2-Dimensional Topological Field Theories

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Notes taken during talk.
Her notes are appended at the end and problem set.

Folklore Theorem.

2-dim'l topological quantum field theories
$\cong$ comm. Frobenius algebras

NOTE: Here, we take our algebras to be associative and unital.

[Reference: book by Kock]

RHS of thm:

Def. Fix a field k. An algebra A is a vector space over k with a multiplication $\cdot : A \otimes A \to A$

Ex. $M_n(k)$ ($n \times n$ matrices) is a non-commutative algebra

Ex. Let G be a group. Then $kG \ni \sum a_i g_i$ (the group algebra) is a commutative algebra if G is commutative

Def. An algebra is Frobenius if it is finite-dim'l and has a non-degenerate trace $\varepsilon : A \to k$ (equivalently: an associative pairing $\langle , \rangle : A \otimes A \to k$)

For a trace, non-degenerate means no non-trivial left or right ideal in $\ker(\varepsilon)$

$\rightsquigarrow$ Pairing:
$$\begin{tikzcd} A\otimes A \arrow[rr] \arrow[dr, "\cdot"'] & & k \\ & A \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

Def. A pairing is associative if
$$\langle a\cdot b, c\rangle = \langle a, b\cdot c\rangle$$

NOTE. Trace $\rightsquigarrow$ Assoc. pairing
from $\langle\,,\rangle$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & k \\ & 1\otimes A \arrow[ur, "\langle\,,\rangle"'] & \end{tikzcd}$$

non-degen. $\longleftrightarrow$ non-degen

LHS of THM

2-diml TQFTs:

Classification of surfaces:

connected, orientable, compact surfaces are classified (up to diffeomorphism) by their genus & number of boundary components

Define a category 2-Cob w/ objects $\mathbb{N}$ and morphisms $2\text{Cob}(n,m) = \{\text{cobordisms } \coprod_n S^1 \to \coprod_m S^1\}/\cong$

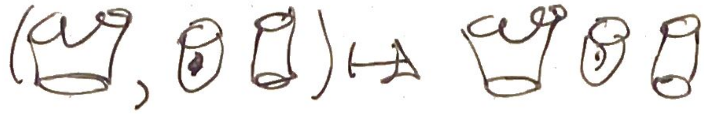
$\cong \{\text{cpct., orientable surfaces w/ } n+m \text{ labelled boundaries}\}/\cong$

Composition is by gluing

2-Cob is a symmetric monoidal category:

i.e. $2\mathrm{Cob} \times 2\mathrm{Cob} \xrightarrow{\oplus} 2\mathrm{Cob}$

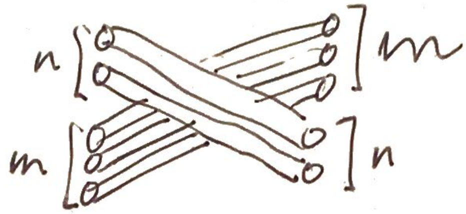
$$(n, m) \longmapsto n+m$$



(disjoint union)

$$(n', m') \longmapsto n'+m'$$

Symmetric: $\sigma_{n,m}$:



Def. A 2-dim'l TQFT (top. quantum field theory) is a symmetric monoidal functor

$$F: (2\mathrm{Cob}, \oplus, \sigma) \longrightarrow (\mathrm{Vect}, \otimes, \tau)$$

s.t. $F(n+m) \cong F(n) \otimes F(m)$

$F(S \amalg T) \cong F(S) \otimes F(T)$

$F(\sigma_{n,m}) = \tau_{F(n), F(m)}$.

Digression: Operads & Props

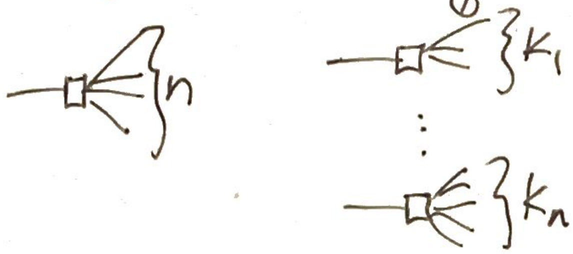
Def. A (topological) operad is a sqnc. of spaces $\mathcal{O}=\{\mathcal{O}(n)\}_{n\geq 0}$ together w/ comp. maps

$\boxed{\circlearrowright_{\Sigma_n}}$ symmetric action

comp maps:

$$\underset{\in}{\mathcal{O}(n)}\times\underbrace{\mathcal{O}(k_1)\times\dots\times\mathcal{O}(k_n)}_{\in}\to\mathcal{O}(k_1+\dots+k_n)$$

s.t. ...



Def: An $\mathcal{O}$-algebra is a sp. for which these operations make sense:

X s.t. $\mathcal{O}(n)\times_{\Sigma_n} X^n \to X$ s.t. ...

