

SYNTHETIC ∞ -CATEGORY THEORY

INTRODUCTORY WORKSHOP :
HIGHER CATEGORIES & CATEGORIFICATION
MSRI , FEB 2020

Dominic Verity
CENTRE OF AUSTRALIAN
CATEGORY THEORY
MACQUARIE UNIVERSITY
SYDNEY, AUSTRALIA

&

Emily Riehl
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE MD, USA

AIM: GIVE A PRECISE AND MODEL
INDEPENDENT ACCOUNT OF
 ∞ -CATEGORY THEORY.

RESOURCES

" ∞ -CATEGORY THEORY FROM SCRATCH"

https://journals.mq.edu.au/index.php/higher_structures/article/view/38

"ELEMENTS OF ∞ -CATEGORY THEORY"

<http://www.math.jhu.edu/~eriehl/elements.pdf>

$(\infty, 1)$ - CATEGORIES

SCHEMATIC

"DEFINITION"

AN $(\infty, 1)$ -CATEGORY IS
A CATEGORY WEAKLY
ENRICHED IN SPACES

∞ -GROUPOIDS

MODELS

QUASI-CATEGORIES
COMPLETE SEGAL SPACES
SEGAL CATEGORIES
MARKED QUASI-CATEGORIES

ALSO FIBRED
VARIANTS OF
THESE NOTIONS

∞ - CATEGORIES

AUXILIARY AIM: BUILD A FRAMEWORK THAT CAN BE EXTENDED TO GIVE FOUNDATIONS FOR THE CATEGORY THEORY OF (∞, n) - CATEGORIES

MODELS:

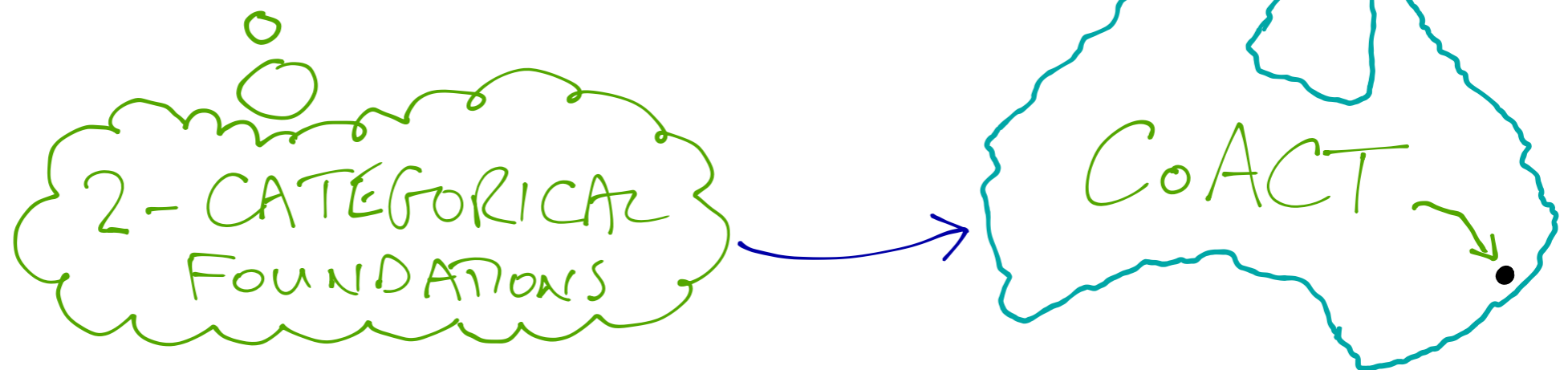
COMPLICIAL SETS
ITERATED SEGAL
 \mathbb{H}_n -SETS AND SPACES

CATEGORIES WEAKLY ENRICHED IN $(\infty, n-1)$ -CATEGORIES

WE WILL USE THE TERM ∞ -CATEGORY TO MEAN ANY STRUCTURE THAT INHABITS AN INSTANCE OF THE FRAMEWORK WE BUILD HERE.

BENEFITS

-) GIVE **SYNTHETIC** PROOFS OF ∞ -CATEGORICAL RESULTS, WHICH THEN APPLY IN ALL MODELS.
-) TRANSFER CATEGORICAL RESULTS DERIVED **ANALYTICALLY** IN ONE MODEL TO RELATED MODELS.
-) ADAPT AND APPLY INTUITIONS FROM "META-CATEGORY" THEORY.



PLAN

PART 1: AN AXIOMATIC FRAMEWORK FOR ∞ -CATEGORY THEORY.

PART 2: LIMITS AND ADJUNCTIONS.

PART 3: HOMOTOPY COHERENT MONADS AND BECK MONADICITY.

DIVING IN: 2-CATEGORIES

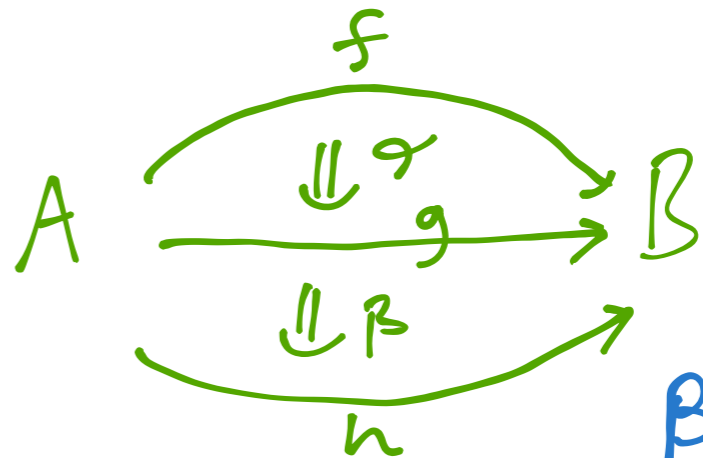
Modeled on the totality of categories, functors + natural transformations

OF THE STRICT VARIETY

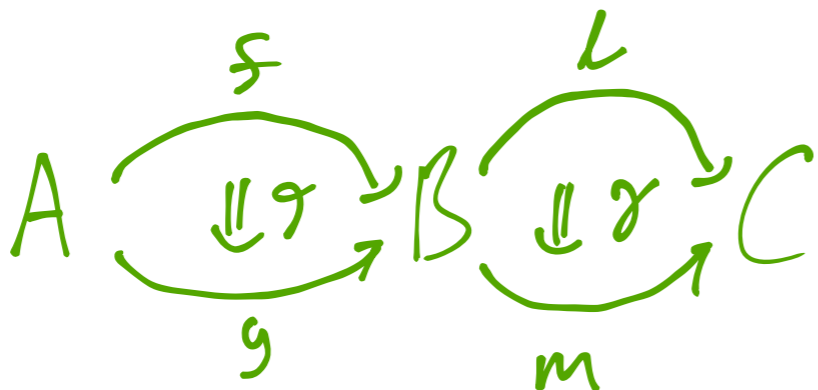
AKA CATEGORIES ENRICHED IN $(\text{Cat}, \times, \mathbb{1})$

- OBJECTS A, B, C
- 1-CELLS $A \xrightarrow{f} B$
- 2-CELLS $A \begin{matrix} \xrightarrow{f} \\ \Downarrow \gamma \\ \xrightarrow{g} \end{matrix} B$

