Intro to Higher Cats., Dualizability, & Applications to TFTs.

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Day 2

Yesterday:

* dualizability/adjoints
* bicategories
* ∞-groupoids $= (\infty, 0)$-category

We have k-morphisms for all k

all k-morphisms for $k>0$ are invertible

Def: An ∞-groupoid is a space (i.e. Kan complex)

Now we try to define $(\infty,1)$-category $\mathcal{C}$

- k-morphs for all k
- they are invertible for $k>1$.

How do we do this?

Idea: want category "enriched in" $(\infty,0)$-cat. $\mathcal{C}$

So, for fixed obj., X, Y, we take

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to be an ∞-groupoid (i.e. space)

Def 1. An $(\infty,1)$-category is a category enriched in spaces.

NOTE. We have associativity on the nose!

Ex: (0) any (ordinary) category
View $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ as a discrete space
(1) $2\mathcal{C}ob^{top}$ in Nathalie's talk
(2) Spaces, Chain Complexes (Ch)
These are: take any model cat. It gives a space-enriched category.

Def 2. An $(\infty,1)$-category is a (fibroid) relative category, i.e., it is a pair of categories $(\mathcal{C}, W)$ with $W \subseteq \mathcal{C}$ containing all objects

Ex. (Spaces, w.hom.eq), (Ch, q.iso)

	Pros	Cons
Def 1	natural examples	- doesn't incorporate everything - hard to do cat. theory
Def 2	intuition	hard to do cat. theory

Def 3. An $(\infty,1)$-category is a quasi-category, which is a simplicial set $X_\bullet$ with certain "horn lifting properties"

e.g. lowest cond.

$$\begin{tikzcd}
\Lambda^2_1 = \left(0 \arrow[r] & 1 \arrow[r] & 2 \right) \arrow[r, hook] \arrow[d] & \Delta^2 \arrow[dl, dashed, "\exists \text{ lift (not unique)}"] \\
& & X_\bullet &
\end{tikzcd}$$

simplicial sets

FACT. higher horn lifting conditions imply that there is a contractible space of possible compositions.

Ex. Given a category, its nerve $N\mathcal{C}_\bullet$ is a simplicial set $N\mathcal{C}_0 = \mathrm{Ob}\,\mathcal{C}$

$$N\mathcal{C}_n = \mathrm{mor}\,\mathcal{C} \underset{\mathrm{ob}\,\mathcal{C}}{{}_t\times_s} \mathrm{mor}\,\mathcal{C} \underset{\mathrm{ob}\,\mathcal{C}}{\times} \dots \underset{\mathrm{ob}\,\mathcal{C}}{\times} \mathrm{mor}\,\mathcal{C}$$

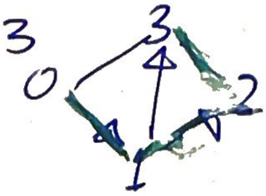
(n composable morps)

is a quasicategory (q-cat)

pro: can do category theory

Def 4. An $(\infty, 1)$-category is a complete Segal space, i.e.

- simplicial space $X_\bullet : \Delta^{op} \to \text{Space}$ $\simeq$ Kan cplxs

s.t. $X_n = X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1 \leftarrow$ $n=3$ 

induced by $[1] \to [n]$

$0 \mapsto i-1 \qquad 1 \le i \le n$

$1 \mapsto i$

is a weak equivalence

Ex. $N\mathcal{C}$ from before :)

- X_0 encodes underlying ∞-groupoid of X

Is the nerve $N\mathcal{C}$ complete?

If in $\mathcal{C}$, every isom. is an identity, yes

Generally, no.

$\rightsquigarrow$ modification Rezk: $N(\mathcal{C}, \mathrm{Iso}\,\mathcal{C})$ instead of $N(\mathcal{C}, W)$

Pros: · useful for bordisms
· has natural generalizations to (∞, n)-cats

Examples Bord_n (Lurie, Calaque-S)

Span_n (Haugseng)

Alg_n - Calaque-S

- Haugseng

- Johnson-Freyd-S

Which definition is the "right" one?

Unicity Thm. Toen, Barwick-Schommer-Pries, many others

Up to an action of $(\mathbb{Z}/2\mathbb{Z})^n$, "all" models of (∞,n)-cats are equivalent.

i.e., model ~~cats~~ / homotopy theories of $(\infty,1)$-cats $\simeq$

☆ So, we can choose whichever one works best in our context ☆

As we return to dualizability, we will use definition 4.

Homotopy Category

$\mathcal{C} \longmapsto h_1(\mathcal{C})$ cat

① cat. enriched in space $\longmapsto \pi_0 \mathrm{Hom}_{\mathcal{C}}(X,Y)$

In any model, $(\infty,1)$-cat $\rightsquigarrow h_1\mathcal{C}$ taking
iso. classes
path. comps.
of morphisms

④ $h_1(X_\bullet)$ obj: underlying set of X_0

Segal space

$\mathrm{Hom}_{h_1(X)}(Z,Y) = \pi_0(\{Z\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{Y\})$

<u>Def./Construction</u> Given a sym. mon. $(\infty,1)$-cat $\mathcal{C}$, the homotopy cat. $h_1(\mathcal{C})$ has a symm. mon. structure (<u>hint</u>. exercise 7 on sheet)

An object X in $\mathcal{C}$ is <u>dualizable</u> if it has a dual in $h_1(\mathcal{C})$.