Ex. $\mathcal{O}(n)=\{*\}\ \forall n$

$\mathcal{O}$-alg. has $\{*\}\times X^n\to X$

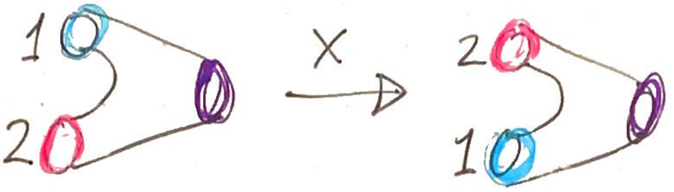
$n=2$ mult, comm

It's the operad of commutative algebras Com

Ex. $\mathcal{O}(n) = \Sigma_n \circlearrowleft \Sigma_n$ mult

is the operad of associative algebras.

Ex. $\mathcal{O}(n) = \Big\{ n\Big[$  $\Big\}\Big/\cong$ $= \{$ connected cpct surface w/ n+1 boundaries $\}/\cong$

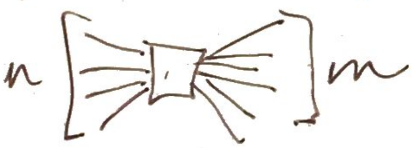


$\exists$ a diffeo taking $1 \to 1$ $\quad 2 \to 2$ $\quad 0 \to 0$

NOTE. $\mathcal{O}_0 \subset \mathcal{O}$ given by genus 0 surfaces only is a suboperad. NOTE: $\mathcal{O}_0(n) = \{*\} \cong$ Com $\Rightarrow$ any $\mathcal{O}$-alg. is a comm. alg.

[Props will relate this back to our theorem.]

Props idea: generalization of operads w/ operations w/ n inputs & m outputs $\forall n, m \geq 0$

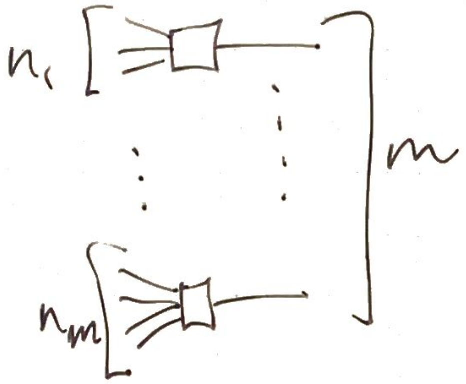


DEF. A prop. is a symm. monoidal category ($\oplus$) w/ objects the natural numbers st. $\oplus = +$ on $\mathbb{N}$

Ex. 2Cob is a prop

Ex. $\mathcal{O}$ operad, then

$$S_{\mathcal{O}} = \begin{cases} \mathrm{Obj} = \mathbb{N} \\ \mathrm{mor}_{S_{\mathcal{O}}}(n,m) = \coprod_{n_1+\dots+n_m=n} \mathcal{O}(n_1) \times \dots \times \mathcal{O}(n_m) \end{cases}$$

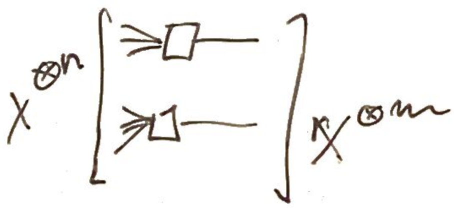


$\oplus = \coprod$ disjoint union

Def. An ***algebra over a prop*** $\mathcal{P}$ is a symm. monoidal functor $F : \mathcal{P} \to (\mathrm{Vect}, \otimes, \tau)$

Lemma. $\{$algebras over an operad $\mathcal{O}\}$ $\cong$ $\{$algebras over the prop $S_{\mathcal{O}}\}$

Pf.
$$F : S_{\mathcal{O}} \xrightarrow{X^{\otimes}} \mathrm{VECT}$$
$$\underset{\underset{1+\dots+1}{\shortparallel}}{n} \mapsto X^{\otimes n}$$



Thm (rephrased). 2Cob is the prop of commutative Frobenius algebras.

Remark. We have seen

$$\begin{array}{ccc} \mathcal{C}omm & \hookrightarrow & \mathcal{O}_{surfaces} \\ & & \downarrow \wr \\ & & S_{\mathcal{O}_{surf}} \hookrightarrow 2\mathcal{C}ob \end{array}$$

(follows from Lemma)

