Intro to Higher Cats., Dualizability, & Applications to TFTs.

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Two motivating situations:

① **Def.** Let $\mathcal{M} = \mathrm{Vect}_{\mathbb{C}}$ be a monoidal category.
A vector space V has a (left) <u>dual</u> if $\exists\ V^\vee$ a vector space, and linear maps

$ev_V : V^\vee \otimes V \longrightarrow \mathbb{C}$ (evaluation)

$coev_V : \mathbb{C} \longrightarrow V \otimes V^\vee$ (coevaluation)

such that:

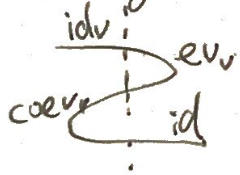
① $V \cong \mathbb{C} \otimes V \xrightarrow{coev_V \otimes id_V} V \otimes V^\vee \otimes V \xrightarrow{id_V \otimes ev_V} V \otimes \mathbb{C} \cong V$

is the identity

② $V^\vee \xrightarrow{id \otimes ev_V} V^\vee \otimes V \otimes V^\vee \xrightarrow{ev_V \otimes id_{V^\vee}} V^\vee$

is the identity

Picture this:



② **Def.** A functor $R : \mathcal{C} \longrightarrow \mathcal{D}$ has a <u>left adjoint</u> if $\exists$ a functor $L : \mathcal{D} \to \mathcal{C}$ and ntrl. transformations

$c : L \circ R \longrightarrow id_{\mathcal{C}}$ (counit)

$u : id_{\mathcal{D}} \longrightarrow R \circ L$ (unit) such that:

① $R = \mathrm{id}_{\mathcal{D}} \circ R \xrightarrow{u \circ \mathrm{id}_R} R \circ L \circ R \xrightarrow{\mathrm{id}_R \circ c} R \circ \mathrm{id}_{\mathcal{D}} = R$

②' $L \xrightarrow{\mathrm{id}_L \circ u} L \circ R \circ L \xrightarrow{c \circ \mathrm{id}_L} L$

are both the identity

NOTE. $(V^\vee, ev_V, coev_V)$ (dualizability data) is unique up to isomorphism.

(L, c, u) same.

Generalize these:

① Just let $\mathcal{M}$ be a monoidal cat.

An object $V \in \mathcal{M}$ .... $\exists\, V^\vee$ obj. in $\mathcal{M}$ & morphisms

$ev_V : V^\vee \otimes V \to \mathbb{1}$ $\quad coev_V : \mathbb{1} \to V \otimes V^\vee$

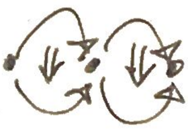
s.t. ....

② First ... concept 2-cat or bicategory (intuition)

objects: $\bullet\ \bullet$ categories

1-morphisms: $\bullet \to \bullet$ functors

2-morphisms  ntrl transformations

compose in 2 ways:  OR 

To generalize ② Let $\mathcal{B}$ be a ~~Bi~~cat, $\mathcal{C}, \mathcal{D}$ ~~$\in$~~ $\mathrm{Ob}\,\mathcal{B}$.

A 1-morphism $R : \mathcal{C} \to \mathcal{D}$ .... if $\exists$ a 1-morph $L : \mathcal{C} \to \mathcal{D}$,

s.t. ......

Today: Use this motivation as a reason for defining

- bicategories
- "higher" categories
- some $(\infty,1)$-categories

$\rightsquigarrow$ Dualizability more generally.

Tomorrow. Applications to TFTs:

- (fully) extended TFTs
- Cobordism Hypothesis.

Lets start with enriched categories.

Let $(\mathcal{S}, \otimes)$ be a monoidal category

(ex: $(\mathrm{Set}, \times)$, $(\mathrm{Cat}, \times)$, $(\mathrm{Space}, \times)$, $(\mathrm{Vect}_k, \otimes)$)

Def. An $\mathcal{S}$-enriched category $\mathcal{C}$ consists of

(O) set of objects $\mathrm{Ob}\,\mathcal{C}$

(M) $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$, an object in $\mathcal{S}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ "morphisms"

(C) composition: $\forall X, Y, Z$ in $\mathrm{Ob}\,\mathcal{C}$, a morphism in $\mathcal{S}$ $\mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$,

("$g \otimes f \mapsto g \circ f$")

(1) $\forall\ X \in \mathrm{Ob}\,\mathcal{C}$, a morph. in $\mathcal{S}$

$\mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(X,X)$ satisfying associativity

