2-Dimensional Topological Field Theories

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[her notes appended at the end]

From Yesterday:

PROP = Symm. monoidal cat. w/ obj. = $\mathbb{N}$

Algebra = Symm. monoidal functor

Operad = collection of Obj. $\{\mathcal{O}(n)\}_{n \geq 0} \curvearrowleft \Sigma_n$

w/ comp. etc.

Algebra is X s.t. $\mathcal{O}(n) \times X^n \to X$ s.t. ....

$\mathcal{O}$ operad $\rightsquigarrow$ Prop $S_{\mathcal{O}}$

Alg $\leftrightarrow$ Alg

Why do we still talk about operads?

Because props are bad!

Ex. Can construct a complicated prop w/ only the trivial algebra!

Contrast. For any operad, $\exists$ a free algebra over it! ☺

Vocab. We say "the operad/prop of XX algebras" meaning "an operad/prop whose algebras are XX algebras" moving on.....

2-dim'l topological (conformal) field theories

More "fancy" version of 2Cob:

$$2\mathrm{Cob}^{top} = \begin{cases} \mathrm{Obj} = \mathbb{N} & (n \leftrightarrow \coprod_n S^1) \\ \mathrm{Mor}(n,m) = \coprod_{\Sigma} \mathcal{M}(\Sigma) \end{cases}$$

Σ: Cob $\coprod_n S^1 \to \coprod_m S^1$

$\mathcal{M}(\Sigma)$ = Modulispace of Riemann surfaces of type Σ

$= \{$Riem. structure of $\Sigma\} / \text{diff.}$

= space of conformal classes of Riem. structures

$\simeq$ isometry classes of cst. structures on Σ (hyperbolic struct)

$\simeq$ biholomorphic classes of cpx. structures

$\simeq$ classifying space for smooth bundles w/ fiber Σ

$\simeq B\mathrm{Diff}^{+}(\Sigma, \partial)$ ...

Def. $2\mathcal{C}ob^{Ch} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = C_*(Mor_{2\mathcal{C}ob^{top}}(n,m)) \end{cases}$

$2\mathcal{C}ob^{H_*} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = H_*(Mor_{2\mathcal{C}ob^{top}}(n,m)) \end{cases}$

Def. A <u>2-dim</u> <u>TCFT</u> (top. conformal field theory) is a symm. monoidal functor

$F : 2\mathcal{C}ob^{C_*} \to Chains$

A <u>2-dim</u> <u>HCFT</u> (homological " " ") is a symm. monoidal functor

$F : 2\mathcal{C}ob^{H_*} \to GVect$

<u>Questions</u>

1. What is this as an algebraic structure?
2. Does it exist?
3. Is it interesting?

<u>Answers</u>

1. Will say later
2.
   - less struct. $\rightsquigarrow$ more examples
   - more complicated $\rightsquigarrow$ fewer examples
   - expected examples from physics, symplectic topo, ...

③ • Having alg. structure is often useful

• A field theory is a repn of $C_*(\mathcal{M}(\Sigma))$ or $H_*(\mathcal{M}(\Sigma))$ $\rightsquigarrow$ learn about $\mathcal{M}(\Sigma)$.

$V^{\otimes n} \longrightarrow V^{\otimes m}$

① What sort of structure is an HCFT?

Degree 0 part (H_0)

$\mathcal{M}(\Sigma)$ connected $\Rightarrow H_0(\mathcal{M}(\Sigma)) = k$

and $2\mathcal{C}ob^{H_0}$ is the linearization of $2\mathcal{C}ob$

$\Rightarrow$ any HCFT is in particular a TQFT

i.e. a commutative Frobenius algebra.