Now we have seen part of the theorem:

$$2D\ TQFT = 2\mathcal{C}ob\text{-alg} \Rightarrow \mathcal{C}om\text{-alg}$$

To get a Frobenius alg: we need a non-degt trace $A \to k$

$F : 2\mathcal{C}ob \to \mathrm{Vect}$ TQFT

$\bigcirc \mapsto k$

$1 \mapsto A$

 $\mapsto [A \otimes A \to A]$

 $\mapsto [A \to k]$

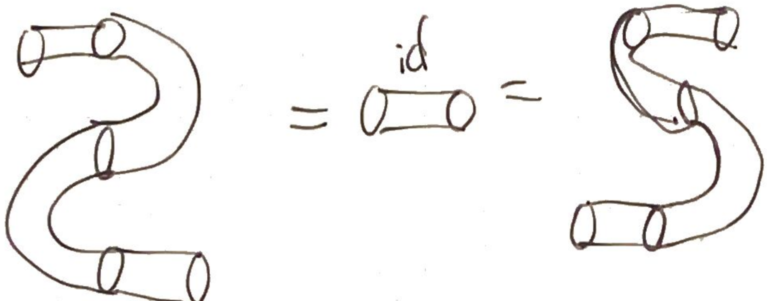
pairing $= F($  $)$

$=$ 

What about other surfaces?

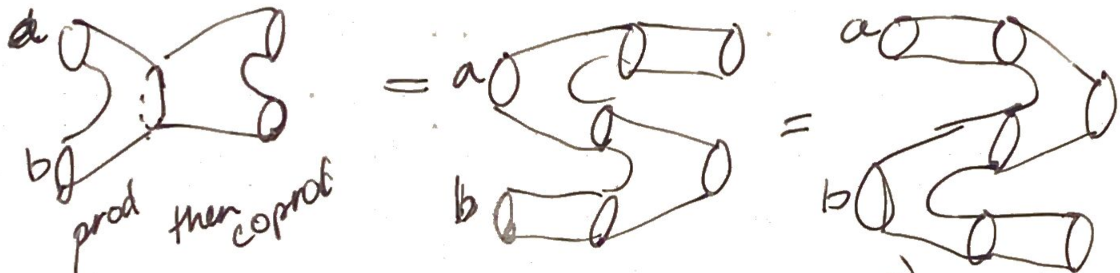
FACTS: (of algebras)

- A pairing is non-degt iff $\exists$ a pairing $[k \to A \otimes A]$ s.t.



(finite dim'l)

- An algebra is Frobenius iff it has a coproduct $[A \to A \otimes A]$ s.t.



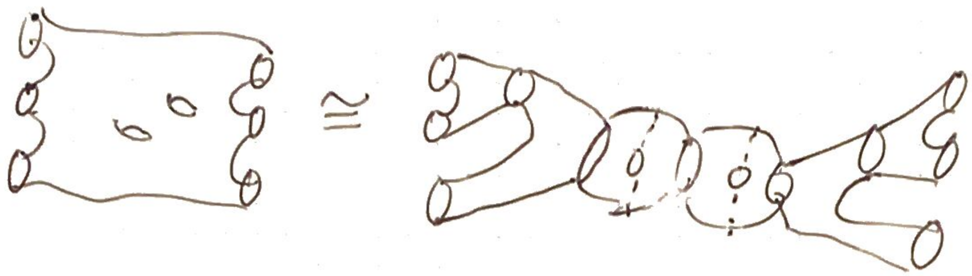
(Frobenius identity)

PROOF OF THEOREM

$$2TQFT \rightleftarrows \text{Comm. Frob. Alg.}$$

$$F \longmapsto (A = F(1), F(\;), F(\;))$$

$$n \mapsto A^{\otimes n} \longleftarrow\!\!\!\!\mid$$



2-dim TOPOLOGICAL QUANTUM FIELD THEORIES

MSRI Feb 20

FOLKLORE THEOREM:

$$\boxed{\text{2d-TQFT} = \text{Comm. FROBENIUS ALGEBRAS}}$$

ALGEBRA = VECTOR SPACE WITH A PRODUCT

COMMUTATIVE IF THE PRODUCT IS COMM.

EX: k, $k[x]$ FOR k FIELD

NON-COMM EX: $M_n(k)$

FROBENIUS = WITH A NON-DEGENERATE TRACE / (ASSOCIATIVE) PAIRING

$\varepsilon : A \to k$

WITH NO NON-TRIVIAL LEFT/RIGHT IDEAL

$$\begin{tikzcd} \langle -,- \rangle : A\otimes A \arrow[dr, "\cdot"'] \arrow[rr] & & k \\ & A \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

s.t.

$$\begin{tikzcd} & A\otimes A\otimes A \arrow[dl, "m\otimes 1"'] \arrow[dr, "1\otimes m"] & \\ A\otimes A \arrow[dr, "\langle\rangle"'] & & A\otimes A \arrow[dl, "\langle\rangle"] \\ & k & \end{tikzcd}$$

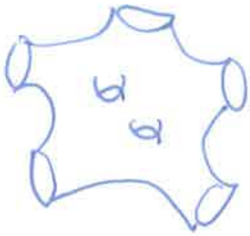
EXERCISE 1) SHOW THAT $M_n(k)$ WITH THE USUAL TRACE IS A NON-COMMUTATIVE FROBENIUS ALGEBRA.