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{C}}(Z,W) \otimes \mathrm{Hom}(Y,Z)) \otimes \mathrm{Hom}(X,Y) \arrow[r, phantom, "\cong"] \arrow[d, "\circ\otimes id"] & - \otimes (- \otimes -) \arrow[d, "id\otimes\circ"] \\
& \mathrm{Hom}(Z,W) \otimes \mathrm{Hom}(X,Z) \arrow[d, "\circ"] \\
\mathrm{Hom}(Y,W) \otimes \mathrm{Hom}(X,Y) \arrow[r, "\circ"] & \mathrm{Hom}(X,Y)
\end{tikzcd}$$

(2) similar for identities

<u>Def</u>. A <u>2-category</u> is a cat. enriched in $(\mathrm{Cat}, \times)$

What does this mean?

$\forall\ X, Y$, we have <u>category</u> $\mathrm{Hom}_{\mathcal{C}}(X,Y)$

Ob : 1-morphisms

morphs : 2-morphisms

comp: 

But we <u>also</u> have comp (C) which is a functor of  categories

Example CAT 2-category
obs: categories
1morphs: functors
2 morphs: ntrl transformations.

PROBLEM! Associativity

We require the diagram to commute.
often not satisfied i
$\rightsquigarrow$ we weaken this condition.

Def. A bicategory is (O), (M), (C), (I) + invertible 2-morphism α in (1)
+ similarly modify for (2)
satisfying axioms.

Axioms. pentagon and two triangle identities

Remark. Every bicat is equiv. to a strict one. (2-cat)

Examples:

① ALG^{bi}: obs: $\mathbb{C}$-algebras $\ni A, B$
1mos: $A \to B$ $\approx$ (A,B)-bimodule ${}_A M_B$
2mos: homom. of bimodules

$${}_A M_B \xrightarrow{f} {}_A N_B$$

(C) comp: ${}_B N_C \circ {}_A M_B := {}_A (M \otimes_B N)_C$

② SPAN_2^{bi} 2-spans of sets

obs: sets $\ni S, T$

1mos: $S \to T$ is $(f,g): S \xleftarrow{f} X \xrightarrow{g} T$

2mos:

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
S & \Sigma \arrow[u, "a"'] \arrow[d, "b"] \arrow[l, phantom, "\circlearrowright"] \arrow[r, phantom, "\circlearrowright"] & T \\
 & Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

isom class of

isom: $\Sigma \to \Sigma'$ making diagram commute

comp: Assume we have chosen pullbacks

$$\begin{tikzcd}
 & & X \times_T X' \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dll, bend right] \arrow[drr, bend left] & & \\
S & X \arrow[l] \arrow[r] & T & X' \arrow[l] \arrow[r] & U
\end{tikzcd}$$

The two comps of 2-morphs are also given by pulling back

③ (informally. For details see Schommer-Pries)

$2Cob^{ext}$

obj: finite sets of points (0-dim'l manifolds)

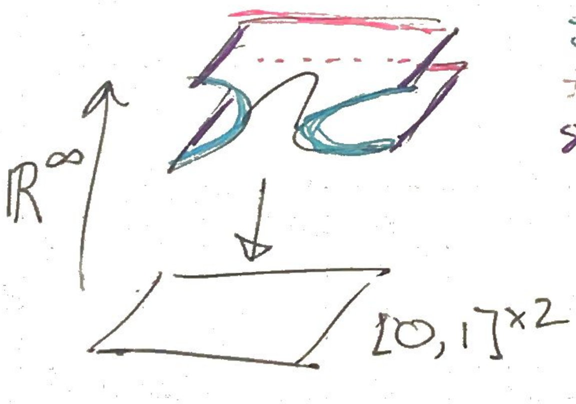
1mos: 1-dim'l manifold M w/ bdry ∂

· together w/ diffeo $\partial M \cong \underset{\text{source}}{X} \sqcup \underset{\text{target}}{Y}$

2mos: isomorphism classes of 2 dim'l "Bordisms"

2-dim'l "bordism": 2 dim'l manifold w/ corners

which can be embedded into $[0,1]^2 \times \mathbb{R}^\infty$



source: front X

target: back ⊐⊐

sides are id bordisms

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Now our def. of having adjoints makes sense.

Q: Can we "combine" having duals and having adjoints?