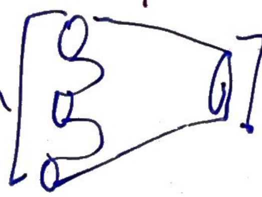
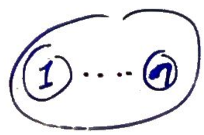
$H_0(\mathcal{M}(\Sigma)) \otimes F(1)^{\otimes 2} \longrightarrow F(1)$

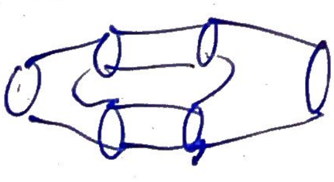
$\Downarrow$

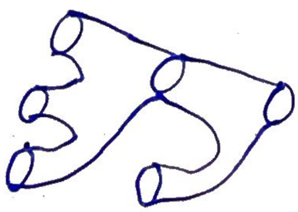
$[*]$ generator

Genus 0 "operadic" part

(or "coop")

Σ of form n  1 $\simeq$ 





LECTURE II (LEFT OVER FROM YESTERDAY)

• PROP = SYMMETRIC MONOIDAL CATEGORY WITH OBJECTS $\mathbb{N}$

ALGEBRA OVER A PROP = SYMM. MONOIDAL FUNCTOR.

• OPERAD = COLLECTION $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ + COMPOSITION MAPS
$\circlearrowright \Sigma_n$

X IS AN $\mathcal{O}$-ALGEBRA : $\mathcal{O}(n) \times X^n \longrightarrow X$ s.t ...

• ANY OPERAD DEFINES A PROP SO THAT THE ALGEBRAS ~~AGREE~~.

Q: WHY DO I / PEOPLE STILL TALK ABOUT OPERADS WHEN PROPS ARE ~~SO~~ MUCH SIMPLER AND MORE GENERAL AT THE SAME TIME ??

ANSWER: BECAUSE PROPS ARE MUCH LESS WELL-BEHAVED THAN OPERADS!

eg: CAN CONSTRUCT A COMPLICATED PROP WITH NO NON-TRIVIAL ALGEBRA

BY CONTRAST: ANY OPERAD HAS A FREE ALGEBRA OVER ANY GIVEN THING!

LANGUAGE: SAYING "THE OPERAD / PROP OF XX ALGEBRAS" IS USED TO SAY "AN OPERAD/PROP WHOSE CATEGORIES OF ALGS IS $\equiv$

⚠ EVEN FOR OPERADS, THIS IS NOT UNIQUE.
(THOUGH FOR OPERADS THERE MIGHT BE A LARGER

2-d TOPOLOGICAL CONFORMAL FIELD THEORIES

MORE FANCY VERSION OF THE COBORDISM CATEGORY:

$$2\mathcal{C}ob^{top} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = \text{MODULI SPACE OF RIEM. COB.} \end{cases}$$

$$\coprod_n S^1 \xrightarrow{\Sigma} \coprod_m S^1$$

$$\simeq \coprod_\Sigma B\mathrm{Diff}^+(\Sigma)$$

$\mathcal{M}(\Sigma) = \{\text{RIEM METRICS ON } \Sigma\}/\mathrm{Diff}(\Sigma)$

= SPACE OF CONFORMAL CLASSES OF RIEM. METRICS ON Σ (UP TO SCALING)

= BIHOLOMORPHIC CLASSES OF CPX STRUCTURES ON Σ

= ISOMETRY CLASSES OF HYPERBOLIC STRUCT ON Σ

= CLASSIFYING SPACE FOR SMOOTH BDLS WITH FIBER Σ

SPACE OF HIGH INTEREST BECAUSE OF ITS MANY DESCRIPTIONS!

DEF: $$2\mathcal{C}ob^{Ch} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = C_*\left(Mor_{\mathcal{C}_2^{top}}(n,m)\right) \end{cases}$$

$$2\mathcal{C}ob^{H_*} = \begin{cases} Obj = \mathbb{N} \\ Mor(n,m) = H_*(\ \text{———}\ ) \end{cases}$$

SYMM. MON. STRUCT INDUCED BY $\coprod$ AS BEFORE

DEF: A 2-dim TCFT (TOPOLOGICAL CONFORMAL FIELD THEORY) is A SYMM MONOIDAL FUNCTOR

$$F: 2\mathcal{C}ob^{Ch} \longrightarrow Ch = \text{Chain CPX}$$

A 2-d HCFT (HOMOLOGICAL CONFORMAL FIELD THEORY) is

$F: 2\mathcal{C}ob^{H_*} \longrightarrow g\mathrm{Vect}$ = GRADED VECT SPACES.