2) LET G BE A GROUP AND kG THE ASSOCIATED GROUP ALGEBRA. CAN kG BE MADE INTO A FROBENIUS ALG?

GRADED EXAMPLE: $A = H^*(M)$ FOR M CLOSED ORIENTED.

$$\varepsilon = \begin{cases} - \cap [M] & \text{IN TOP DEGREE} \\ 0 & \text{ELSE} \end{cases}$$

2-d TQFTs



CLASSIFICATION OF SURFACES:
CONNECTED, ORIENTABLE SURFACES ARE CLASSIFIED BY THEIR GENUS AND NUMBER OF BOUNDARY COMPONENTS.

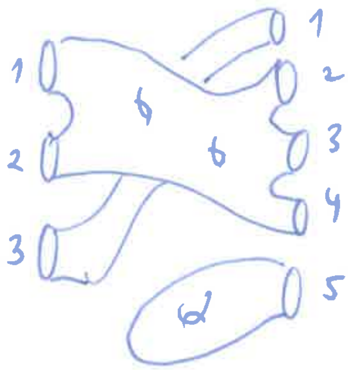
DEFINE A CATEGORY 2-Cob WITH OBJECTS $\mathbb{N}$, WHERE

$n \leftrightarrow \coprod_n S^1$

$0 \leftrightarrow \emptyset$

AND $\mathrm{Mor}_{\mathcal{C}_2}(n, m)$ = COBORDISM $\coprod_n S^1 \xrightarrow{\Sigma} \coprod_m S^1$

= SURFACE Σ WITH $\coprod_{n+m} S^1 \xrightarrow{\cong} \partial\Sigma$



$3 \longmapsto 5$

TODAY: Σ CONSIDERED UP TO DIFFEOMORPHISM REL ∂

$\rightarrow \mathrm{Mor}_{2\mathrm{Cob}}(n, m)$ = SET OF DIFFEO CLASSES

TOMORROW:
REPLACE $\mathrm{Mor}_{2\mathrm{Cob}}$ BY A SPACE OF SURFACES.

COMPOSITION = GLUING OF SURFACES:

IDENTITY = CYLINDER

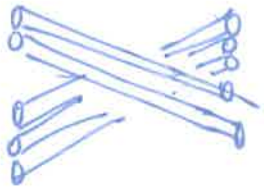
NOTE: 2-Cob IS A SYMMETRIC MONOIDAL CATEGORY:

$2\mathrm{Cob} \times 2\mathrm{Cob} \xrightarrow{\oplus} 2\mathrm{Cob}$ INDUCED BY $\coprod$

$(n, m) \longrightarrow n+m$

$(\Sigma_1, \Sigma_2) \qquad \Sigma_1 \sqcup \Sigma_2$

$(n', m') \qquad n' + m'$

SYMMETRY : $\sigma_{n,m}$: $\begin{matrix} n \\ + \\ m \end{matrix}$  $\begin{matrix} m \\ + \\ n \end{matrix}$

Definition: A 2dim-TQFT is a symmetric monoidal functor $F : (2\mathrm{Cob}, \oplus, \sigma) \longrightarrow (\mathrm{Vect}, \otimes, \tau)$ i.e. a functor such that

$$F(n+m) \cong F(n) \otimes F(m)$$

$$F(\Sigma_1 \amalg \Sigma_2) \cong F(\Sigma_1) \otimes F(\Sigma_2)$$

$$F(\sigma_{n,m}) = \tau_{n,m} : F(n) \otimes F(m) \longrightarrow F(m) \otimes F(n).$$

TWIST MAP.

Exercise: ③ • Show that $H_*: (\mathrm{Spaces}, \times, \tau) \longrightarrow (g\mathrm{Vect}, \otimes, \tau)$

$H'_*: (\mathrm{Spaces}, \amalg, \tau) \longrightarrow (g\mathrm{Vect}, \oplus, \tau)$

are symm. monoidal functors.

• Can you make reduced homology as a sym. monoidal functor?

④ Find a functor $F: \underset{\oplus, \sigma}{2\mathrm{Cob}} \longrightarrow \underset{\otimes, \tau}{\mathrm{Vect}}$ which is not symmetric monoidal.

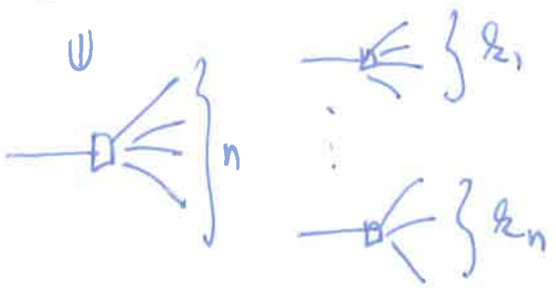
DIGRESSION: OPERADS AND PROPS

DEF: AN OPERAD IS A COLLECTION $\mathcal{O} = \{\mathcal{O}(n) \circlearrowleft \Sigma_n\}_{n \geq 0}$

OF SETS/SPACES/VECTOR SPACES/... (OBJECTS IN A SYMM. MONOIDAL CAT)

TOGETHER WITH COMPOSITION MAPS

$$\mathcal{O}(n) \times \mathcal{O}(k_1) \times \dots \times \mathcal{O}(k_n) \longrightarrow \mathcal{O}(k_1 + \dots + k_n)$$



SATISFYING

- UNITARY ($1 \in \mathcal{O}(1)$)
- ASSOCIATIVITY CONDITIONS.

