Categorifications & Lie algebra actions on categories arising from representation theory – I

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Motivation/Big Picture:

Knot invariants: (recall Lauda's talks)

$\dot{U}_q(\mathfrak{sl}_m) \curvearrowright V_m^{\otimes n} \curvearrowleft$ braid group

- varying n, this gives invariants of tangles (in particular of knots)
- Skew Howe duality: $\dot{U}_q(\mathfrak{sl}_m) \curvearrowright \Lambda(V_m \otimes V_n) \curvearrowleft U_q(\mathfrak{sl}_n)$

$\Rightarrow$ knot invariants are controlled by Lie algebra $U_q(\mathfrak{g})$ actions!

$\rightsquigarrow$ Categorified knot inv. are controlled by categorified Lie alg. / $U_q(\mathfrak{g})$ action

Have: $\dot{U} \curvearrowright W \leftarrow$ vect. sp. (w/ weight sp. decomp.)

algebra (w/ idempotents) $\cong$ category

$W \cong$ category

categorified: 2-cat $\mathcal{U}$ 2cat

$1_\lambda \longmapsto W_\lambda$

$E_i 1_\lambda \longmapsto e_{i,\lambda} : W_\lambda \to W_{\lambda + \alpha_i}$

THE 2-Category $\boxed{Alg_2(\mathbb{C})}$

<u>Objects</u>: (finite diml) $\mathbb{C}$-~~algebras~~

<u>morphs</u>: $Hom(A,B) = (B,A)$-bimodules ${}_BM_A$

<u>composition</u>: ${}_CN_B \circ {}_BM_A := {}_CN_B \otimes_B {}_BM_A$

<u>Identity morph</u> $id_A = {}_AA_A$

<u>2-morphs</u>: bimodule maps

- Depending on $\mathcal{U}$, want to restrict the obj., 1-mor, 2 morphs

- <u>Assignment</u>. $A \longmapsto A\text{-mod}$

$${}_BM_A \longmapsto {}_BM_A \otimes_A -$$

$$\varphi \longmapsto \text{ntrl. trans.}$$

gives an action of $Alg_2(\mathbb{C})$

Baby Example: $IndRes_2(\mathbb{C})$

<u>Objects</u>: finite groups G (cor. gp. alg. $\mathbb{C}G$)

<u>1-morphs</u>: "compositions of inductions/restrictions"

Zigzags of finite groups:

$$\begin{tikzcd}[column sep=small]
G_n & & \cdots\cdots G_3 & & G_2 & & G_1 = G \\
& H_{n-1} \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] & & H_2 \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] & & H_1 \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

bimod: $\mathbb{C}G_n \otimes_{\mathbb{C}H_{n-1}} \cdots \otimes_{\mathbb{C}H_2} G_2 \otimes_{\mathbb{C}H_1} G_1$ bimod. maps

$\underline{\text{TODAY}}$: Categorify $\underbrace{\dot{U}(\mathfrak{h})}_{\substack{\shortparallel \\ H}}$

$\mathfrak{h}$ = Heisenberg $\overset{\text{lie}}{\text{alg}}$ $\longrightarrow$ $[p,q] = \hbar = c$ ← central

$U(\mathfrak{h})$ = associative alg. $/\mathbb{k}$

· gen'd by P_+, P_-

· relation $P_+P_- - P_-P_+ = 1$

idempotent version:

$\boxed{\dot{H}}$ objects: $\mathbb{N}$

morphs: $n \to m$ span of $P_{\varepsilon_1}P_{\varepsilon_2}\cdots P_{\varepsilon_r}$

where $\sum \varepsilon_i = m-n$

Can also allow infinitely many generators

P_{+i}, P_{-i} $(i \in \mathbb{Z}_{>0})$ w/ relations

$[P_i, P_j] = 0 = [P_{-i}, P_{-j}]$ $\forall\, i,j > 0$

$[P_i, P_{-j}] = \delta_{ij} 1 \sim H_\infty$

Want. $P_+ P_- \mathbb{1}_n - P_- P_+ \mathbb{1}_n = 1\mathbb{1}_n$

categorified ⇝ $P_+ P_- = P_- P_+ \oplus \mathrm{ID}$ where P_-, P_+ are certain bimodules/functors

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Graphical Calculus.

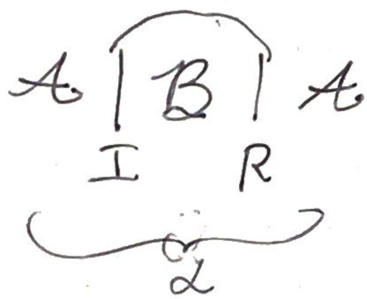
given adjoint functors $\mathcal{A} \underset{R}{\overset{I}{\rightleftarrows}} \mathcal{B}$ (I, R) $\mathcal{A}, \mathcal{B}$ abelian categories

ex. $(I = \underbrace{\mathbb{C}[G] \otimes_{\mathbb{C}[H]} -}_{\text{induction}}$, $R = \mathbb{C}[G] \otimes_{\mathbb{C}[G]} -)$

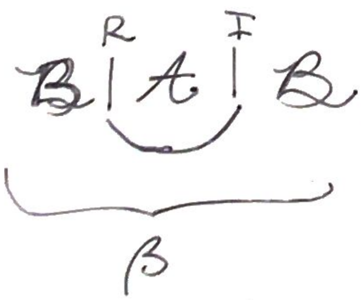
$\underbrace{\mathbb{C}[H]}_{\text{restriction}}$ $H < G$