QUESTIONS 1) WHAT IS THIS, AS AN ALGEBRAIC STRUCTURE?

2) DOES THIS EXIST?

3) IS IT OF ANY INTEREST??

ANSWERS: 1) ?? — WILL SAY SOMETHING IN A BIT.
(NO CLEAN ANSWER LIKE IN TQFT CASE YESTERDAY)

2) • IT'S LESS "STRICT", SO THERE SHOULD BE MORE EXAMPLES.
• IT'S MORE COMPLICATED, SO THERE SHOULD BE FEWER EXAMPLES!
• THERE ARE A GOOD NUMBER OF EXPECTED EXAMPLES FROM PHYSICS, SYMPLECTIC TOPOLOGY, ... (NOT EASY IN PRACTICE!)

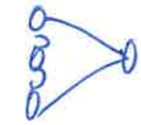
3) • KNOWING THAT A OBJECT ADMITS A STRUCTURE (LIKE A MULTIPLICATION, A TRACE, ...) IS OFTEN USEFUL (eg H^* AND THE CUP PRODUCT, PD, ...)
• ALSO: THE MODULI SPACE ITSELF IS INTERESTING AND A FIELD THEORY GIVEN BY A REPRESENTATION OF ITS HOMOLOGY $\rightsquigarrow$ CAN HELP US TO STUDY THE MODULI SPACE!

SOME ANSWER TO 1): WHAT IS A TCFT/HCFT?

HCFT: ISSUE: WE DON'T KNOW SO MUCH ABOUT $H_*(\mathcal{M}(\Sigma))$ ... BUT WE DO KNOW SOME THINGS:

① DEGREE 0: EACH $\mathcal{M}(\Sigma)$ IS CONNECTED SO WE HAVE ONE GENERATOR OF $H_0(2Cob^{top}(n,m))$ FOR EACH TOP. TYPE OF COBORDISM

$\Rightarrow$ A 2d-HCFT IS IN PARTICULAR A COMM. FROBENIUS ALGEBRA.

② GENUS 0, "OPERADIC" PART $= \mathcal{M}($  $)$

$n \longrightarrow 1$

$\mathcal{M}($  $) = \mathcal{M}\left(\bigcirc\!\!\!\!\!\!\!\!\!\!\!\!\quad 00..0\right) \simeq B\mathrm{Diff}(\ldots) \simeq fD_2(n) = \mathrm{Emb}^f(\coprod_n D^2, D^2)$

$\mathrm{Diff}(D^2, \coprod D^2, \partial) \to \mathrm{Diff}(D^2, \partial) \to \mathrm{Emb}(\coprod_n D^2, D^2)$

$\simeq *$

$\boxed{\text{FRAMED LITTLE DISC OPERAD, COMPOSITION = INSERTION}}$

[GETZLER] $H_*(fD_2(n)) = BV$

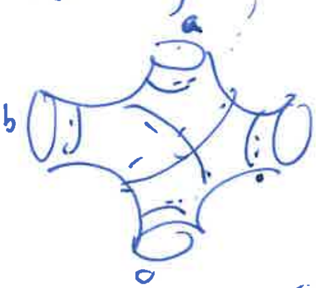
= OPERAD OF BATALIN-VILKOVISKY ALGEBRAS

= COMM. ALG WITH AN OPERATOR Δ OF DEGREE 1 SATISFYING THAT $\Delta(a \bullet b \bullet c) = \Delta(a \bullet b) \bullet c + \ldots - \Delta a \bullet b \bullet c - \ldots$ (*)

$\Delta \Leftrightarrow$ GENERATOR IN $H_1(\mathcal{M}($  $)) \simeq H_1(S^1) \simeq H_1(fD_2(1))$

$\bullet \Leftrightarrow$ GENERATOR IN $H_0(\mathcal{M}($  $))$

(*) $\Leftrightarrow$ "LANTERN RELATION"



$\boxed{\text{+ SAME FOR CO-OPERADIC}}$ 

AND THAT'S ALL WE REALLY UNDERSTAND, (UNLESS WE ADD SOME ASSUMPTIONS)

eg INVERTIBLE FIELD THEORIES?