$\mathcal{O}(n)$ = SPACE/SET OF n-ARY OPERATIONS.

Σ_n PERMUTES THE INPUTS

DEF: AN $\mathcal{O}$-ALGEBRA IS A SPACE/VECT SPACE/... X FOR WHICH THESE OPERATIONS MAKE SENSE, i.e.

$$\mathcal{O}(n) \times_{\Sigma_n} X^n \longrightarrow X \qquad \text{s.t. } \dots$$

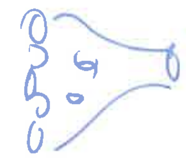
EX: • $\mathcal{O}(n) = \{*\}$ $\forall n$ $\quad \mathcal{O} = \mathcal{C}om$ IS THE COMM. OPERAD.

THEN AN $\mathcal{O}$-ALG IS A COMMUTATIVE ALGEBRA, WITH

$$\{*\} \times_{\Sigma_n} X^n \longrightarrow X \qquad \text{TAKING } (x_1, \dots, x_n) \longmapsto x_1 \dots x_n$$

• $\mathcal{O}(n) = \Sigma_n$ $\forall n$ $\quad \mathcal{O} = \mathcal{A}s$ IS THE ASSOCIATIVE OPERAD

$$(\sigma, x_1, \dots, x_n) \longmapsto x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(n)} .$$

EX: $\mathcal{O}(n) = \left\{ \begin{matrix} \text{CONNECTED} \\ \text{SURFACES WITH} \\ n+1 \text{ BDRY COMP} \end{matrix} \right\} \Big/ \cong \; = \left\{ \right.$  $\left. \right\}$

$n \longrightarrow 1$

$\cup$

$\mathcal{O}_0(n) = \left\{ \begin{matrix} \text{GENUS } 0 \\ \text{SURFACES} \\ \text{WITH } n+1 \\ \text{BDRY} \end{matrix} \right\} = \{*\} = Com(n)$ SUBOPERAD

$\Rightarrow$ AN $\mathcal{O}$-ALGEBRA IS IN PARTICULAR A COMM. ALGEBRA.

EXERCISE 5 WHAT IS AN $\mathcal{O}$-ALGEBRA FOR $\mathcal{O}$ = THE SURFACE OPERAD?

PROP = GENERALISATION OF OPERADS, ALLOWING OPERATIONS $n \longrightarrow m$ FOR ANY n, m, INSTEAD OF JUST $n \longrightarrow 1$.

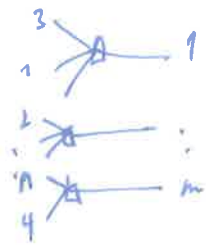
DEF: A PROP IS A SYMMETRIC MONOIDAL CATEGORY WITH OBJECTS $\mathbb{N}$.

⚠ EASY DEFINITION, BUT PROPS ARE MUCH MORE TRICKY THAN OPERADS IN PRACTICE!
eg: $\exists$ FREE ALG OVER ANY OPERAD, BUT NOT OVER PROPS!

EX: • 2Cob IS A PROP.

• ANY OPERAD $\mathcal{O}$ DEFINES A PROP $S_{\mathcal{O}}$ BY SETTING

$$S_{\mathcal{O}}(n,m) = \bigoplus_{k_1+\dots+k_m=n} \mathcal{O}(k_1) \otimes \dots \otimes \mathcal{O}(k_m)$$

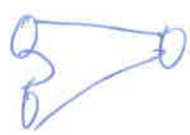


Def: An algebra over a prop $\mathcal{P}$ is a symm. monoidal functor $F : \mathcal{P} \longrightarrow \mathrm{Vect}$ / symm. mon. cat

Exercise 6: For $\mathcal{O}$ an operad, show that $\mathcal{O}$-algebras are the same thing as $S_{\mathcal{O}}$-alg

Thm reformulated: $2\mathrm{Cob}$ is the prop of comm. Frobenius algebras.

