Categorifications & Lie Algebra Actions on categories arising from representation theory II

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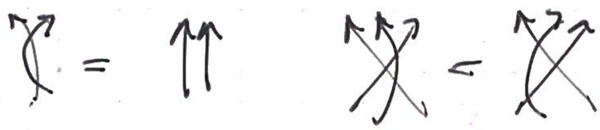
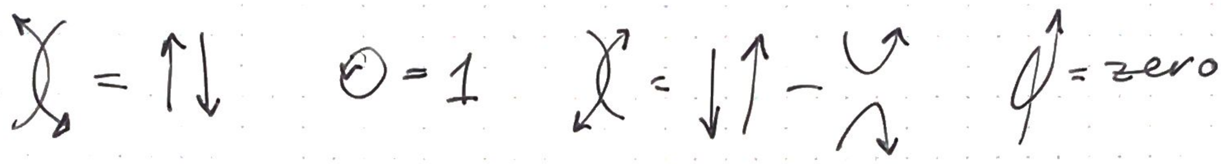
<u>last time:</u>

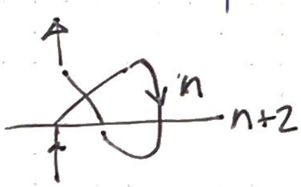
Prop. "Natural" natural transformations/bimodule maps between induction & restriction for $S_n \hookrightarrow S_{n+1}$ satisfy:

(1)  (symmetric gp. relns)

(2) 

(Mackey/Heisenberg relations)

What is $\uparrow$ explicitly?



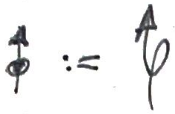
$$\mathbb{C}[S_n]_n \ni g \mapsto \underbrace{\sum_{i=1}^{n} g\, s_i s_{i+1} \cdots s_{n-1} \overset{S_n}{\otimes} s_{n-1} s_{n-2} \cdots s_i}_{\in\, \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]}$$

$$\mapsto g \underbrace{\sum_{i=1}^{n} s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_i}_{\shortparallel} \in \mathbb{C}[S_n]$$

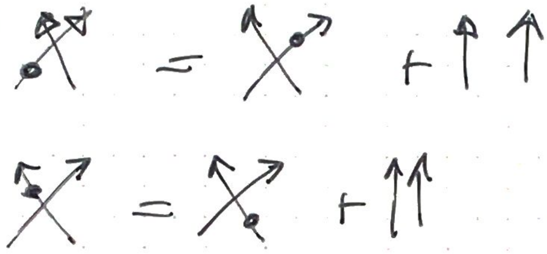
$$\boxed{\xi_{n+1}} := \sum_{i=1}^{n} (i, n+1) \quad (n+1)\text{st Jucys-Murphy element}$$

Together w/ $\mathbb{Z}[\mathbb{C}[S_{n+1}]]$, S_{n+1} generates

$$\mathbb{C}[S_{n+1}]^{S_n} - \mathrm{End}\left(\begin{matrix}\uparrow_n \\ \downarrow_n\end{matrix}\right)$$

Denote: 

Relations:

(3) 

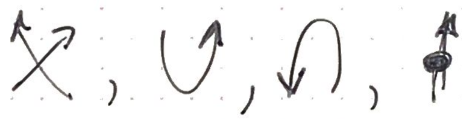
$\boxed{\text{(dAHA relations)}}$

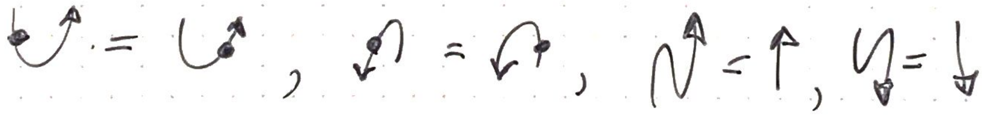
Def. (Khovanov)

The 2-category $\mathcal{H}'$ is the monoidal category given by

Objects: $\mathbb{N} \longleftrightarrow \mathbb{1}_n \qquad \begin{matrix}\wedge n \\ \text{"}\end{matrix} \qquad \begin{matrix}\vee n \\ \text{"}\end{matrix}$

1-morphs: gen'd by $\underset{n \to n+1}{P_+ \mathbb{1}_n}$, $\underset{n \to n-1}{P_- \mathbb{1}_n}$

2-morphs: gen'd by  subject to relations (1), (3)

cyclicity 

ex. 2-morphisms

Remark:

End($P_+^d 1_n$) given by diagrams w/ d strands composed from $\times$ and $\uparrow$ is isomorphic to Dunfeld's

Degenerate affine Heck

$dAHA := \mathbb{C}[S_d] \times \mathbb{C}[x_1, \dots, x_d]$

$w \in S_d \mapsto w \qquad x_i \mapsto s_i \qquad \downarrow \mathbb{C}[S_d]$

 $\longleftrightarrow x_1^3 x_2^2 x_3^2$

Thm. (Khovanov, Brundan-Savage-Webster)

$\mathring{H}_\infty \xrightarrow{\sim} K_0(\mathcal{H})_\mathbb{C}$ isom of $\mathbb{C}$-linear cat.

ex

$Alg_2(\mathbb{C}) \rightsquigarrow K_0(Alg_2(\mathbb{C}))$ bimodules $K_0^{split}((B,A)\text{ bimod}) \otimes_\mathbb{Z} \mathbb{C}$

objects: algebras A

morphs: $A \to B$ split Grothendieck gp. of (B,A)

$$\dot{H}_{\infty} \xrightarrow{\sim} K_0(\dot{\mathcal{H}})_{\mathbb{C}}$$

$$P_d \mathbb{1}_n \longmapsto [\Lambda^d_{+,n}] \qquad \Lambda^d_{+,n} = \Lambda^d(P_+ \mathbb{1}_n)$$

$$P_{-d} \mathbb{1}_n \longmapsto [S^d_{-,n}] \qquad S^d_{-,n} = S^d(P_- \mathbb{1}_n)$$

$d \in \mathbb{Z}_{>0}$

where $\dot{\mathcal{H}}$ is the karoubian closure of $\dot{\mathcal{H}}'$.