EX:
- A COMM. FROB ALG (FACTORS THROUGH $H_0(\mathcal{M})$)
- $HH_*(A, A)$ FOR A SYMM. (eg COMM) FROB. ALG

Δ NO TRACE

(FACTORS THROUGH $H_*(\mathcal{M})$ ...) $\Leftrightarrow H_*(\mathcal{M})$

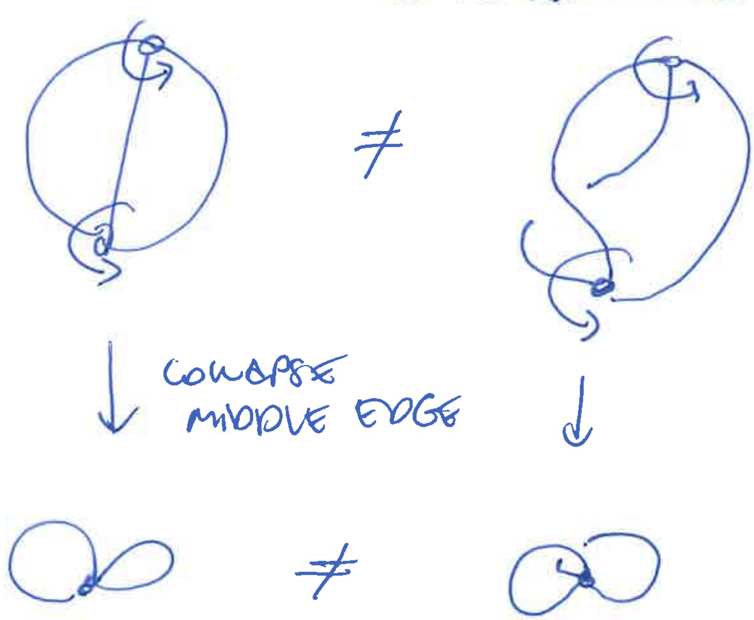
CONNES

WHAT is $C_*(\mathcal{M}(\Sigma))$? A PRIORI THIS IS THE SINGULAR CHAIN — BUT THAT IS FAR TOO BIG A MODEL TO GIVE A DESCRIBABLE STRUCTURE!

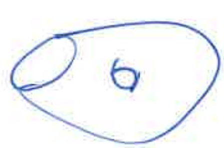
SMALLER MODEL: FAT GRAPHS.

DEF: A FAT GRAPH IS A GRAPH TOGETHER WITH A CYCLIC ORDERING OF THE HALF-EDGES AT EACH VERTEX:

EX:



A FAT GRAPH CAN BE FATTENED TO A SURFACE, REPLACING EACH EDGE BY A RIBBON AND EACH VERTEX  BY A SPIDER 

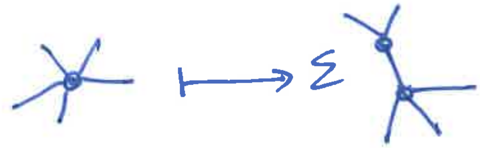
EX:  AND  ...