Note: We have already seen that the "operadic part" of $2\mathrm{Cob}$ contains Com. Hence $2\mathrm{Cob}$-alg are commutative algebras with mult given by

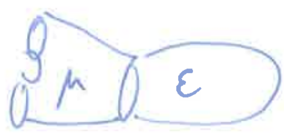


$2 \longrightarrow 1$

Need a trace $1 \longrightarrow 0$ $\qquad V \longrightarrow k$

inner product $2 \longrightarrow 0$ $\qquad V \otimes V \longrightarrow k$, $\mu : V \otimes V \to V$, $\varepsilon : V \to k$

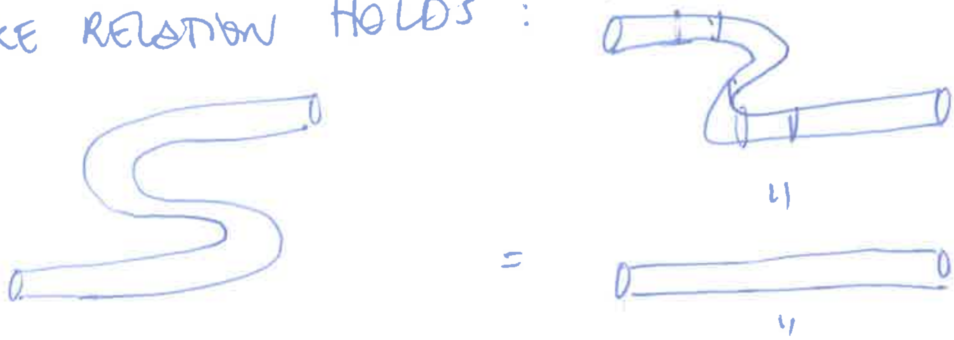
$=$



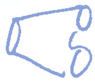
WHAT ABOUT ALL THE OTHER SURFACES?

FACTS OF ALGEBRA:

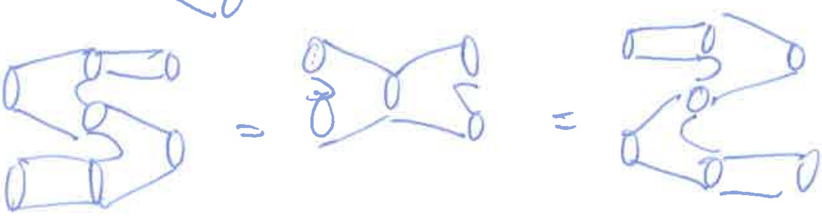
- AN [illegible], PAIRING IS NON-DEGENERATE IFF $\exists$ COPAIRING $k \to V^{*} \otimes V$ S.T. THE SNAKE RELATION HOLDS: $V \hookrightarrow V \otimes V^{*} \otimes V \longrightarrow V$



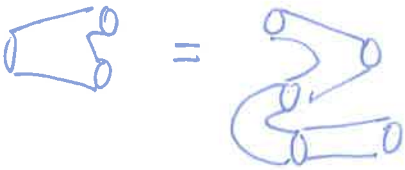
- AN ALGEBRA IS FROBENIUS IFF IT HAS A COUNITAL / COASSOCIATIVE COPRODUCT $A \longrightarrow A \otimes A$ S.T. THE FROBENIUS RELATION



HOLDS:



MOREOVER,



EXERCISE: PROVE SOME OF THIS ?

PROOF OF THE THEOREM

$$\text{2-d TQFT} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \text{Comm. Frob Algebras}$$

<u>DEFINITION OF Φ</u>: LET $F: 2Cob \to Vect$ BE A SYM. MON FUNCTOR.

$\Phi(F) = (A, \mu, \varepsilon)$ FOR $A = F(1)$

$\mu = F($  $)$ $\quad 1 = F($  $)(1)$

$\varepsilon = F($  $)$

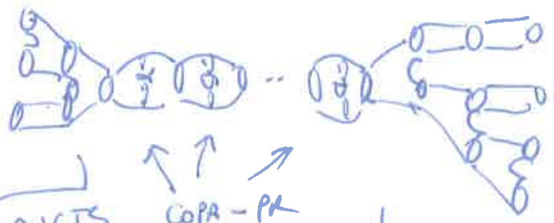
<u>CLAIM</u>: THE RELATIONS OF SURFACES $\Rightarrow (A, \mu, \varepsilon)$ IS A COMMUTATIVE FROBENIUS ALGEBRA.

<u>DEFINITION OF Ψ</u>: LET (A, μ, ε) BE A COMM. FROB ALG.

LET $\Delta: A \to A \otimes A$, $u: k \to A$ BE THE ASS. COPRODUCT AND THE UNIT. DEFINE

$$\Psi(A) : \mathcal{C}_2 \longrightarrow Vect$$

$$n \longmapsto A^{\otimes n}$$

$\Sigma \cong$  $\longmapsto$ CORRESPONDING COMP. IN A.

<u>CLAIM</u>: (MORE DIFFICULT) THIS IS WELL-DEFINED, AND IS A FUNCTOR.

<u>CHECK</u>: Ψ AND Φ ARE MUTUAL INVERSES.