2-CELLS COMPOSE VERTICALLY



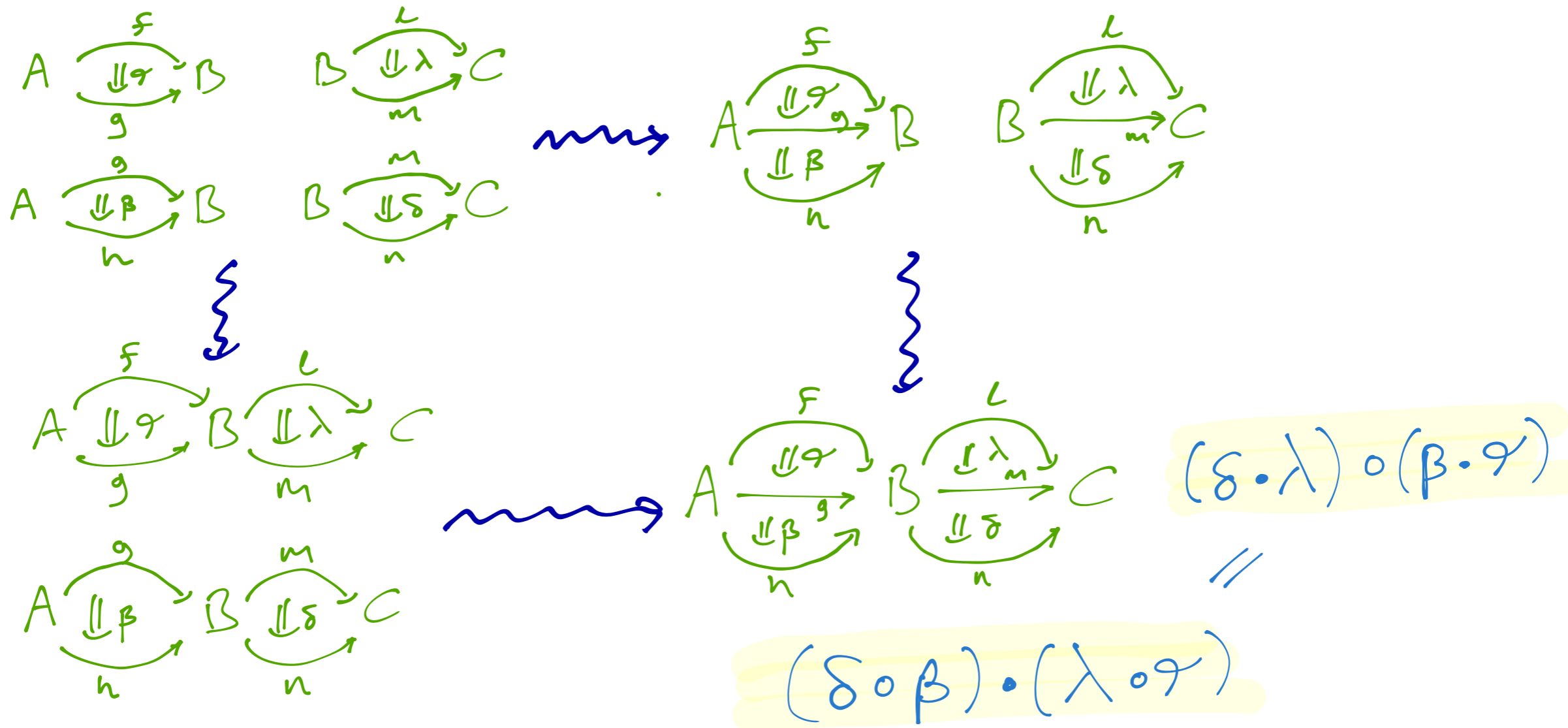
HORIZONTAL



$\gamma \circ g$

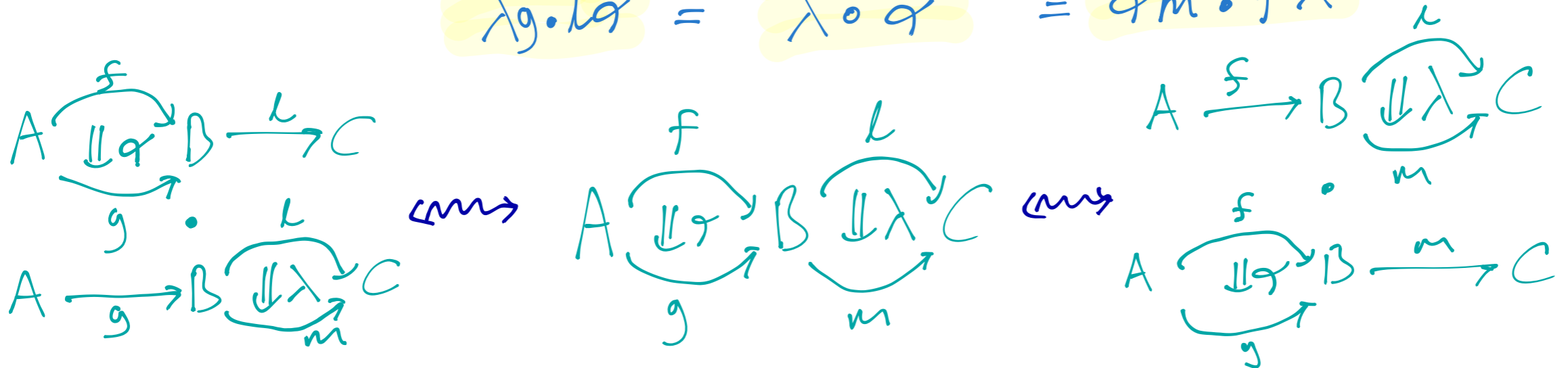
BOTH STRICTLY ASSOCIATIVE

MIDDLE-4-INTERCHANGE

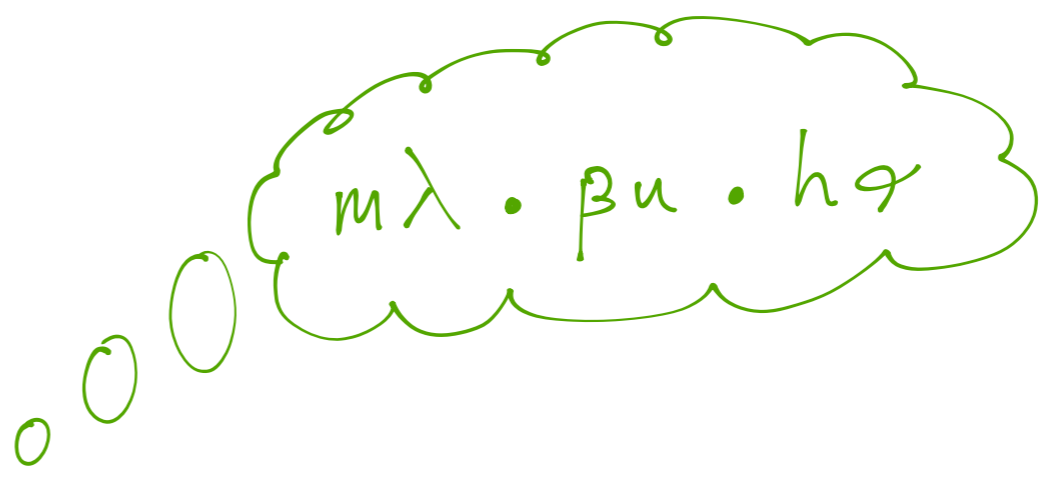
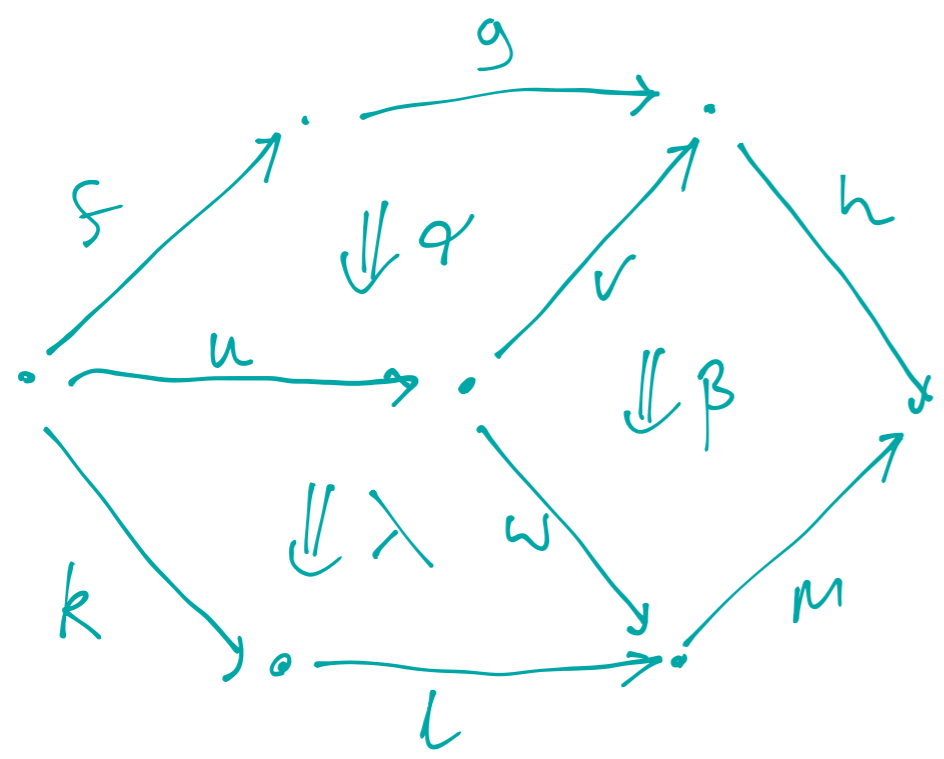


WHISKERING

$$\lambda g \cdot l \alpha = \lambda \circ \alpha = \alpha m \cdot f \lambda$$

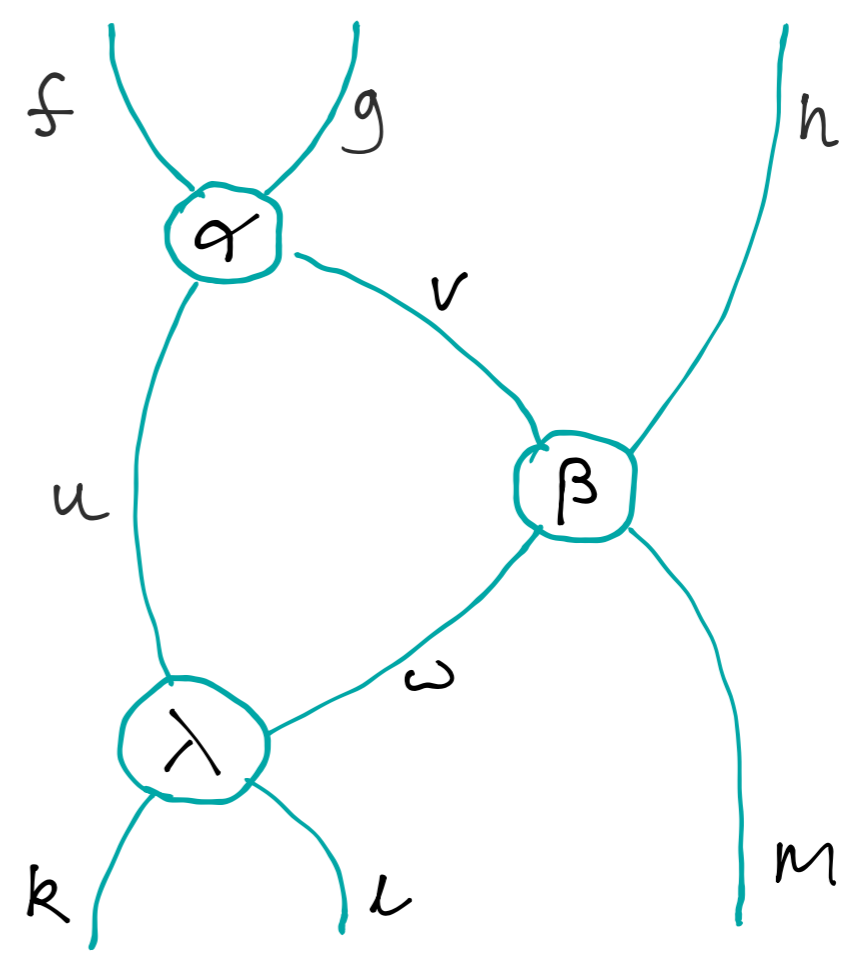
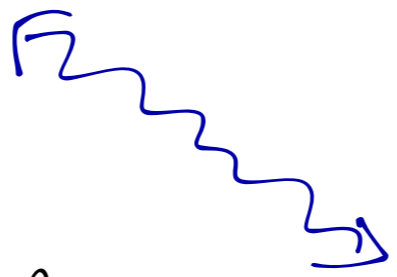