[illegible] ([illegible]) [illegible] THE HOMOLOGY [illegible]

THERE IS A CHAIN CPX

$$Fat_* = \left(\mathbb{Z}\langle \text{ORIENTED FAT GRAPHS}\rangle,\ d = \sum \text{VERTEX EXPLOSION}\right)$$

DEGREE$(\Gamma) = \sum |v| - 3$



VALENCE ≥ 4

THM [BOWDITCH–EPSTEIN, HARER, PENNER ...]

$$H_*(Fat_*) \cong H_*\left(\coprod \mathcal{M}(\Sigma)\right)$$

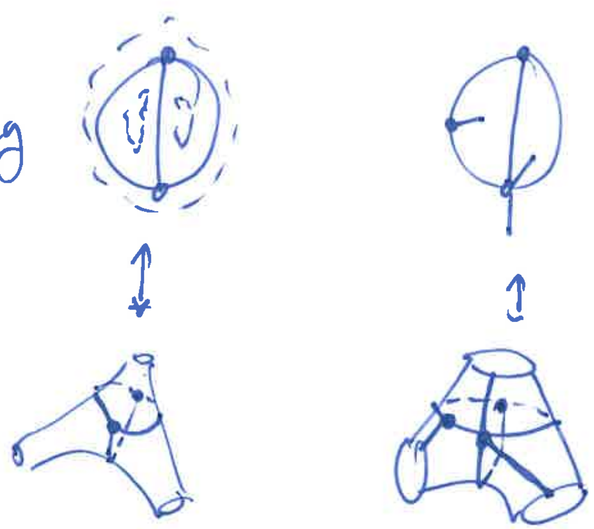
Σ PUNCTURED SURFACE

[PENNER, COSTELLO, GODIN] ADD LEAVES TO BOUNDARY CYCLES

$\updownarrow$

FIXED BOUNDARY CIRCLES

eg



THM: [GODIN, EGAS] GLUING OF SURFACES ALONG BDRY COMPONENTS CAN ALSO BE DEFINED THROUGH A GRAPH MODEL.

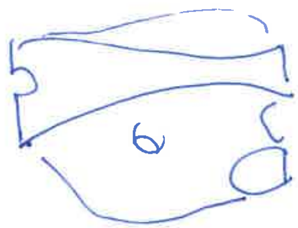
[WIT TOPOLOGY PROJECT: SUBTLETIES OF THIS GLUING]

APPEARANCE OF ANOTHER WELL-KNOWN/INTERESTING OPERAD HERE:

EASIER: OPEN TCFT = SYM. MON. FUNCTORS FROM

$$2\text{-}Cob^{open} = \begin{cases} \mathbb{N} \leftrightarrow \coprod_n I, \quad n \geq 0 \\ Mor(n,m) = C_*(\mathcal{M}(\Sigma)) \end{cases}$$

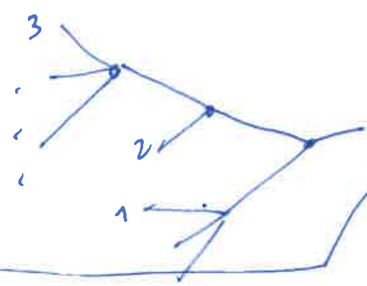
Σ " COBORDISM $\coprod_n I \to \coprod_m I$



MODELED BY FAT GRAPHS WITH LEAVES, GLUING BY ATTACHING LEAVES

GENUS 0 "FLAT" OPERADIC PART = A_∞-OPERAD

TREES = "DISCS"



HH = MACHINE FOR PRODUCING TCFT'S

[COSTELLO, KONTSEVICH-SOIBELMAN]: $HH(A,A)$ is A (2-CLOSED) TCFT, A = OPEN TCFT

[TRADLER-ZEINALIAN, W-WESTERLAND]: $HH(A,A)$ is A "COMPACTIFIED" TCFT, A = SYM. FROB $\equiv$ "$H_0(TCFT)$"

[KLAMT]: $HH(A,A)$ is AN ALG OVER, A = COM. FROB ALG

HARMONIC COMP.