⇝ adjunction maps $\alpha : IR \to \mathrm{ID}$

$\beta : \mathrm{ID} \to RI$

picture: 

← Read bottom → top
right to left

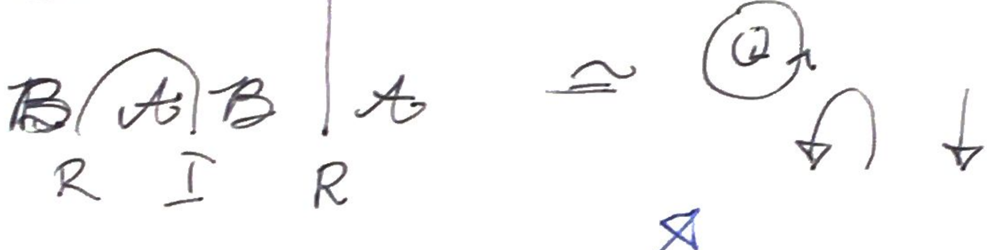


If the functors are biadjoint, have also

$\bar{\alpha} : RI \to ID$, $\bar{\beta} : ID \to IR$

Ex. for natural transformation:

ntrl. transf. of id. functor



Shortcut:

$\uparrow := \underset{I}{|}$ $\qquad$ $\downarrow := \underset{R}{|}$

Check: biadjointness $\Leftrightarrow$ $N = | = U$ in all possible orientations

Depict ntrl. transf. $a : R \to R$ also via 

$a : I \to I$ 

Now: I = induction $\quad R$ = restriction

for $H<G$ finite gps.

Lemma.

1.  $= \bullet$  $= [G:H]$

2. $\mathrm{End}({}_G\mathbb{C}G_H)$ = endomorphs of $I = (\mathbb{C}G)^H$ $= \{a \in \mathbb{C}G \mid ah = ha \ \forall h \in H\}$

   $f \mapsto f(e) =: a$

   $f(gh) = gha = gah \ \forall g, h$

   $\mathrm{End}({}_H\mathbb{C}G_G)$ = endomorphs of $R = (\mathbb{C}G)^H$

   $f \mapsto f(e)$

3.  and 

   for all orientations ($\boxed{a}$ is a ntr. transf.)

Special Case

Consider only the symmetric gps S_n

w/inclusions $S_{n-1} \hookrightarrow S_n \quad \forall n$

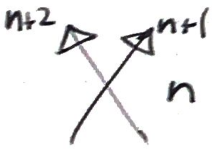
$$s_i = (i, i+1) \mapsto s_i$$

Why consider this case?

Heisenberg Lie algebras act naturally on $\underbrace{\text{Foch space}}$

basis labelled by partitions

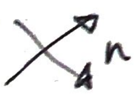
In this case, we have further ntrl. transf.



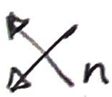
$$\begin{array}{ccc} {}_{n+2}\mathbb{C}[S_{n+1}]_n & \longrightarrow & {}_{n+2}\mathbb{C}[S_{n+1}]_n \\ \cup & & \\ g & \longmapsto & g\, s_n \end{array}$$

We would like to have:

${}_n\mathbb{C}[S_{n+2}]_{n+1} \longrightarrow {}_n\mathbb{C}[S_{n+2}]_{n+2} \quad g \mapsto s_{n+1} g$



$\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} \mathbb{C}[S_n] \longrightarrow {}_n\mathbb{C}[S_{n+1}]_n \quad g \otimes h \mapsto g s_n h$



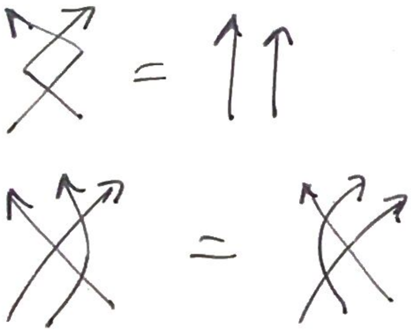
${}_n\mathbb{C}[S_{n+1}]_n \longrightarrow \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} \mathbb{C}[S_n] \quad g \mapsto \begin{cases} 0 & \text{if } g \in S_n \\ x \otimes y & \text{if } g = x s_n y \end{cases}$

— satisfy  ∀ orientations and all n

— compatible w/ "rotations"

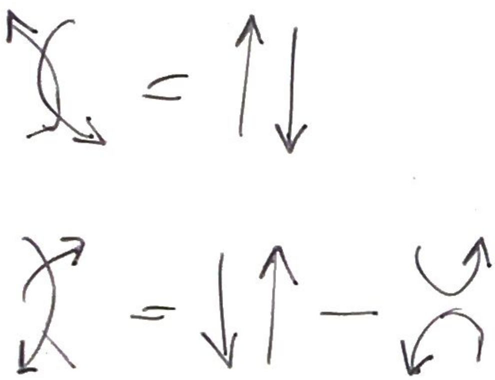
Proposition.

$\boxed{1}$



braid relations

$\boxed{2}$ extra relations



 $= 1$

 = zero

but  ≠ zero

PROOF:

1. "obvious"

2. Special case of Mackey formula

$$ {}_n\mathbb{C}[S_{n+1}]_n \xrightarrow{\sim} \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} \mathbb{C}[S_n] \oplus {}_n\mathbb{C}[S_n]_n $$

as bimodules

$$ RI \cong IR \oplus ID $$