PASTING



STRING DIAGRAMS

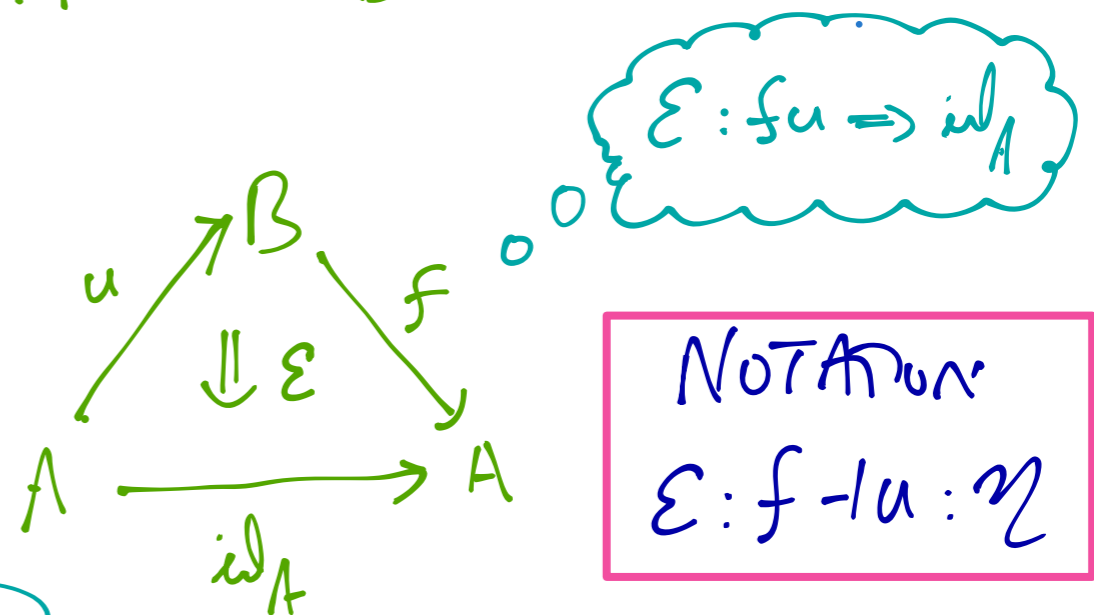
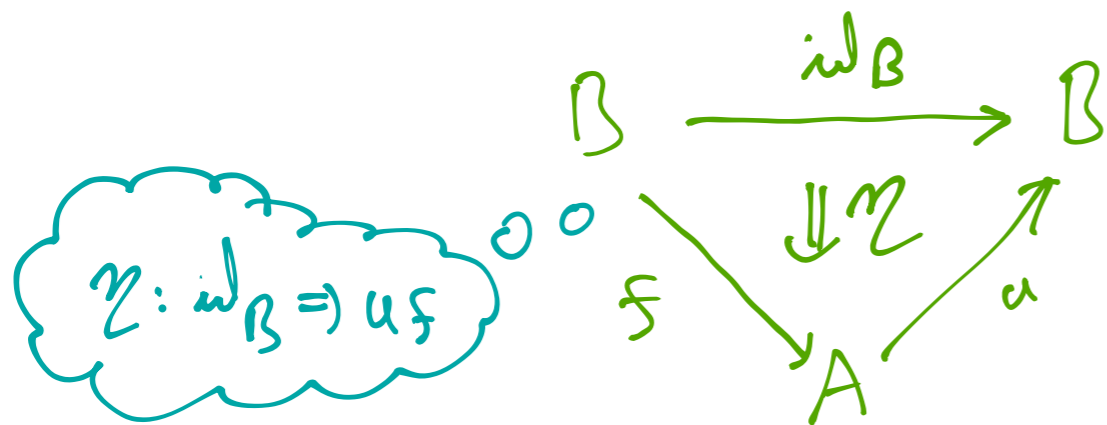
PLANAR
DUAL



ADJUNCTIONS IN 2-CATEGORIES

DEFN AN ADJUNCTION IN A 2-CATEGORY COMPRISES

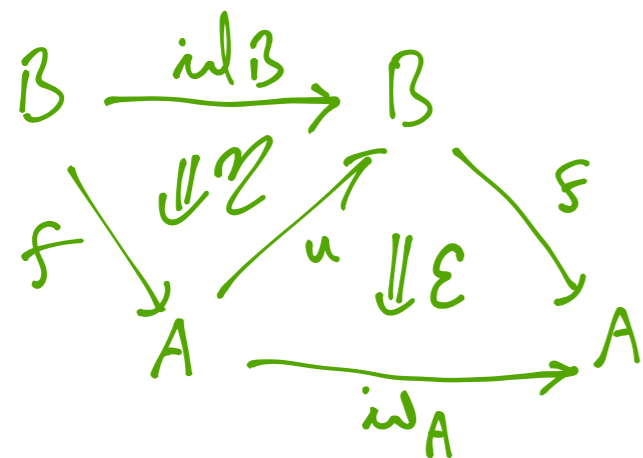
- A PAIR OF OBJECTS A AND B
- A PAIR OF 1-CELLS $A \xrightarrow{u} B$ AND $B \xrightarrow{f} A$
- A PAIR OF 2-CELLS:



NOTATION
 $\epsilon: f \circ u \circ id_A$

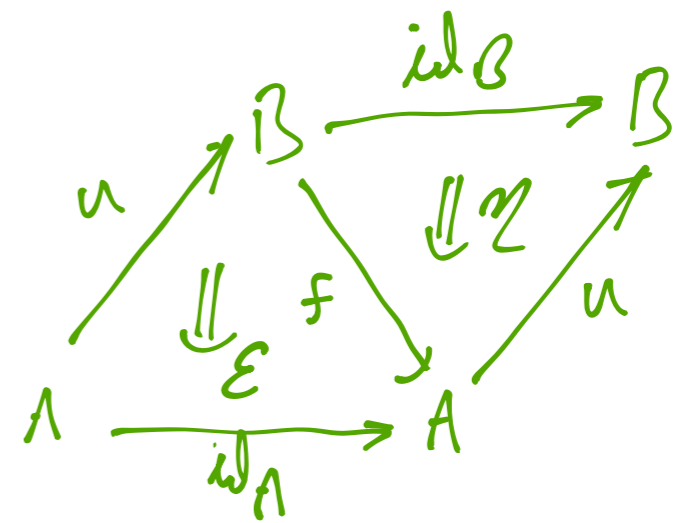
- TRIANGLE IDENTITIES

IDENTITY 2-CELLS



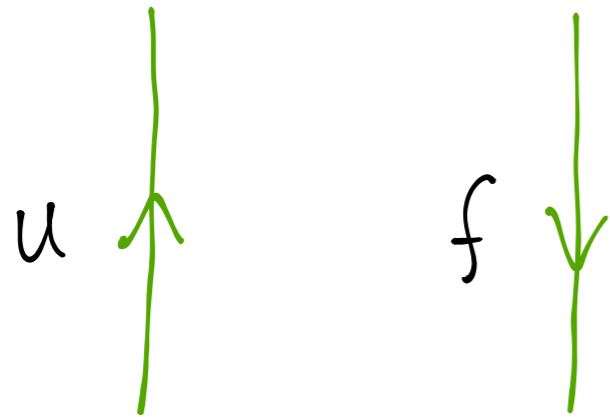
$= id_f$

$= id_u$

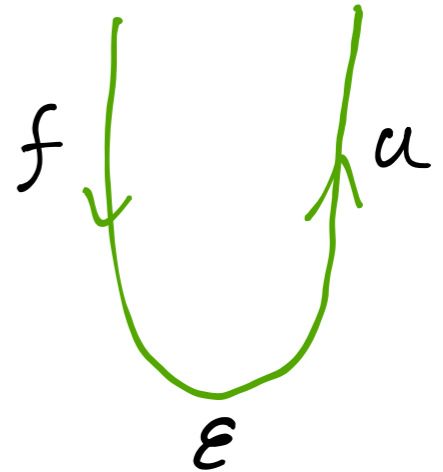
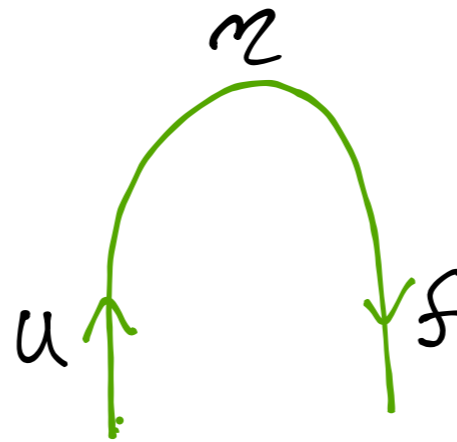