<u>nCob</u>. <u>objects</u>: closed $(n-1)$-dim'l manifolds

<u>mor</u>: diff. cl. of n-dim'l cobordisms

Without definition, a picture of (∞, n)-cat

objects	Ex	$*$
1-mor		$*$
2-mor		$*$
$\vdots$		$\vdots$
n mor		space A
inv: $\sqrt{}$ m+1-mor		
$\vdots$		

to be an (∞, n)-cat

$\Updownarrow$

A ~~E_n~~ algebra

Back to TFTs:

Def. A fully extended n-diml top. field theory is a symmetric monoidal functor of ~~bi~~categories (∞, n)-cats $Z : \underset{\mathrm{Bord}_n}{\cancel{n\mathrm{Cob}^{ext}}} \longrightarrow \cancel{\mathrm{Vect}} = \mathcal{C}^{\otimes}$ ~~$\mathcal{B}^{\otimes}$ simplicial~~ $\mathcal{C}^{\otimes'}$

From $Z : 2\mathrm{Cob}^{top} \longrightarrow \mathcal{C}$

$(\infty, 1)$-cat

get $Z : h_1(2\mathrm{Cob}^{top}) \longrightarrow h_1(\mathcal{C})$

$h_1(2\mathrm{Cob}^{top}) \cong 2\mathrm{Cob}$ — get back usual defn.

Nathalie: oriented 2d TFTs $\xleftrightarrow{1:1}$ comm. Frobenius algebras

$$Z \longmapsto Z(S^1) = Z(1)$$

Try for n=1: oriented 1d TFTs $\xleftrightarrow{1:1}$ $\mathbb{Z}/$ Vect fin'd of groupoids

$\overset{\shortparallel}{\text{framed}}$

$$\frac{\varphi}{Z} \longmapsto Z(\bullet\!\!\rightarrow)$$

Exercise 2 on sheet: $\Rightarrow Z(\bullet\!\!\rightarrow)$ is finite dim'l vector space.

Conversely, given f.d. v.s. V, can define 1-d TFT w/ $Z(\bullet\!\!\rightarrow) = V$

Notice: Ex. 1.b. $\Rightarrow$ V f.d. iff. V has a dual in (Vect, $\otimes$)

– True in any category (replace Vect$^{\text{f.d.}}$ by $\mathcal{C}^{\text{~~fin dualizable~~}}$ ~~as~~ groupoid.

Thm. (... Harpaz) The same works for fully? extended 1-diml TFTs.

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{fr}, \mathcal{C}) \xrightarrow{\simeq} (\mathcal{C}^{\mathrm{dualiz}})^{\sim}$$

$$Z \longmapsto Z(\to) \quad \text{of } \infty\text{-gps}$$

COBORDISM HYPOTHESIS

Generalizations

$\boxed{\mathrm{I}}$ for bicats $\mathrm{Fun}^{\otimes}(2\mathrm{Cob}^{ext}, \mathcal{B})$

[Schommer-Pries, Pstragowski]

$$\xrightarrow{\simeq} (\mathcal{B}^{2\text{-dualiz}})^{\sim}$$

$\boxed{\mathrm{II}}$ for (∞,n)-cats

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow{\simeq} (\mathcal{C}^{n\text{-dual}})^{\sim}$$

generalization of 2-dualizable

[Baez, Dolan, Lurie, Hopkins-Lurie, Schommer-Pries, Ayala-Francis]

Encodes "locality" TFT is fully local.

Exercise 9: $\mathcal{B} = \mathrm{Alg}^{bi}$

$A = k[G]$ G finite

"Dijkgraaf-Witten" finite gauge theory

MSRI Higher Categories and Categorification 2020 — Claudia Scheimbauer
Connections for Women

Introduction to higher categories, dualizability, and applications to topological field theories

This is a long selection of exercises of very different levels and with motivations coming from different areas. I am aware that this list is too long for the problem sessions. Pick the one(s) you find interesting and look up or ask for the precise definitions if needed.

(1) Find the dualizable objects in the following monoidal categories:

(a) vector spaces and direct sum

(b) vector spaces and tensor product

(c) pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product

(d) sets and cartesian product

(e) Span, where objects are sets, a morphism from X to Y is an isomorphism class of spans $X \leftarrow S \rightarrow Y$, composition is pullback, and the monoidal product is the cartesian product

(f) Alg, where objects are $\mathbb{C}$-algebras, a morphism from an algebra A to an algebra B is an isomorphism class of bimodules, composition is relative tensor product,

$${}_BN_C \circ_A M_B = {}_A M_B \otimes_B {}_BN_C;$$

and tensor product over $\mathbb{C}$ as the monoidal structure

(g) nCob and disjoint union

(2) Show that if Z is an n-dimensional topological field theory, then for any closed $(n-1)$-dimensional manifold, $Z(M)$ is finite dimensional.

(3) (a) Show that any small category with a single object is the same data as a monoid.

(b) Let A be a set. Which structure does A need to have so that there is a 2-category with a single object, a single morphism, and A as the set of 2-morphisms?

(4) Look up the details of the definition of a quasi-category. Show the following properties:

(a) Translate the horn-filling conditions for Kan complexes and quasi-categories in dimensions 1, 2, and 3 into categorical content.

(b) Let $\tau_1 : sSet \rightarrow Cat$ be the left adjoint to the nerve functor, called *homotopy category*. Work out/look up an explicit description of τ_1.

(5) (a) Which 1-morphisms have left and/or right adjoints in the following bicategories or $(\infty, 2)$-categories:

(i) Alg^{bi} (*Hint: Look up and use the dual basis lemma from commutative algebra.)*

(ii) Span_2^{bi}

(b) Which objects are 2-dualizable in the following symmetric monoidal bicategories or $(\infty, 2)$-categories (we haven't seen these in detail, but try to figure out the pictures):

(i) $2\mathrm{Cob}^{ext}$ and $2\mathrm{Cob}^{ext,fr}$

(ii) Bord_2 and $\mathrm{Bord}_2^{\mathrm{fr}}$