An Introduction to Categorification of Quantum Groups & Link Invariants I

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Crane & Frenkel 1994:

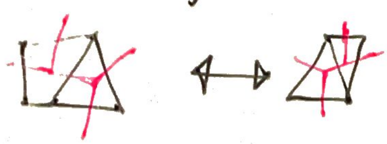
Studying · TQFT (topological quantum field theory)

· invariants of n-manifolds

- take n-manifold & triangulate in

Ex. $n=2$

Basic object:  dual of triangle

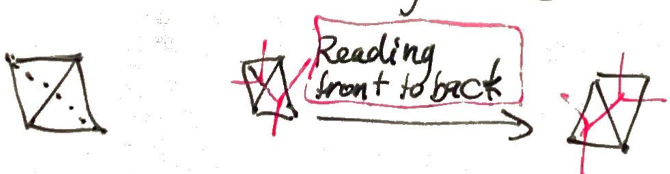
(2-2 move) 

(associativity in schematic mult.)

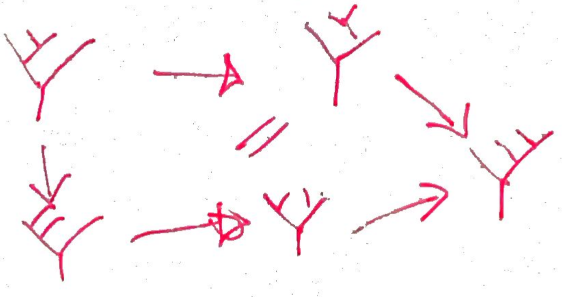
- 2D State Sum TQFT

(linear set + mult. associative)
vect. sp.

- Ex: $n=3$ Basic object is a tetrahedron



(2-3) move for triangulation ~~invariant~~ expressed in terms of dual graphs



- **Observation**: 3D TQFTs requires (linear) cat. & mult.

  4D TQFTs idea: 2-cat & mult

$n=2$	$n=3$	$n=4$
sets	cats	2-cat
mult	mult	mult

- **Examples!** Where do these come from?
  The (linear) categories of interest in 3d TQFTs come from representations of $\underbrace{\text{quantum groups}}_{\text{has pos. basis}}$

★ Case study in categorification ★

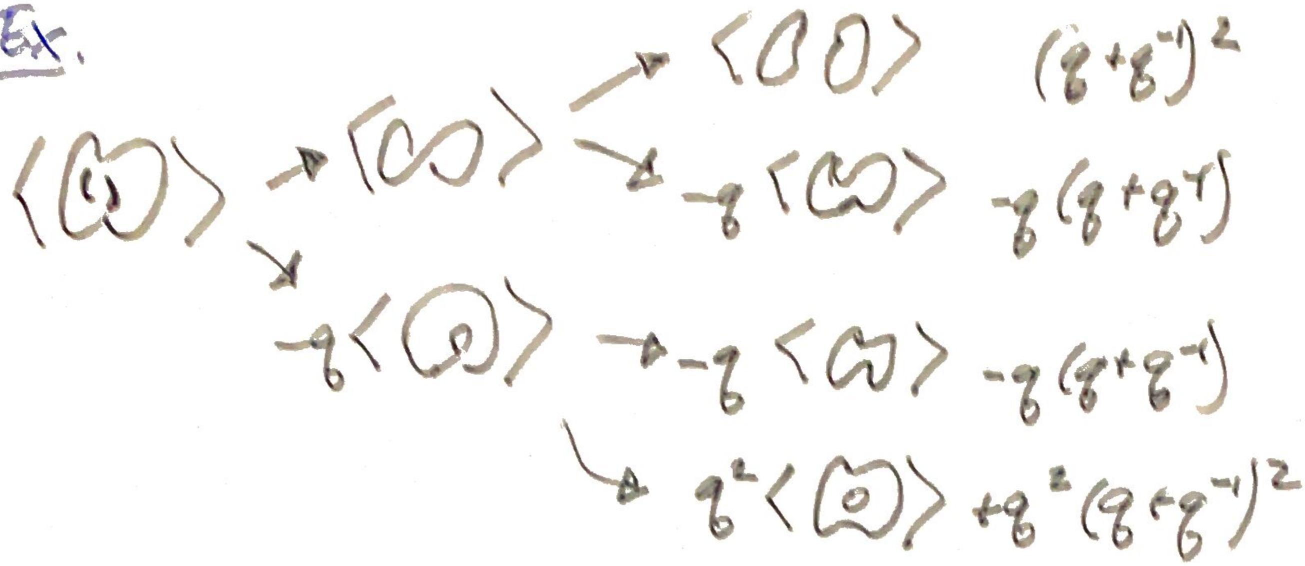
Jones Polynomial $J(K)$

- Take a knot 
- Jones construction produces a Laurent polynom. $\mathbb{Z}[q, q^{-1}]$ (q is a formal parameter)
- Diagramattic (Kauffman Bracket)

$$\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$$

$$\langle O \amalg K \rangle = (q + q^{-1}) \langle K \rangle$$

Ex.