REPRISE WITH STRINGS

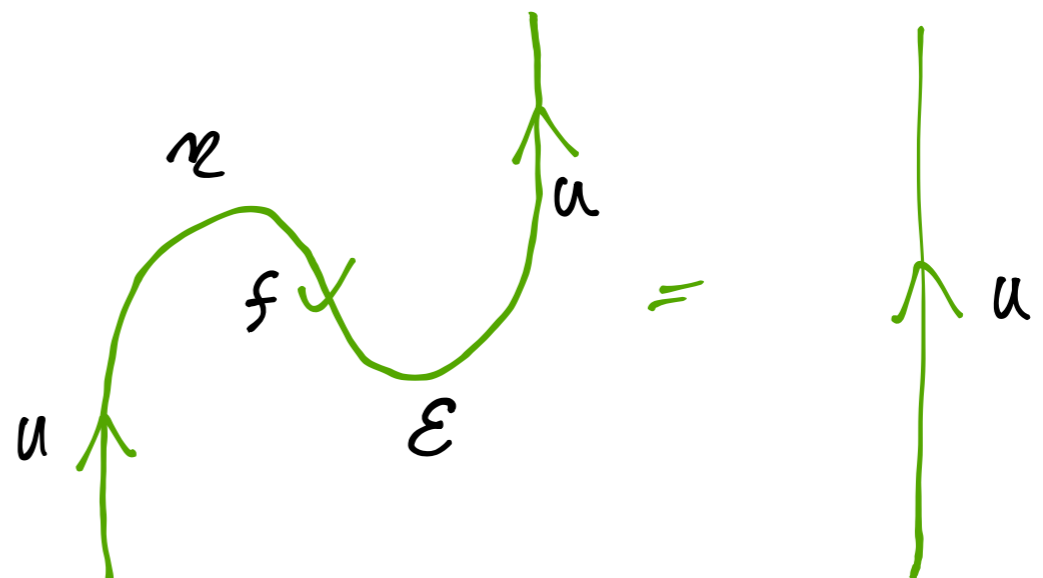
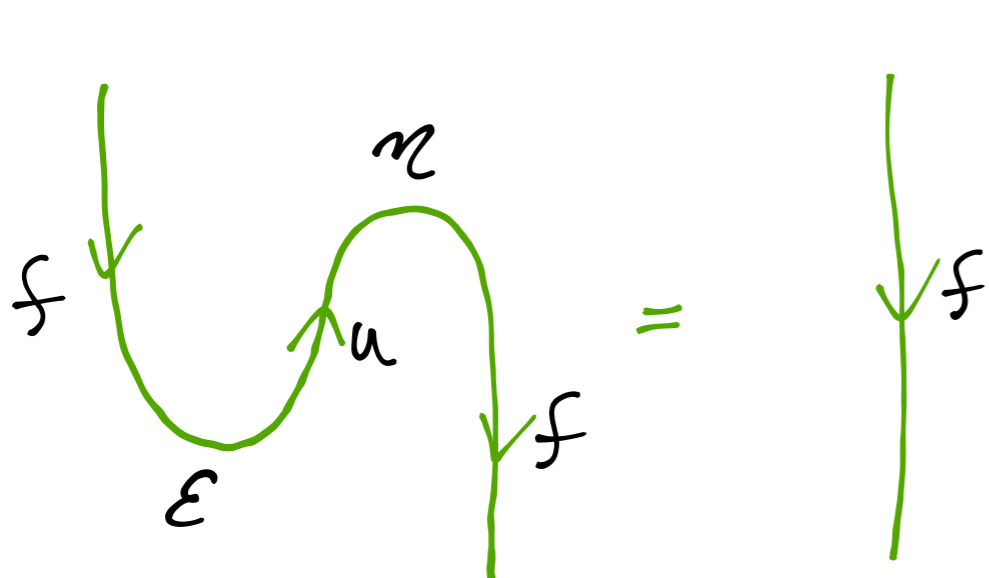
1-CELLS



2-CELLS



TRIANGLE IDENTITIES



SOME CONSEQUENCES

- COMPOSITION OF ADJUNCTIONS
- ADJOINTS ARE UNIQUE UPTO, AND STABLE UNDER, ISOMORPHISM.
- EQUIVALENCES MAY BE PROMOTED TO ADJOINT EQUIVALENCES.
- ADJUNCTIONS ARE STABLE UNDER EQUIVALENCES.

SLOGAN

CATEGORIES LIVE

INSIDE 2-CATEGORIES^{*}

- ⊛ IT IS MORE ACCURATE TO SAY THAT 2-CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP CATEGORY THEORY.

∞ -SLOGAN

∞ -CATEGORIES LIVE

INSIDE $(\infty, 2)$ -CATEGORIES^{*}

- * IT IS MORE ACCURATE TO SAY THAT $(\infty, 2)$ -CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP ∞ -CATEGORY THEORY.

$(\infty, 2)$ -CATEGORIES REALLY?!!

THE TRICK HERE IS TO:

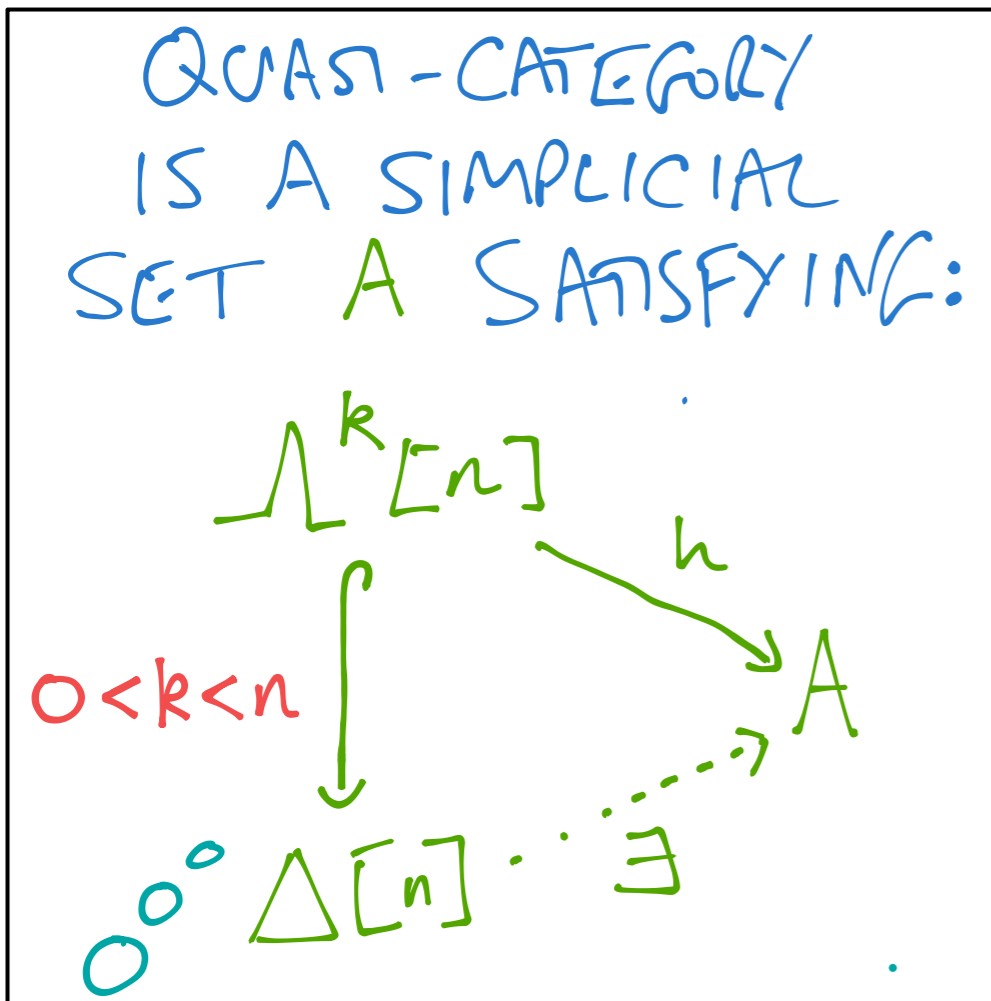
- PICK A MODEL OF $(\infty, 2)$ -CATEGORIES THAT IS FAMILIAR.
- SIMPLIFY MANY ARGUMENTS BY WORKING, WHERE POSSIBLE, IN THE HOMOTOPY 2-CATEGORY ASSOCIATED WITH OUR AMBIENT $(\infty, 2)$ -CATEGORY.

COMMITTING TO AN $(\infty, 2)$ -CATEGORICAL MODEL

AN ∞ -COSMOS IS, IN ESSENCE,
A CATEGORY OF FIBRANT
OBJECTS ENRICHED IN THE
JOYAL MODEL STRUCTURE

QUASI-CATEGORIES

RECALL:

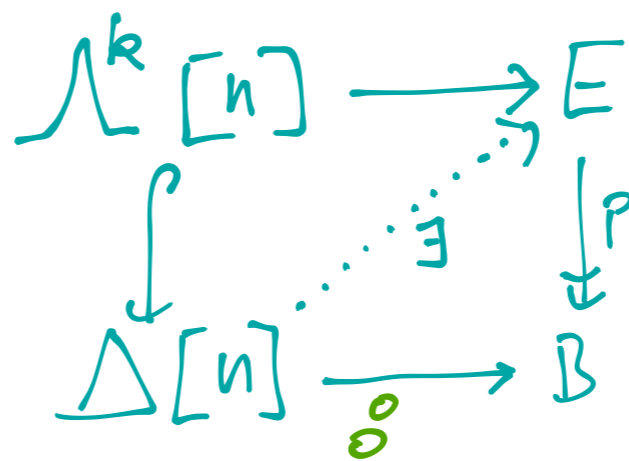


INNER HORN

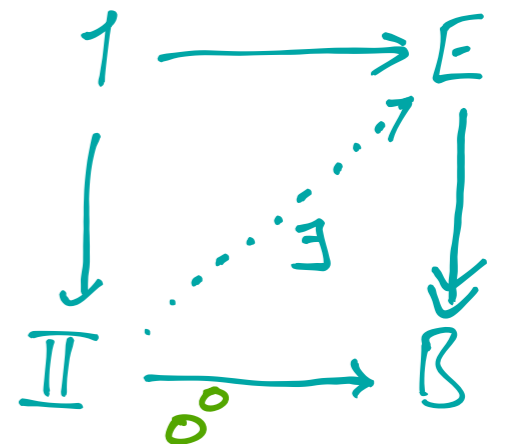
JOYAL MODEL STRUCTURE

ON SSet HAS

-) FIBRANT OBJECTS THE QUASI-CATEGORIES
-) FIBRATIONS, CALLED ISOFIBRATIONS



LIFTING OF COMPOSITES



LIFTING OF ISOMORPHISMS
 $II := \{ \bullet \cong \bullet \}$

-) COFIBRATIONS \equiv MONOS

∞-COSMOS IN ONE PAGE...