Let W be a vector space. We can define $\Lambda^d W$ & $S^d W$

$$W \text{ v.sp} \mapsto \Lambda^d W = e_{sgn} W^{\otimes d}$$

$$W \text{ v.sp} \mapsto S^d W = e_{triv} W^{\otimes d} \qquad e_{triv} = \frac{1}{|S_d|} \sum_{g \in S_d} g$$

Note. $\mathbb{C}[S_d] \to \mathrm{End}(P^d_{\underset{(-)}{+}} \mathbb{1}_n)$

$$\to \Lambda^d(P_{\pm} \mathbb{1}_n)$$

$$\to S^d(P_{\pm} \mathbb{1}_n)$$

Problem. $S^d_- \otimes \Lambda^{d'}_+ \cong \Lambda^{d'}_+ \otimes S^d_- \otimes \Lambda^{d'-1}_+ \otimes S^{d-1}_-$

So $P_d, \hat{P}_{-d}$ are slightly different generators of $\mathring{H}_\infty$

Connections to $\mathring{U}$-categorification?

THM. (sloppy version)

categorified Heisenberg actions $\longleftrightarrow$ categorified $\mathring{U}_q(\mathfrak{g})$ for appropriate $\mathfrak{g}$

Idea. i-induction/i-restriction.

Jucys-Murphy element $\xi_n \in \mathbb{C}[S_n]^{S_{n-1}}$

$\Rightarrow R\mathbb{1}_n = \bigoplus_i \underbrace{R_i \mathbb{1}_n}$ = restriction followed by generalized proj. onto i-eigenspace of ξ_n

Also: $I\mathbb{1}_n \cong \bigoplus_{i \in \mathrm{spec}(\xi_{n+1})} I_i \cdot \mathbb{1}_n$

Fact: $\mathrm{Spec}(\xi_n) \subseteq \mathbb{Z}$

Given a common eigenspace for $J_1, J_2, \dots J_n$
inside a $\mathbb{C}[S_n]$-module M
w/ given eigenvalues $i_1, i_2, \dots i_n$
& multiplicities $m_i :=$ multiplicity of $i \in \mathbb{Z}$

$\rightsquigarrow \lambda := \sum_{i \in \mathbb{Z}} m_i \alpha_i$

$\mathbb{1}_\lambda :=$ projection onto all such eigenspaces

λ can be interpreted as a weight for the Lie algebra $\mathfrak{gl}_\mathbb{Z}$ corresponding to Dynkin diagram

Set $\mathcal{F}_i \mathbb{1}_\lambda := R_i \mathbb{1}_{\lambda, n}$

$\mathcal{E}_i \mathbb{1}_\lambda := I_i \underbrace{\mathbb{1}_{\lambda, n}}_{\mathbb{1}_n \mathbb{1}_\lambda}$

This gives an action of $\dot{U}_q(\mathfrak{gl}_\mathbb{Z})$ given an action of $\mathring{\mathcal{H}}$

Remarks.

- Working w/ Hecke algebras instead of symmetric groups gives spectrum of the form $v^{\#}$ for v a root of unity
  $\rightsquigarrow$ Dynkin diagram  # verts = order of v

- KLR-algebras are endomorphism algebras of compositions of $\mathcal{F}_i 1_\lambda, \mathcal{E}_i 1_\lambda$'s
  (Explicit Relations!!)

- Other concrete incarnation of $\mathring{\mathcal{U}}(\mathfrak{gl}_m)$-action uses $SBIM_2(W = S_n, m \in \mathbb{N})$
  Take $R := \mathbb{C}[x_1, \dots, x_n] \circlearrowleft S_n$ permutations of x_i's
  Objects: parab. subgps. $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_r} < S_n$
  $$\sum_i \lambda_i = n$$
  1-morphs: opposite zig-zags
  $$W_r \supseteq H_r \subseteq W_{r-1} \ \cdots \ \supseteq H_2 \subseteq W_2 \supseteq H_1 \subseteq W_1$$
  $\rightsquigarrow$ bimodule: $R^{W_r} \otimes_{R^{H_r}} \cdots \otimes_{R^{H_2}} R^{W_2} \otimes_{R^{H_1}} R^{W_1}$ $\leftarrow$ invariants

2-morphs bimodule maps

Soergel bimodules

Special Bimodules ("correspondences")

$$S_{\lambda_1} \times \dots \times S_{\underbrace{\lambda_i}_{\substack{a-1 \\ \text{or } \boxed{a+1}}}} \times S_{\underbrace{\lambda_{i+1}}_{\substack{b+1 \\ \boxed{b-1}}}} \times \dots \times S_{\lambda_m}$$

$$S_{\lambda_1} \times \dots S_{\lambda_{i-1}} \times S_a \times S_b \times S_{\lambda_{i+1}} \times \dots \times S_{\lambda_m}$$

$$S_\mu = S_{\lambda_1} \times \dots \times S_{\lambda_{i-1}} \times \underset{\boxed{a}}{S_{a-1}} \times \underset{\boxed{1}}{S_1} \times \underset{\boxed{b-1}}{S_b} \times \dots \times S_{\lambda_m}$$

⇝ Corresponding bimodules which define action of $E_i 1\!\!1_\lambda$ resp. $F_i 1\!\!1_\lambda$