2-d TOPOLOGICAL CONFORMAL FIELD THEORIES

MORE FANCY VERSION OF THE COBORDISM CATEGORY:

$$2\mathcal{C}ob^{top} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = \text{MODULI SPACE OF RIEM. COB.} \end{cases}$$

$$\coprod_n S^1 \xrightarrow{\Sigma} \coprod_m S^1$$

$$\simeq \coprod_{\Sigma} BDiff^+(\Sigma)$$

$\mathcal{M}(\Sigma)$ = {RIEM METRICS ON Σ}/$Diff(\Sigma)$
= SPACE OF CONFORMAL CLASSES OF RIEM. METRICS ON Σ
= BIHOLOMORPHIC CLASSES OF CPX STRUCTURES ON Σ

= ISOMETRY CLASSES OF HYPERBOLIC STRUCT ON Σ
= CLASSIFYING SPACE FOR SMOOTH BDLS WITH FIBER Σ

SPACE OF HIGH INTEREST BECAUSE OF ITS MANY DESCRIPTIONS!

DEF: $2\mathcal{C}ob^{Ch} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = C_*\left(Mor_{\mathcal{C}_2^{top}}(n,m)\right) \end{cases}$

$2\mathcal{C}ob^{H_*} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = H_*(\text{———}) \end{cases}$

SYMM. MON. STRUCT INDUCED BY $\coprod$ AS BEFORE

DEF: A 2dim TCFT (TOPOLOGICAL CONFORMAL FIELD THEORY) is A SYMM MONOIDAL FUNCTOR

$$F: 2\mathcal{C}ob^{Ch} \longrightarrow Ch = \text{Chain CPX}$$

A 2-d HCFT (HOMOLOGICAL CONFORMAL FIELD THEORY) is ...

$$F: 2\mathcal{C}ob^{H_*} \longrightarrow gVect = \text{GRADED VECT SPACES}$$

<u>QUESTIONS</u> 1) WHAT IS THIS, AS AN ALGEBRAIC STRUCTURE?

2) DOES THIS EXIST?

3) IS IT OF ANY INTEREST??

<u>ANSWERS</u>: 1) ?? — WILL SAY SOMETHING IN A BIT.

2) • IT'S LESS "STRICT", SO THERE SHOULD BE MORE EXAMPLES.

• IT'S MORE COMPLICATED, SO THERE SHOULD BE FEWER EXAMPLES!

• THERE ARE A GOOD NUMBER OF EXPECTED EXAMPLES FROM PHYSICS, SYMPLECTIC TOPOLOGY, ... (NOT EASY IN PRACTICE!)

3) • KNOWING THAT A OBJECT ADMITS A STRUCTURE (LIKE A MULTIPLICATION, A TRACE, ...) IS OFTEN USEFUL (eg H^* AND THE CUP PRODUCT, PD, ...)

• <u>ALSO</u>: THE MODULI SPACE ITSELF IS INTERESTING AND A FIELD THEORY GIVEN BY A REPRESENTATION OF ITS HOMOLOGY $\rightsquigarrow$ CAN HELP US TO STUDY THE MODULI SPACE!

<u>SOME ANSWER TO 1)</u>: WHAT IS A TCFT / HCFT?

<u>HCFT</u>: ISSUE: WE DON'T KNOW SO MUCH ABOUT $H_*(\mathcal{M}(\Sigma))$ ... BUT WE DO KNOW SOME THINGS:

1 <u>DEGREE 0</u>: EACH $\mathcal{M}(\Sigma)$ IS CONNECTED SO WE HAVE ONE GENERATOR OF $H_0(2\mathcal{C}ob^{top}(n,m))$ FOR EACH TOP. TYPE OF COBORDISM

$\Rightarrow$ A 2d-HCFT is in particular a comm. Frobenius algebra.

② Genus 0, operadic part $= \mathcal{M}(\quad)$ $\quad n \longrightarrow 1$

$$\mathcal{M}(\quad) = \mathcal{M}\big((OO..O)\big) \simeq BRib_n \simeq fD_2(n)$$

$\qquad$ ‖

Framed little disc operad

[Getzler] $H_*(fD_2(n)) = BV$

= Operad of Batalin-Vilkovisky algebras

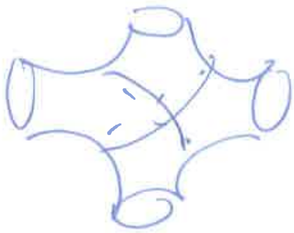
= Comm. Alg with an operator Δ of degree 1 satisfying that $\Delta(a*b*c) = \Delta(a*b)*c + \dots$

(*) $\qquad - \Delta a * b * c - \dots$

$\Delta \Longleftrightarrow$ generator in $H_1(\mathcal{M}(\quad)) \simeq H_1(S^1)$

$(* \Longleftrightarrow$ generator in $H_0(\mathcal{M}(\quad))$ $)$

(*) $\Longleftrightarrow$ "Lantern relation"



And that's all we really understand, (unless we add some assumptions)

eg invertible field theories?