A SIMPLICIAL
CATEGORY K

- EQUIPPED WITH
CLASSES OF
- ISOFIBRATIONS
 - WEAK EQUIVALENCES

PRODUCTS AND
PULLBACKS OF
ISOFIBRATIONS

COFIBRANT
REPLACEMENT
 $A_c \xrightarrow{\sim} A$

ALL OBJECTS
FIBRANT $!A \rightarrow 1$

FIBRATION NOTIONS
STABLE UNDER
PULLBACK

ALL LIMITS
SIMPLICIALLY
ENRICHED

COTENSORS SATISFYING "SM7"
WRT JOYAL MODEL STRUCTURE

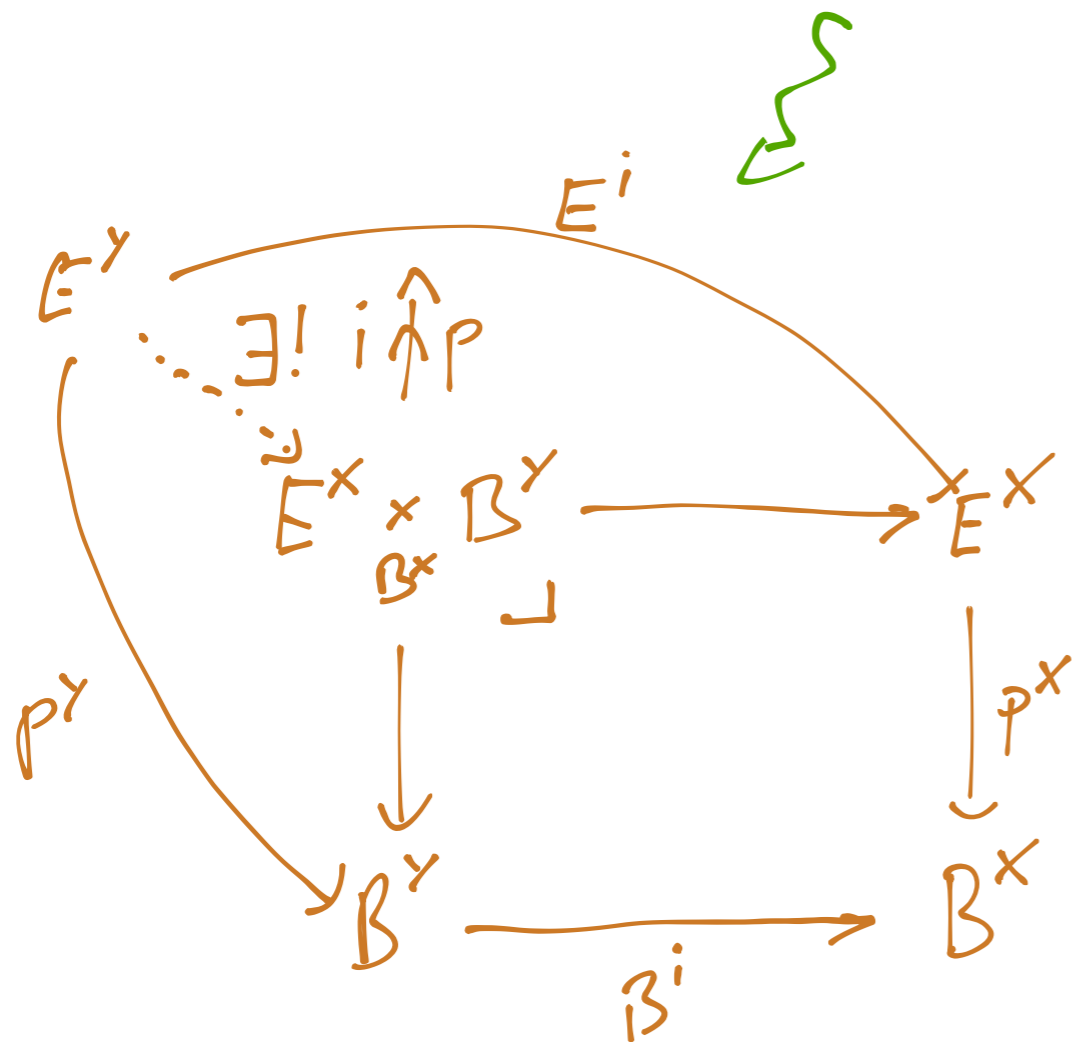
FUNCTOR SPACE

AXIOM "JM7"

$$\text{Fun}_{\mathcal{K}}(A, B)^X \cong \text{Fun}_{\mathcal{K}}(A, B^X)$$

COTENSOR

INCLUSION $i: X \hookrightarrow Y \in \underline{\text{Set}}$
 ISOFIBRATION $p: E \rightarrow B \in \mathcal{K}$



"JM7"

$i \hat{\wedge} p$ IS AN ISOFIBRATION AND IT IS A TRIVIAL FIBRATION IF i IS A WEAK EQUIVALENCE IN THE JOYAL MODEL STRUCTURE OR p IS A TRIVIAL FIBRATION IN \mathcal{K} .

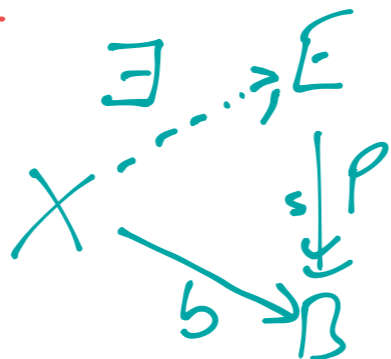
∞ -COSMOS EXAMPLES

$(\infty, 1)$ - MODELS

- QUASI-CATEGORIES
- SEGAL CATEGORIES
- COMPLETE SEGAL SPACES
- MARKED QUASI-CATS

MOST COMMON
EXAMPLES HAVE
ALL OBJECTS

COFIBRANT



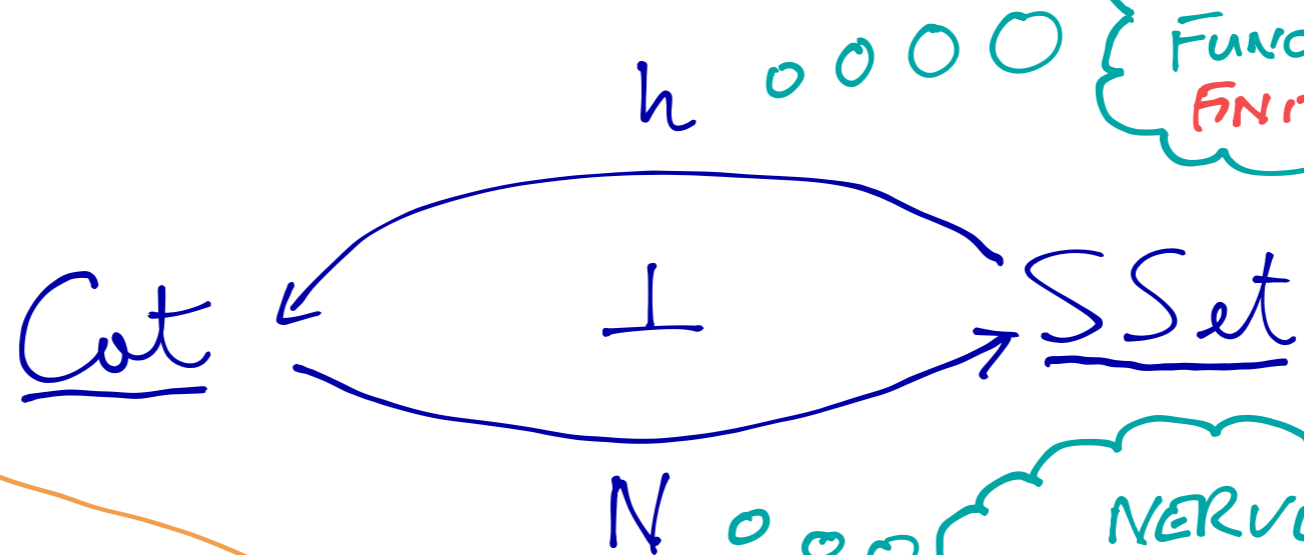
(∞, n) - MODELS

- SEGAL (∞, n) -SPACES
- \mathbb{H}_n - SETS + SPACES
- n -COMPLICIAL SETS

DERIVED EXAMPLES

- SLICES
- ∞ -CATEGORIES FIBRED OVER A BASE.
- (CO)COMPLETE ∞ -CATS

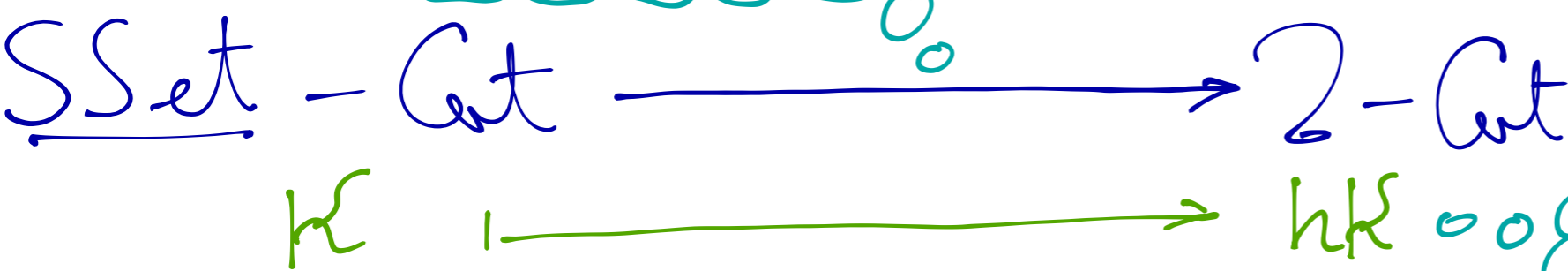
HOMOTOPY 2-CATEGORIES



HOMOTOPY CATEGORY
FUNCTOR - PRESERVES
FINITE PRODUCTS

NERVE
"ALL CATEGORIES
ARE SIMPLICIAL SETS"

APPLY h TO
HOM-SPACES



THE HOMOTOPY
2-CATEGORY OF K

HOMOTOPY 2-CATEGORIES OF

∞ -COSMOS

WHEN A IS COFIBRANT
IN \mathcal{K} THEN ANY HOMSPACE
 $\text{Fun}_{\mathcal{K}}(A, B)$ IS A QUASI-CAT



WE SHALL RESTRICT
THE HTY 2-CAT hK
OF AN ∞ -COSMOS
TO COFIBRANT OBJECTS

EXPLICIT DESCRIPTION
OF hK

- 0-CELLS THE COFIBRANT OBJECTS IN \mathcal{K}
- 1-CELLS ARROWS BETWEEN COF OBJECTS IN \mathcal{K}
- 2-CELLS HOMOTOPY CLASSES OF 1-SIMPLICES, THAT IS OF $(\infty-)$ NATURAL TRANSFORMATIONS, IN $\text{Fun}_{\mathcal{K}}(A, B)$.