Framed Little Disc operad fD_2

$fD_2(n) = \mathrm{Emb}^{fr}(\amalg_n D^2, D^2)$

$\cong$

$\mathcal{M}(\text{ⓞⓞⓞ})$

And composition in fD_2 (insertion) is compatible with the ~~...~~ composition from $2\mathrm{Cob}^{top}$

<u>Ex.</u> $fD_2(1) \simeq S^1$

$H_1(\mathcal{M}(\text{◎})) \cong k$

$\Rightarrow \Delta = \text{generator}$

[Getzler] $H_*(fD_2) = BV$ (Batalin-Vilkovisky)

$\Rightarrow$ an HCFT is in partical BV-alg

i.e., it has a commutative product (same as for Frob.)

+ a degree 1 operator Δ

s.t. $\Delta(a*b*c) = \Delta(a*b)*c \pm a*\Delta(b*c)$

$\pm \ldots \pm \Delta(a)*b*c \pm \ldots$

[Salvatore-W].

Ex. • A is a commutative Frobenius alg., then it defines a HCFT:

$2Cob^{H_*} \twoheadrightarrow 2Cob^{H_0} \xrightarrow{F} GVect$

• $HH_*(A,A)$ for A symm. (e.g. comm.) Frob. alg. defines a HCFT* with $F(1) = HH_*(A,A)$ where Δ = Connes B-operator.

But where HCFT* is a functor from $2Cob^{H,+}$ = morph only for surfaces w/ non-$\emptyset$ outgoing boundary.

(in particular: No trace!)

In some cases, $HH_*(A,A) \simeq H^*(LM)$ M manifold

Where does this come from?

Comes from TCFTs (sometimes working only after $H_*(\,)$)

2-dim TCFT = $F: 2Cob^{C_*} \to Ch_*$?

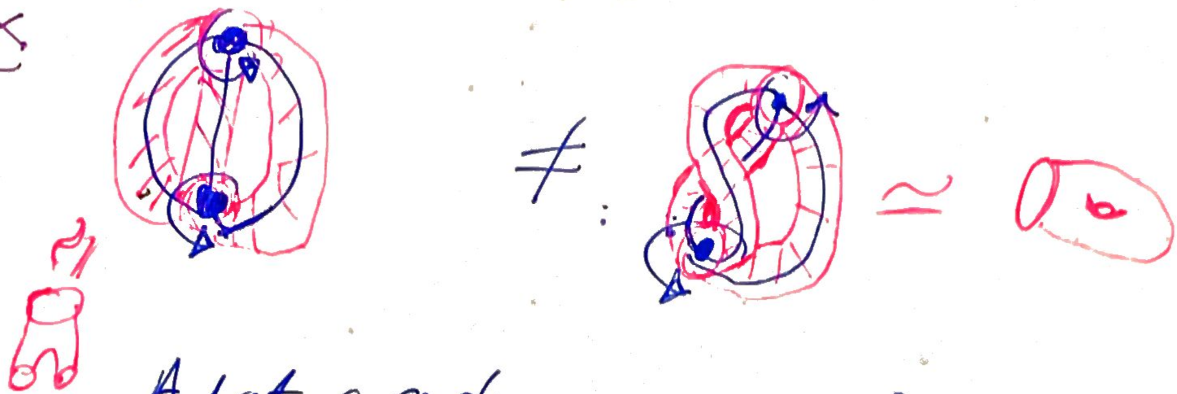
$Mor_{2Cob^{C_*}}(n,m) = \bigoplus_{\Sigma} C_*(M(\Sigma))$

A priori, C_* = singular chains.

HUGE! So we often use a smaller model via graphs.

Def. A *fat graph* is a graph w/ a cyclic ordering of the half-edges at all the vertices

EX.



A fat graph can associate a surface

edge $\rightsquigarrow$ ribbon

vertex $\rightsquigarrow$ disc

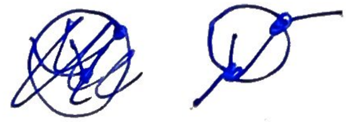
THM. [Bodwitch–Epstein, Penner, Harer]

There exists a chain complex linearly generated by fat graphs (with orientations, differential = vertex explosion) whose homology is that of $\coprod_{\Sigma} \mathcal{M}(\Sigma)$ (moduli spaces) and gluing of surfaces. Can be modelled.

