Higher Categorical Traces in geometric representation theory II

Nick Rozenblyum

GOAL. • We want an action of $\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$.

• Want trace of Frob on this action should give "spectral decomposition" of automorphic forms.

We will mostly work in the Betti setting, as this is where the cat. theory is the cleanest.

We saw. In Betti setting:

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{G^\vee}(X)) = \int_{X(\mathbb{C})} \mathrm{Rep}(G^\vee)$$

Consider: $\mathcal{A}$ is a symm. monoidal DG cat.

$Y \in \mathrm{Spc}$ finite CW complex

GOAL: understand $\int_{\mathcal{Y}} \mathcal{A} =: \mathcal{A}^{\otimes \mathcal{Y}}$

(joint work w/ D. Gaitsgory, V. Kazhdan, Y. Varshavsky)

THM. Let $\mathcal{B}$ be a ([symm.] monoidal) DG category. A ([symm.] monoidal) functor $\mathcal{A}^{\otimes \mathcal{Y}} \longrightarrow \mathcal{B}$ is category equivalent to a family of (–·) functors

$$\mathcal{A}^{\otimes I} \longrightarrow \mathcal{B} \otimes \underbrace{\mathrm{Shv}(\mathcal{Y}^I)}_{= \mathrm{Fun}(\mathcal{Y}^I,\, \mathrm{Vect})}$$

ntrl in $I \in \mathrm{Fin}$.

Proof Sketch. As ordinary categories, this is equivalent to:

$$\operatorname*{colim}_{(I \to J) \in \mathrm{Tw}(\mathrm{Fin})} \mathcal{A}^{\otimes I} \otimes \mathrm{Shv}_{le}(\mathcal{Y}^J) \to \mathcal{A}^{\otimes \mathcal{Y}}$$

is an equiv.

As functors of $\mathcal{Y}$, both sides commute with sifted colimits

$\Rightarrow$ enough to check when $\mathcal{Y} = A$, a finite set.

Exercise. Let $\mathcal{C}$ be a category w/ colimits, $\Phi : \mathcal{D} \to \mathcal{C}$ a functor.

For $d \in \mathcal{D}$, the natural map

$$\Phi(d) \longrightarrow \operatorname*{colim}_{(d_1 \to d_2) \in \mathrm{TW}(\mathcal{D})} \Phi(d_1) \otimes_{\mathcal{C}} \mathrm{Maps}(d_2, d_1)$$

is an iso.

(More or less a version of Yoneda)

So we have: Hecke stack. X alg. curve/$\mathbb{C}$

$$\mathrm{Hecke}_I = \{ P_1, P_2, \underline{x} = (x_1, \dots, x_I) \in X^I, \; P_1|_{X-\underline{x}} \simeq P_2|_{X-\underline{x}} \}$$

$$\begin{tikzcd} & \mathrm{Hecke}_I \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times X^I \end{tikzcd}$$

Geometric Satake:

$\exists$ monoidal functors

$$\mathrm{Sat}_I : \mathrm{Rep}(G^\vee)^{\otimes I} \to \mathrm{Shv}(\mathrm{Hecke}_I)$$

natural in $I \in \mathrm{Fin}$.

We obtain a family of functors

$\mathrm{Rep}(G)^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$

natural in $I \in \mathrm{Fin}$.

We have thm (Nadler, Yun): $\exists$ functors

$$\mathrm{Rep}(G^\vee)^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \left[\begin{array}{l}\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\ \otimes \mathrm{Shv}_{lc}(X^I)\end{array}\right.$$

$$\subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$$

$\therefore \exists$ action

$\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}^{\mathrm{Betti}}) \circlearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

Categorical set-up. $\mathcal{A}^{\otimes} \circlearrowright \mathcal{M}$

Want to take trace of Frobenius.

Categorical Traces

Say $\mathcal{O}$ is a symm. monoidal 2-category $\to (\infty, 2)$

$o \in \mathcal{O}$ dualizable object.

$F_o : o \to o$

$\mathrm{tr}(F_o, o) \in \underline{\mathrm{End}(\mathbb{1}_o)}$

it is functorial:

$\mathcal{O}' \in \mathcal{O}$ dualizable object

w/ $F' : \mathcal{O}' \to \mathcal{O}'$

and a map $t : \mathcal{O} \to \mathcal{O}'$ admitting a right adjoint & a 2-morph.

$\alpha : t \circ F \to F' \circ t$.