∞ -CATEGORIES

$(\infty-)$ FUNCTORS

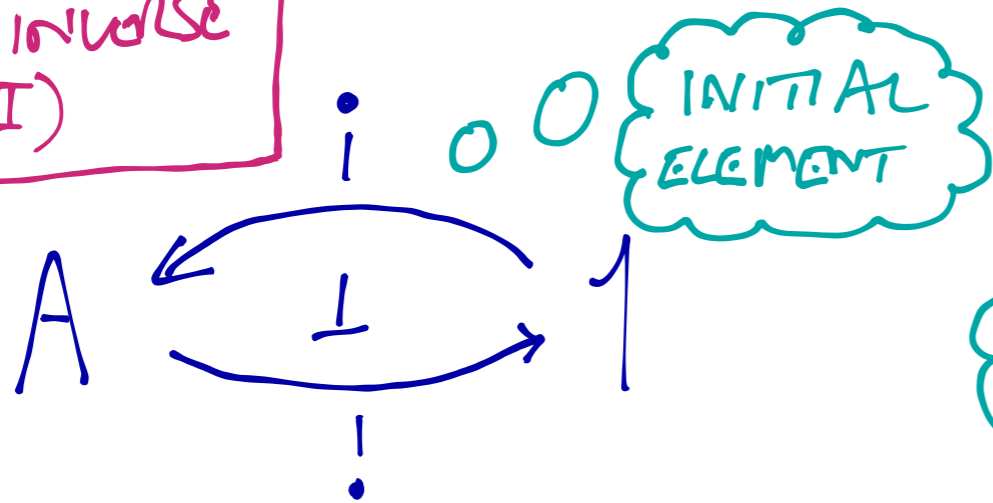
A WILD CONJECTURE!

2-CATEGORICAL ADJUNCTIONS
IN THE HOMOTOPY 2-CATEGORY
 hK OF AN ∞ -COSMOS K ARE
THE CORRECT NOTION WITH RESPECT
TO ∞ -CATEGORY THEORY.

FIRST STEPS

... TOWARDS ∞ -CATEGORY THEORY.

LEFT ADJOINT
RIGHT INVOLVE
(LARI)



RIGHT ADJOINT
RIGHT INVOLVES
(RARI)

MINIMAL DATA PRESENTING A TERMINAL ELEMENT

(i) AN ELEMENT $t: 1 \rightarrow A$

(ii) A 2-CELL $A \xrightarrow{id_A} A$

NATURAL FAMILY
OF MAPS FROM THE
ELEMENTS OF A TO t

(iii) $\epsilon t = id_t$

WHOSE COMPONENT
AT t IS THE IDENTITY

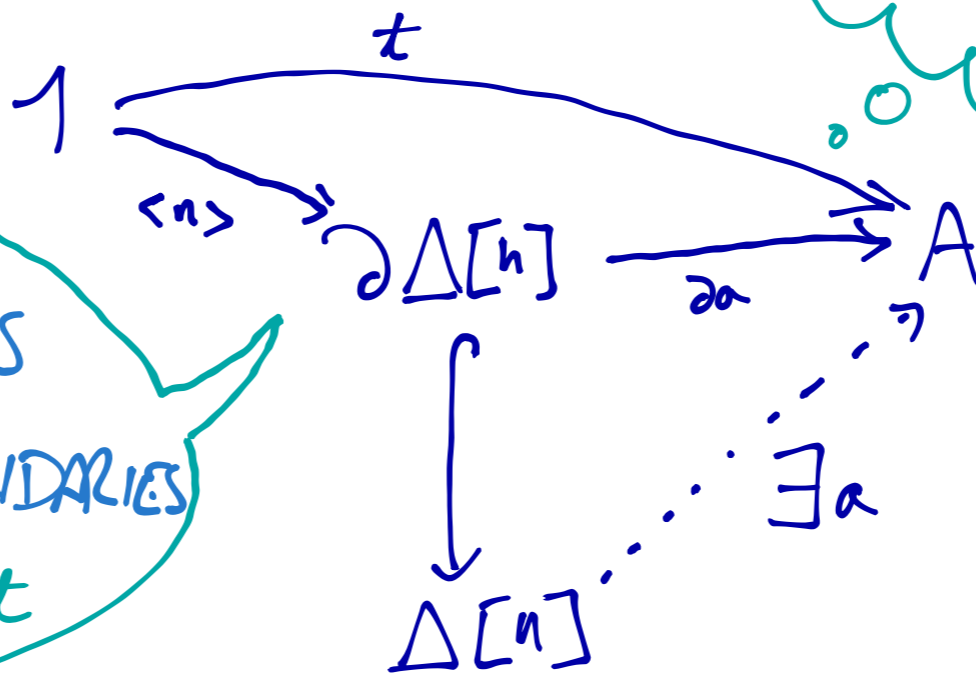
EXERCISE

SUPPOSE THAT A IS A QUASI-CATEGORY;
AN OBJECT IN THE ∞ -COSMOS QC_{Cat} .
LET t BE A VERTEX OF A , THEN

$t : 1 \rightarrow A$ IS A TERMINAL ELEMENT



THIS IS JOYAL'S
TERMINAL OBJECT
NOTION



A ADMITS FILLERS
FOR SIMPLEX BOUNDARIES
WITH LAST VERTEX t

IMMEDIATE CONSEQUENCES

- TERMINAL ELEMENTS PRESERVED BY RIGHT ADJOINTS.

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B AND FUNCTORS $f: B \rightarrow A$ WE HAVE

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \searrow \downarrow \exists! & \nearrow \tau \\ & 1 & \end{array} \text{ IN } \mathcal{K}.$$

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B THE GENERALISED ELEMENT $B \xrightarrow{!} 1 \xrightarrow{\tau} A$ IS TERMINAL IN THE QUASI-CAT $\text{Fun}_{\mathcal{K}}(B, A)$.

LIMITS AND COLIMITS

$$\begin{array}{ccc} \underline{\text{SSet}}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \times & & A \longmapsto A^x \end{array}$$

$$\begin{array}{ccc} \mathcal{K}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \cup & & A \longmapsto A^u \end{array}$$

COTENORS

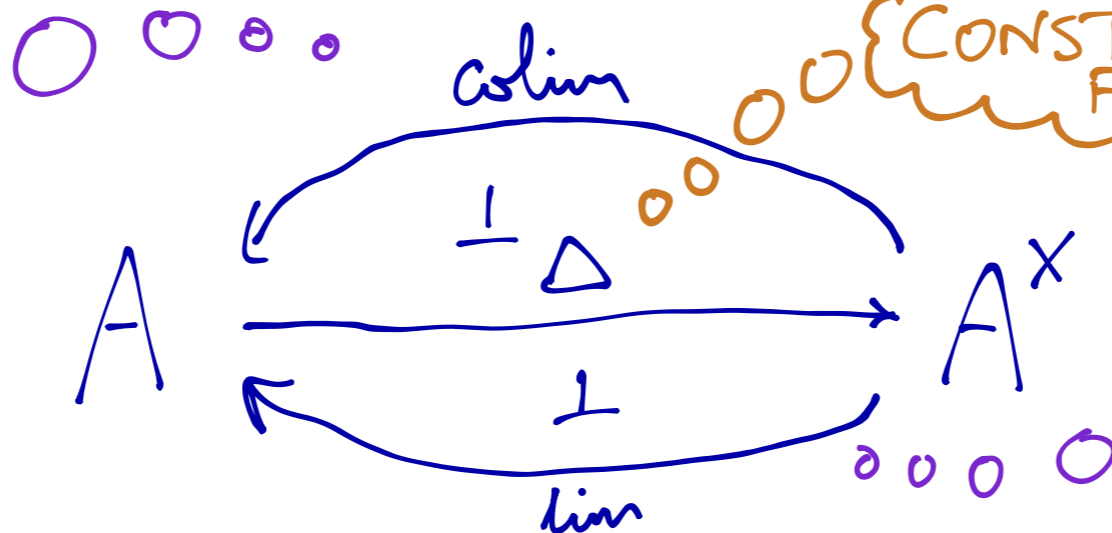
(CO)LIMITS OF SIMPLICIAL SET SHAPED DIAGRAMS

CLOSURE

(WHEN \mathcal{K} CLOSING IS CLOSED)

LIMITS OF DIAGRAMS WHOSE SHAPES ARE ∞ -CATEGORIES IN \mathcal{K}

COLIMITS OF DIAGRAMS OF SHAPE X



CONSTANT DIAGRAM FUNCTOR

LIMITS OF DIAGRAMS OF SHAPE X

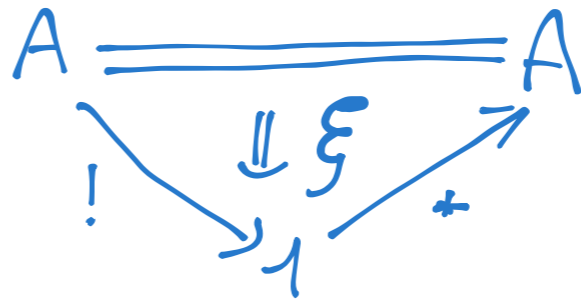
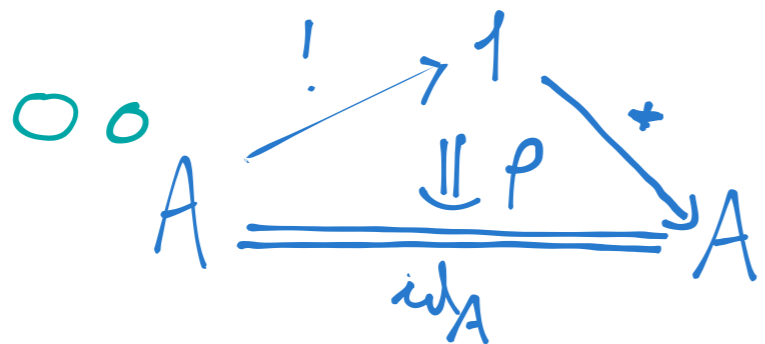
FAMILIES OF DIAGRAMS

WE MAY WISH TO ASK THAT AN ∞ -CATEGORY HAS SOME, BUT NOT ALL, LIMITS OF A GIVEN SHAPE.