Easier: Open-TCFTs

$2Cob^{open, C_*} = \begin{cases} Obj = \mathbb{N} & (n \leftrightarrow \coprod_n I) \\ Mor(n,m) = \bigoplus_{\Sigma} C_*(\mathcal{M}(\Sigma)) \\ \qquad \Sigma: \coprod_n I \to \coprod_m I \end{cases}$

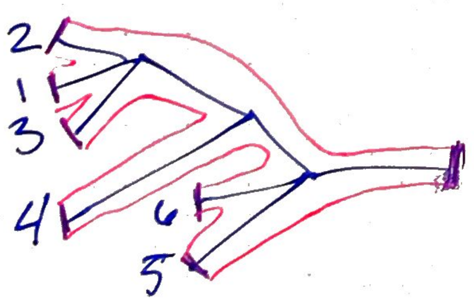
model this with fat graphs w/ leaves



Gluing & attaching @ leaves

Rem: "Genus 0 operadic part" + discs = trees is exactly the A_∞-operad

i.e. an open TCFT is in particular an A_∞-alg.



$6 \longrightarrow 1$

☆ HH = machine for producing 2-d TCFTS ☆

[Costello, Kontsevich-Soibelman]

$A = F(I)$ for A an open TCFT then $HH(A,A)$ is a 2-d TCFT.

[Tradler-Zeinalian, W-Westerland]

A is a symmetric Frobenius alg., $HH_*(A,A)$ is a compactified version of 2d-TCFT.

[Klamt] $HH_*(A,A)$, A comm. Frob. is another flavor of TCFTS

$\boxed{\text{2d TCFTs}}$

$\uparrow$ HH

extends to compactified version

Open TCFTS $\supset$ Symm. Frob. $\supset$ Comm. Frob.

extends to "loop diagrams"

Problem 1. *Show that the $n \times n$-matrices $M_n(k)$ over a field k with the usual trace is a non-commutative Frobenius algebra, i.e. that the trace is non-degenerate in the sense that its kernel has no non-trivial left or right ideal.*

Problem 2. *Let G be a group and kG the associated group algebra. Can kG be made into a Frobenius algebra?*

Problem 3. *Let* Spaces *denote the category of topological spaces and* GVect *the category of graded vector spaces.*

(1) *Show that homology as a functor*
$$H_*\colon (\text{Spaces}, \times, \tau) \to (\text{GVect}, \otimes, \tau)$$
and also as a functor
$$H'_*\colon (\text{Spaces}, \sqcup, \tau) \to (\text{GVect}, \oplus, \tau)$$
is symmetric monoidal.

(2) *Can you make reduced homology into a symmetric monoidal functor between appropriate categories?*

Problem 4. *Find a functor $F\colon (2\text{Cob}, \oplus, \sigma) \to (\text{Vect}, \otimes, \sigma)$ which is not symmetric monoidal.*

Problem 5. *Let $\mathcal{O}$ be the surface operad, with $\mathcal{O}(n)$ the set of topological types of connected surfaces with $n+1$ boundary components. Give a concise description of what it means to be an $\mathcal{O}$-algebra.*

Problem 6. *For an operad $\mathcal{O}$, let $S_{\mathcal{O}}$ be its associated prop. Show that the category of $\mathcal{O}$-algebras is isomorphic to the category of $S_{\mathcal{O}}$-algebras.*

Problem 7. *We have seen that Frobenius algebras could be defined in terms of a non-degenerate trace, an associative pairing, or a coproduct satisfying the Frobenius identity. Prove the equivalence between some of these different definitions of Frobenius algebras.*

Problem 8. *Show that the Euler characteristic defines a functor $2\text{Cob} \to \mathbb{Z}$, seen as a category with one object. Can it be made into a symmetric monoidal functor by choosing appropriately the symmetric monoidal structures on both sides?*