"Def." Let $\mathcal{B}$ be a symmetric monoidal bicategory. (so left & right are same.) (*def of this is too long.)

A object X in $\mathcal{B}$ is 2-dualizable if:

① it has a dual in the underlying monoidal cat.

② An ev_X, $coev_X$ from ① both have left & right adjoints

What is the underlying monoidal category?

$\mathcal{B}$ bicat $\rightsquigarrow h_1(\mathcal{B})$ $= \tau_1(\mathcal{B})$

obj = ob $\mathcal{B}$

mor = 2-isomorphism (inv. 2 morphs) classes of 1-morphs

So, if $\mathcal{B}$ is symmetric monoidal, then $h_1(\mathcal{B})$ is symmetric monoidal.

Q: Can we do the "same" to get "n-categories"?

$\boxed{\text{Want:}}$ structure which captures
objects, 1-morphs, 2-morphs, ...
... n-morphs, ...?

Attempt 1. "strict" n-cats

cat enriched in (2Cat, ×)

Problem: *strict associativity ῑ ῑ

* not true that any "weak" n-cat is equiv to strict n-cat

We will return to this tomorrow.

How can we include "k-morphisms ∀ k"
but assume they are all invertible
⇝ "∞-groupoid"

Idea: (Grothendieck)

Given a space X ⇝ $\pi_1(X, x)$ fundamental group

$\pi_1(X,x) \rightsquigarrow \pi_{\leq 1}(X)$

X → $\pi_{\leq 1}(X)$ fundamental groupoid
obs are pts in X
Morphs: homotopy classes of paths

$\pi_{\leq 1}(X) \to \pi_{\leq 2}(X)$

$\pi_{\leq 2}(X)$ fundamental 2-groupoid
obs pts
1mos paths
2mos homotopy classes of path homotopies


reminiscent of comp of bicats

Whatever a good def. of ∞-groupoid is, we should have a "fundamental ∞-groupoid" of a space.

Homotopy Hypothesis (turn above into defn)

Def. An ∞-groupoid is a space.

MSRI Higher Categories and Categorification 2020
Connections for Women

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Introduction to higher categories, dualizability, and applications to topological field theories

This is a long selection of exercises of very different levels and with motivations coming from different areas. I am aware that this list is too long for the problem sessions. Pick the one(s) you find interesting and look up or ask for the precise definitions if needed.

(1) Find the dualizable objects in the following monoidal categories:

(a) vector spaces and direct sum

(b) vector spaces and tensor product

(c) pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product

(d) sets and cartesian product

(e) Span, where objects are sets, a morphism from X to Y is an isomorphism class of spans $X \leftarrow S \rightarrow Y$, composition is pullback, and the monoidal product is the cartesian product

(f) Alg, where objects are $\mathbb{C}$-algebras, a morphism from an algebra A to an algebra B is an isomorphism class of bimodules, composition is relative tensor product,

$$_BN_C \circ_A M_B =_A M_B \otimes_B {}_BN_C;$$

and tensor product over $\mathbb{C}$ as the monoidal structure

(g) nCob and disjoint union

(2) Show that if Z is an n-dimensional topological field theory, then for any closed $(n-1)$-dimensional manifold, $Z(M)$ is finite dimensional.

(3) (a) Show that any small category with a single object is the same data as a monoid.

(b) Let A be a set. Which structure does A need to have so that there is a 2-category with a single object, a single morphism, and A as the set of 2-morphisms?

(4) Look up the details of the definition of a quasi-category. Show the following properties:

(a) Translate the horn-filling conditions for Kan complexes and quasi-categories in dimensions 1, 2, and 3 into categorical content.

(b) Let $\tau_1 \colon sSet \to Cat$ be the left adjoint to the nerve functor, called *homotopy category*. Work out/look up an explicit description of τ_1.

(5) (a) Which 1-morphisms have left and/or right adjoints in the following bicategories or $(\infty,2)$-categories:

(i) Alg^{bi} (*Hint: Look up and use the dual basis lemma from commutative algebra.*)

(ii) Span_2^{bi}

(b) Which objects are 2-dualizable in the following symmetric monoidal bicategories or $(\infty,2)$-categories (we haven't seen these in detail, but try to figure out the pictures):

(i) $2\mathrm{Cob}^{ext}$ and $2\mathrm{Cob}^{ext,fr}$

(ii) Bord_2 and $\mathrm{Bord}_2^{\mathrm{fr}}$