Representation Theoretic Perspective

- Start w/ quantum group $U_q(sl_2)$
  $U_q(sl_2)$ is an algebra over $\mathbb{C}(q)$ (rational fcns in q)
  V = 2-dim'l repns

- Given a local picture of a knot aka a tangle

$t =$  $\mapsto V^{\otimes 5}$

$\uparrow$ map of repns

$\mapsto V^{\otimes 5}$

We can tensor b.c. $U_q(sl_2)$ is a Hopf algebra

- For knots: they have no endpts so

 $\mapsto V^{\otimes 0} = \mathbb{C}[q]$

$\uparrow$ mult by some scalar $= J(K)$

$\mapsto V^{\otimes 0} = \mathbb{C}[q]$

Physics

In SU(2) Gauge Theory

Jones polynomial comes from Wilson loops.
(or homology of a connection around a knot)

Generalizations

- Rep theory: $U_q(\mathfrak{g})$ $\longmapsto$ $\varphi_{\mathfrak{g},V}(k)$

  simple lie algebra

  V - arbitrary rep of $\mathfrak{g}$

  $\boxed{\text{Reshetiken-Turaiev}}$

- Physics: Chern-Simons w/ other Lie groups

- Diagrammatic: Less straight forward $\mathfrak{sl}(n)$
  Requires a detailed knowledge of rep. theory, braidings, or R-matrices. Case-by-case for each.

Diagrammatic description of knot invariants $\longleftrightarrow$ Diagrammatic or gen & rel description of $\mathrm{Rep}(U_q(\mathfrak{g}))$

$\mathfrak{sl}_2$ – Temperly-Leib Algebras (1971)

$\mathfrak{sl}_3$ — Kuperberg (1997)

$\mathfrak{sl}_4$ – Kim (2003)

$\mathfrak{sl}_5$ – Morrison (2007) (conjecture)

Full Solution : 2014
Cautis-Kamnitzer-Morrison

We need two ingredients to understand this:

① Definition of $\dot{U}_q(\mathfrak{gl}_m)$

② Definition of quantum Weyl group actions

Diagrammatic $\mathfrak{sl}_n$-knot invariants.

DEF: $\dot{\mathcal{U}} = \dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ is a $\mathbb{C}(q)$-linear category

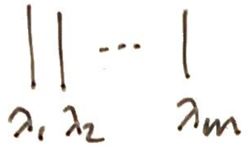
(i.e. $\mathrm{Hom}(\lambda, \lambda') \leftrightarrow \mathbb{C}(q)$-vector space)

<u>Objects</u>: $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$

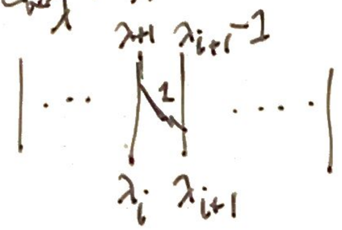
<u>Morphisms</u>: For $1 \le i \le m-1$ $\quad \alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$

i^{th} $\quad (i+1)^{\text{th}}$

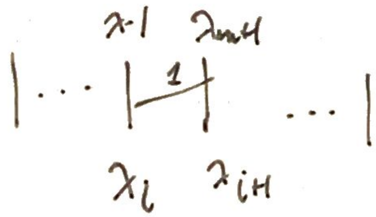
$1_\lambda : \lambda \to \lambda$



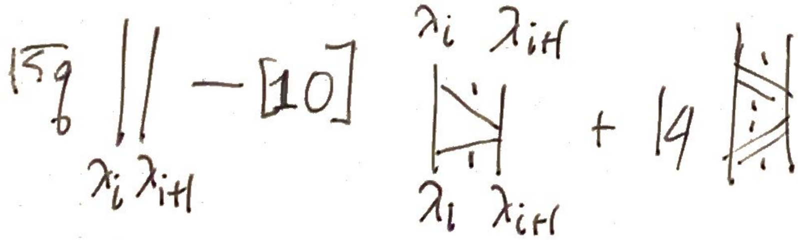
$E_i 1_\lambda : \lambda \to \lambda + \alpha_i$



$F_i 1_\lambda : \lambda \to \lambda - \alpha_i$



eg: $\left[15q 1_\lambda - \left(\frac{q^{10} - q^{-10}}{q - q^{-1}}\right) E_i F_i 1_\lambda + 14 E_i^2 F_i^2 1_\lambda\right] : \lambda \to \lambda$



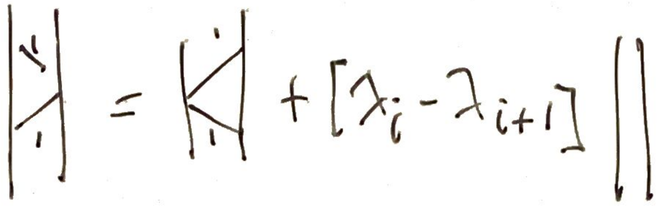
(convention: just draw λ_i, λ_{i+1} since nothing happens elsewhere)

Relations

key: $\mathfrak{sl}_2$-relations

quantum integer $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$

$$E_i F_i 1_\lambda = F_i E_i 1_\lambda + [\lambda_i - \lambda_{i+1}] 1_\lambda$$



Define divided powers:

$$E_i^{(a)} 1_\lambda = \frac{E_i^a 1_\lambda}{[a]!} \qquad [a]! = [a][a-1]\cdots[1]$$

$$F_i 1_\lambda = \frac{F_i^a 1_\lambda}{[a]!}$$

- Serre Relations:

if $j = i \pm 1$: $E_i^{(2)} E_j 1_\lambda - E_i E_j E_i 1_\lambda + E_j E_i^{(2)} 1_\lambda = 0$

$j \neq i \pm 1$ $\quad E_i E_j 1_\lambda = E_j E_i 1_\lambda$

(similar for F_j)