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "F"] \arrow[d, "t"'] \arrow[dr, phantom, "\swArrow\alpha"] & \mathcal{O} \arrow[d, "t"] \\
\mathcal{O}' \arrow[r, "F'"] & \mathcal{O}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\text{unit}"] \arrow[d, equal] \arrow[dr, phantom, "\swArrow"] & \mathcal{O} \otimes \mathcal{O}^{\vee} \arrow[r, "F \otimes id"] \arrow[d, "t \otimes (t^R)^{\vee}"] \arrow[dr, phantom, "\swArrow"] & \mathcal{O} \otimes \mathcal{O}^{\vee} \arrow[r, "\text{counit}"] \arrow[d, "t \otimes (t^R)^{\vee}"] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, equal] \\
\mathbb{1} \arrow[r, "\text{unit}"] & \mathcal{O}' \otimes \mathcal{O}'^{\vee} \arrow[r, "F' \otimes id"] & \mathcal{O}' \otimes \mathcal{O}'^{\vee} \arrow[r, "\text{counit}"] & \mathbb{1}
\end{tikzcd}$$

Suppose $\mathcal{C} \in DGCat$ is dualizable

$F : \mathcal{C} \to \mathcal{C}$

$Tr(F, \mathcal{C}) \in End(1_{DGCat}) = Vect$

$\;\;\|$

$HH(\mathcal{C}, F)$

Consider a functor

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "t"] \arrow[d, equal] & \mathcal{C} \arrow[d, "F"] \\ \mathrm{Vect} \arrow[r, "t"] \arrow[ur, phantom, "\Nearrow"] & \mathcal{C} \end{tikzcd}$$

$t(k) = C$

__Exercise.__ t has a colimit preserving right adjoint iff $c \in \mathcal{C}$ is cpct.

We have $c \in \mathcal{C}$ cpct

$\alpha : C \to F(c)$

This gives $tr(\alpha) \in HH(\mathcal{C}, F)$.

$$\begin{array}{c} tr(\alpha) \\ \shortparallel \\ cl(c, \alpha) \end{array}$$

__Ex.__ X scheme

$\mathcal{C} = QCoh(X) \quad F = id$

$\mathcal{E} \in QCoh(X)$ perfect complex

$\Rightarrow cl(\mathcal{E}) \in HH(QCoh(X))$

$$\begin{array}{c} HH(QCoh(X)) \\ \text{\scriptsize HKR} \ \cong \\ \bigoplus H^*(X, \Omega^i[i]) \end{array}$$

this gives Chern characters.

- Take $\mathcal{O} = \mathrm{Mor}_2$

  <u>objects</u>: monoidal DG categories

  <u>1-morphs</u>: bimodule categories

  <u>2-morphs</u>: functors

Every object $\mathcal{A} \in \mathrm{Mor}_2$ is dualizable.

Dual: $\mathcal{A}^{rev}$: opposite monoidal structure

$F : \mathcal{A} \to \mathcal{A}$ monoidal functor.

$$\begin{array}{c} tr(F, \mathcal{A}) \in End(\mathbb{1}_{\mathrm{Mor}_2}) \\ \| \\ \mathrm{DGCat} \end{array}$$

$$\begin{array}{c} tr(F, \mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A} \otimes \mathcal{A}^{rev}} \mathcal{A}_F \\ \| \\ \mathcal{HH}(\mathcal{A}, F) \end{array}$$

<u>Functorality?</u>:

$\mathcal{A}, \mathcal{B}$ monoidal cats

${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ $\quad \mathcal{A} \otimes \mathcal{B}^{rev}$: module cat

What does it mean to have a right adjoint?

$\exists\ {}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$ $\quad$ $\mathcal{B} \otimes \mathcal{A}^{Rep}$-module cat

s.t. ....

Prop. If $\mathcal{A}, \mathcal{B}$ are rigid, then ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ admits a right adjoint bimodule iff $\mathcal{M}$ is dualizable as a DG category.

Upshot. If $\mathcal{A}$ is a rigid $\otimes$-cat

$F_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ $\quad$ $\otimes$-functor

$\mathcal{M}$ is an $\mathcal{A}$-module cat. s.t. $\mathcal{M}$ is dualizable w/ a twisted endofunctor

$F_{\mathcal{M}}: \mathcal{M} \to F_{\mathcal{A}}^{*}(\mathcal{M})$

$\Rightarrow tr^{en}_{(\mathcal{A}, F_{\mathcal{A}})}(\mathcal{M}, F_{\mathcal{M}}) \in HH(\mathcal{A}, F_{\mathcal{A}})$