EXAMPLE

- POINTED ∞ -CATEGORY HAS AN ELEMENT $*$: $1 \rightarrow A$ WHICH IS BOTH INITIAL + TERMINAL

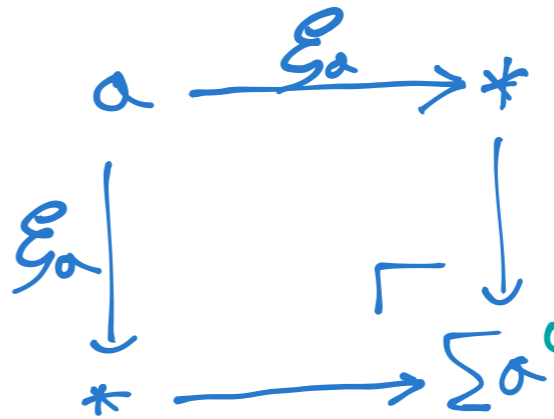
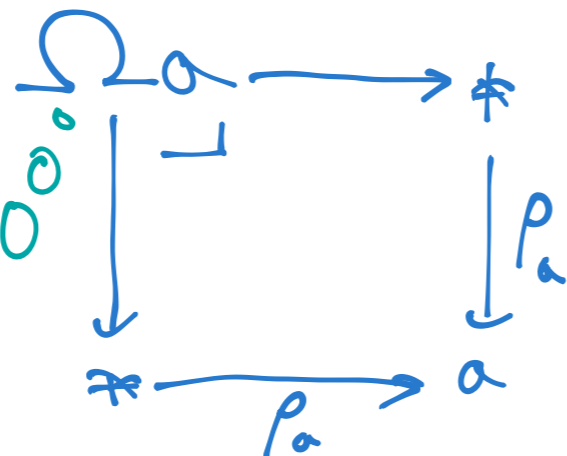
POINTS



COPOINTS

- IT IS COMMON ONLY TO ASK FOR CERTAIN PULLBACKS AND PUSHOUTS

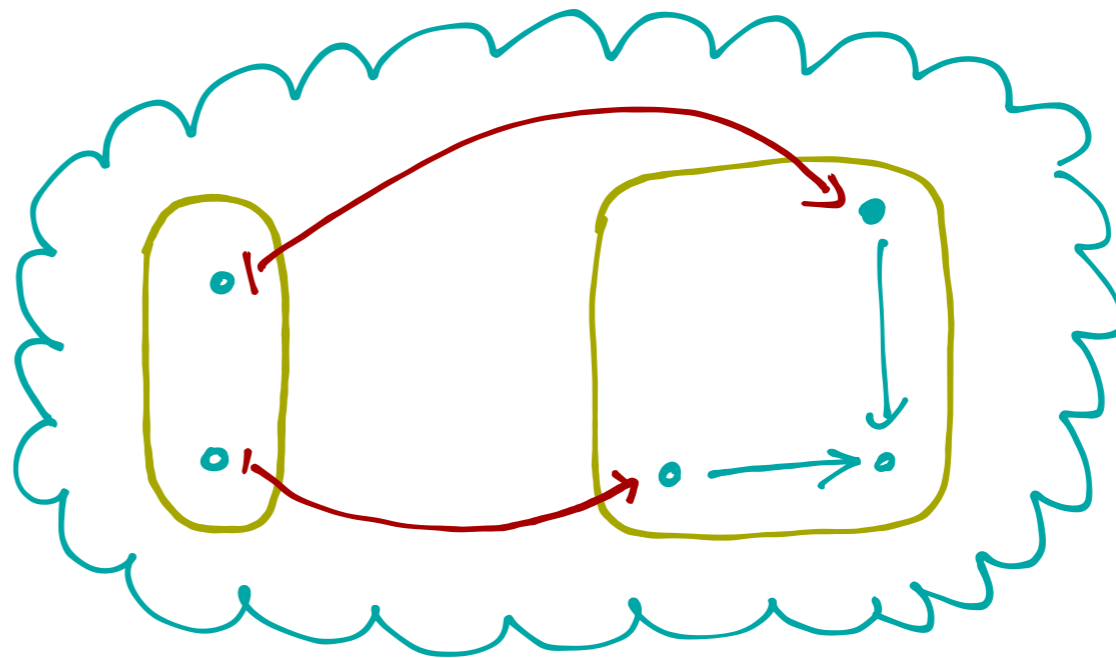
LOOP SPACE



SUSPENSION

LOOP SPACES

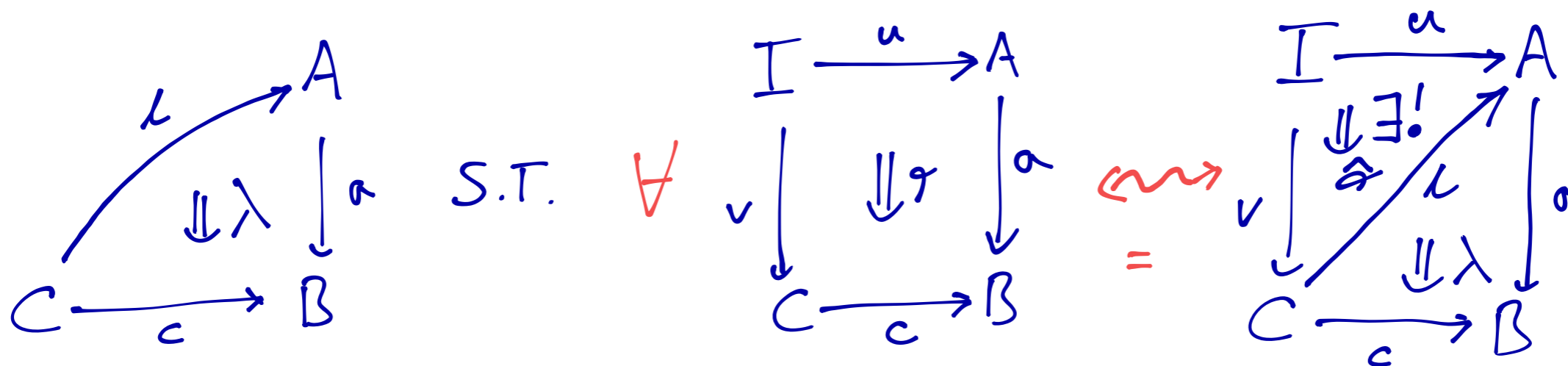
$$\begin{array}{ccc}
 A^{\square} & \xrightarrow{f} & A^{\square} \\
 \downarrow & & \downarrow \\
 1 & \xrightarrow{(*,*)} & A \times A
 \end{array}$$



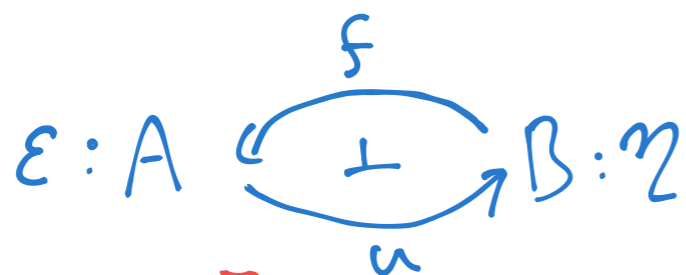
- LOOP SPACES ARE PULLBACKS, SO THEIR UNIVERSAL PROPERTY GIVES IN THE DIAGRAM ∞ -CATEGORY $A^{\square} \dots$
- \dots BUT WE ONLY WANT TO ASK FOR LIMITS OF DIAGRAMS IN A^{\square}
- $A^{\square} \xrightarrow{f} A^{\square}$ SPECIFIES A FAMILY OF DIAGRAMS IN A^{\square} WHOSE LIMITS ARE LOOP SPACES.

LIMITS OF FAMILIES

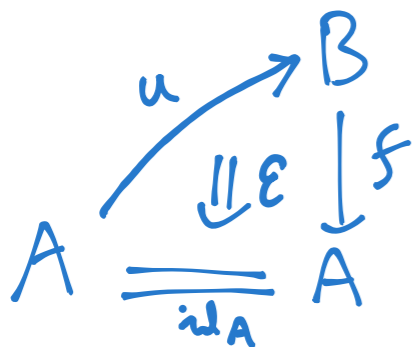
ABSOLUTE RIGHT LIFTING (IN A 2-CATEGORY)



