043–1064, 2016. https://arxiv.org/abs/1510.03475. 2

[GI19] Pinhas Grossman and Masaki Izumi. Infinite families of potential modular data related to quadratic categories. 2019. https://arxiv.org/abs/1906.07397. 3

[JFS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. *Adv. Math.*, 307:147–223, 2017. https://arxiv.org/abs/1502.06526. 2

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics*, 123(2):177–254, Jun 1989. 1

[MS17] Michaël Mignard and Peter Schauenburg. Modular categories are not determined by their modular data. 2017. https://arxiv.org/abs/1708.02796. 3