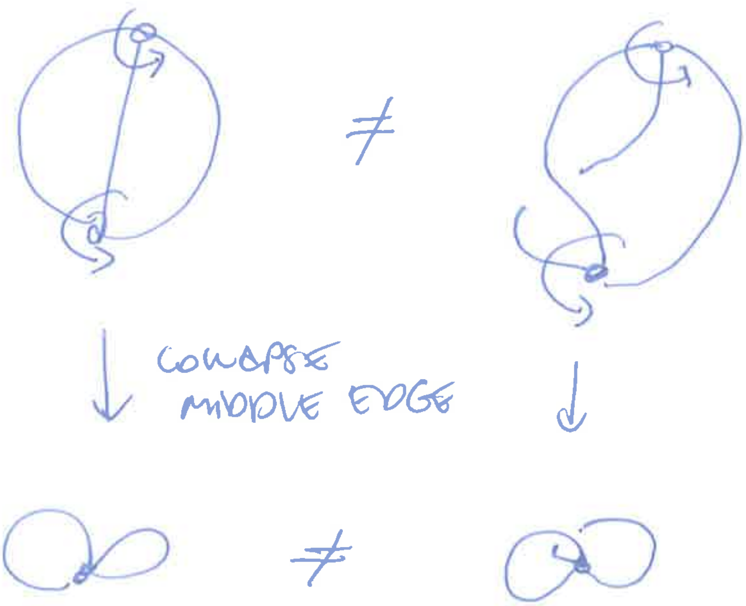
2-d TCFTs

What is $C_*(M(\Sigma))$? A priori this is the singular chains — but that is far too big a model to give a describable structure!

Smaller model: Fat graphs.

DEF: A fat graph is a graph together with a cyclic ordering of the half-edges at each vertex:

Ex:



A fat graph can be fattened to a surface, replacing each edge by a ribbon and each vertex  by a spider 

Ex:  and  ...

Thm (Penner, Harer, ...) Collapsing/expanding edges defines a differential on $\mathbb{Z}\{\text{fat graphs}\}$ and the homology of that chain cpx is M(punctured surfaces)).

- OPEN TCFT'S AND A_∞-ALGEBRAS

- CLOSED TCFT'S AND OUR WIT-PROJECT

Additional Exercises

(A) Show that ~~use the~~ Euler characteristic ~~to~~ defines a functor

$$\mathcal{C}_2 \longrightarrow \mathbb{Z} = \left\{ \begin{matrix} * \\ \mathbb{Z} \end{matrix} \right.$$

Can it be made symmetric monoidal?

[Apparently, $\mathcal{C}_2[\mathcal{C}_2^{-1}] \xrightarrow{\simeq} \mathbb{Z}$ induced by this functor!]

See Søren's notes +
[Juer - Tillmann]

Problem 1. *Show that the $n \times n$-matrices $M_n(k)$ over a field k with the usual trace is a non-commutative Frobenius algebra, i.e. that the trace is non-degenerate in the sense that its kernel has no non-trivial left or right ideal.*

Problem 2. *Let G be a group and kG the associated group algebra. Can kG be made into a Frobenius algebra?*

Problem 3. *Let* Spaces *denote the category of topological spaces and* GVect *the category of graded vector spaces.*

(1) *Show that homology as a functor*

$$H_*\colon (\mathrm{Spaces}, \times, \tau) \to (\mathrm{GVect}, \otimes, \tau)$$

and also as a functor

$$H'_*\colon (\mathrm{Spaces}, \sqcup, \tau) \to (\mathrm{GVect}, \oplus, \tau)$$

is symmetric monoidal.

(2) *Can you make reduced homology into a symmetric monoidal functor between appropriate categories?*

Problem 4. *Find a functor $F\colon (\mathrm{2Cob}, \oplus, \sigma) \to (\mathrm{Vect}, \otimes, \sigma)$ which is not symmetric monoidal.*

Problem 5. *Let $\mathcal{O}$ be the surface operad, with $\mathcal{O}(n)$ the set of topological types of connected surfaces with $n+1$ boundary components. Give a concise description of what it means to be an $\mathcal{O}$-algebra.*

Problem 6. *For an operad $\mathcal{O}$, let $S_{\mathcal{O}}$ be its associated prop. Show that the category of $\mathcal{O}$-algebras is isomorphic to the category of $S_{\mathcal{O}}$-algebras.*

Problem 7. *We have seen that Frobenius algebras could be defined in terms of a non-degenerate trace, an associative pairing, or a coproduct satisfying the Frobenius identity. Prove the equivalence between some of these different definitions of Frobenius algebras.*

Problem 8. *Show that the Euler characteristic defines a functor $\mathrm{2Cob} \to \mathbb{Z}$, seen as a category with one object. Can it be made into a symmetric monoidal functor by choosing appropriately the symmetric monoidal structures on both sides?*