EXAMPLE

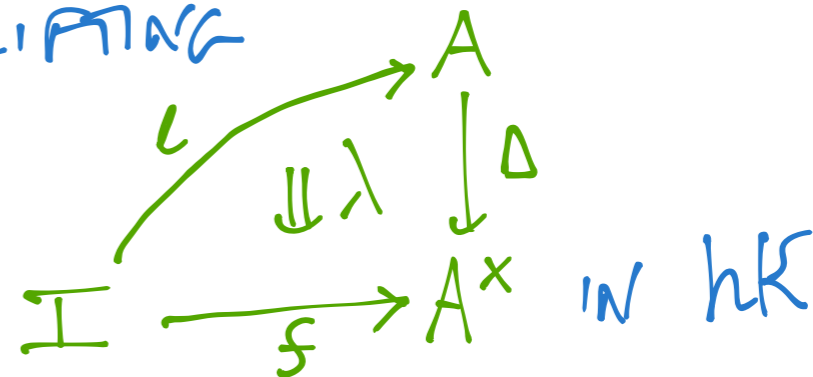


IS AN ABSOLUTE
RIGHT LIFTING



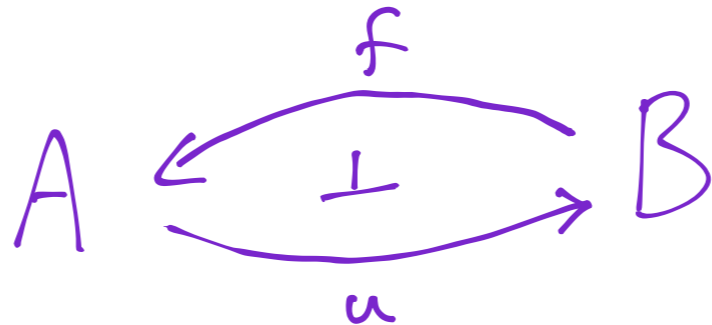
DEFN A FAMILY OF DIAGRAMS

$f: I \rightarrow A^X$ ADMITS A LIMIT
IF THERE IS AN ABSOLUTE
RIGHT LIFTING

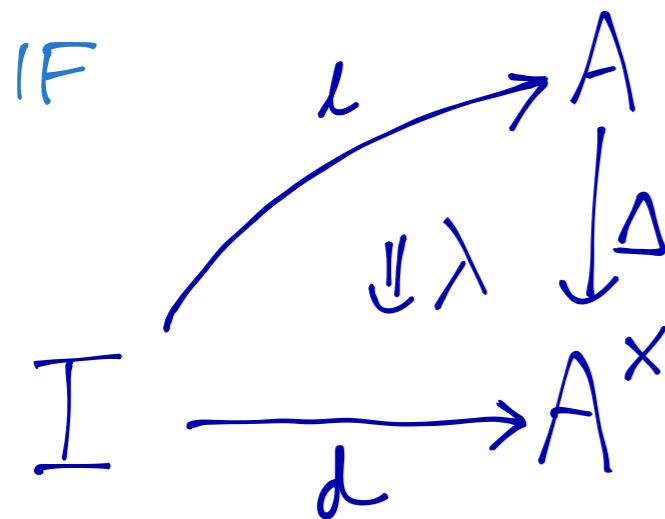


PRESERVATION

RIGHT ADJOINTS PRESERVE LIMITS OF FAMILIES

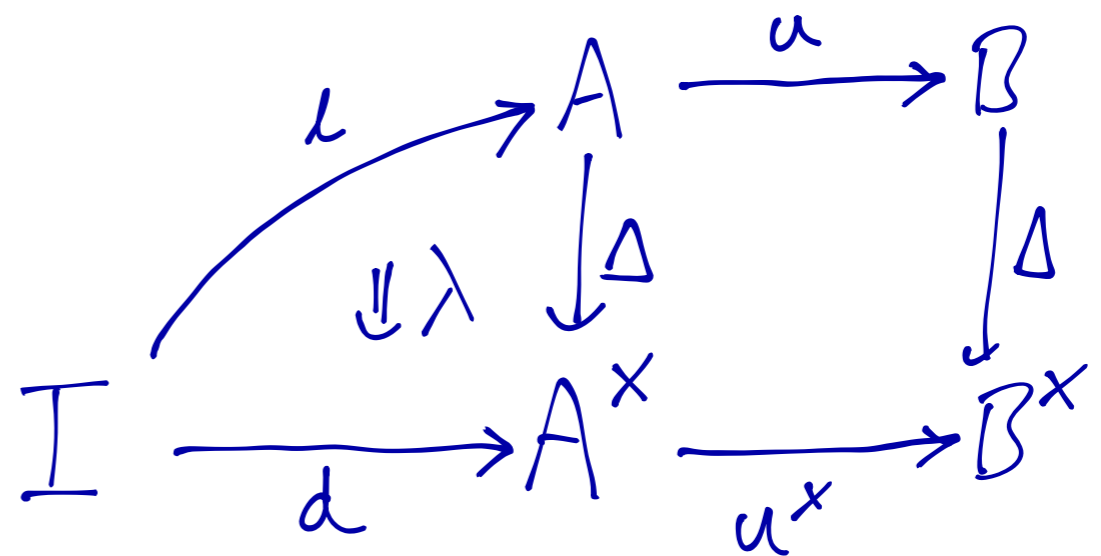


GIVEN AN ADJUNCTION



IS AN ABSOLUTE
RIGHT LIFTING

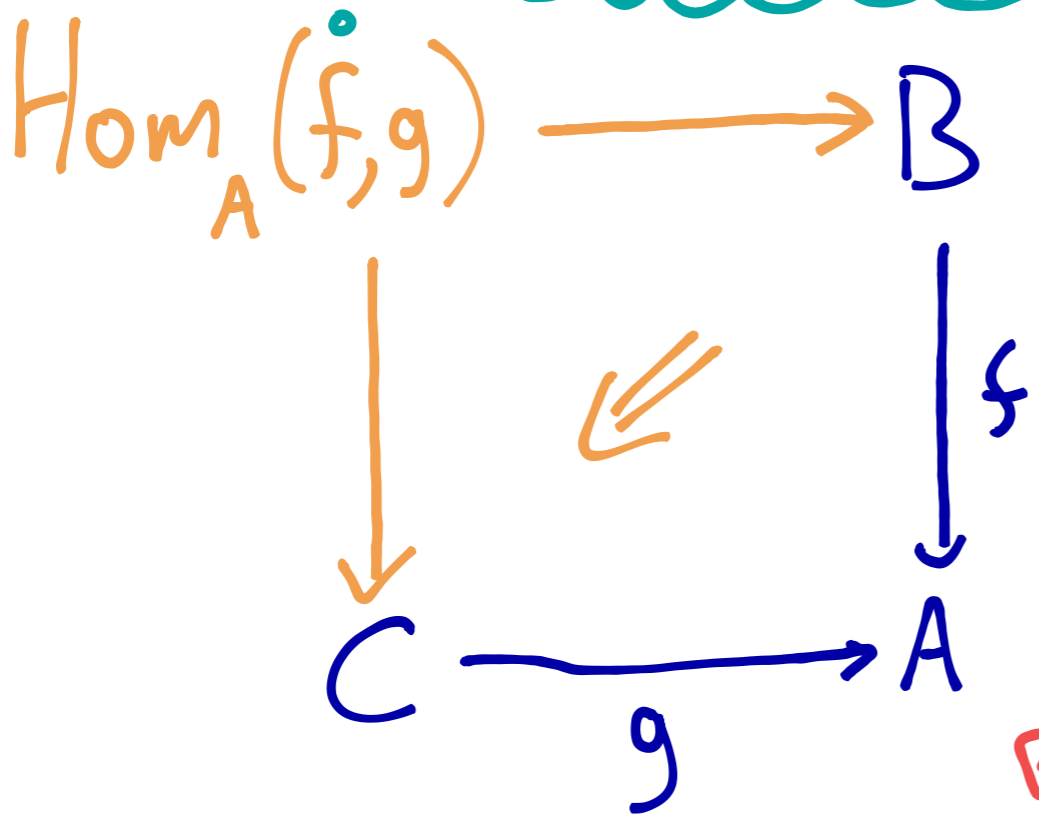
\Rightarrow



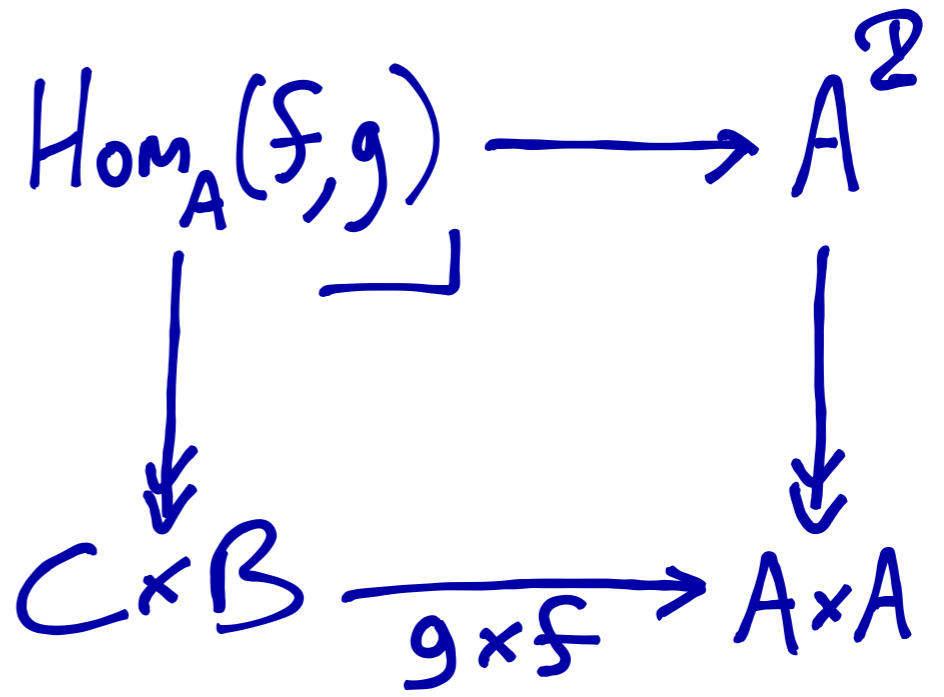
IS ALSO AN ABSOLUTE
RIGHT LIFTING.

Hom - SPACES

THE HOM SPACE FROM f TO g



A PARAMETERISED OBJECT IN THE ∞ -CATEGORY A



WEAK COMMA OBJECTS

HOM-SPACES POSSESS A **WEAK** 2-UNIVERSAL PROPERTY IN THE HOMOTOPY 2-CATEGORY.

SPECIFICALLY FOR EACH X THE INDUCED FUNCTOR

$$hK(X, \text{Hom}_A(f, g)) \longrightarrow hK(X, f) \downarrow hK(X, g)$$

IS

- SURJECTIVE ON OBJECTS,

- FULL, AND

- CONSERVATIVE.

AKA
SMOTHERING

WE SAY THAT $\text{Hom}_A(f, g)$ IS THE **WEAK COMMA** OF THE FUNCTORS f AND g IN hK .

HOM-WISE CHARACTER OF ADJUNCTIONS

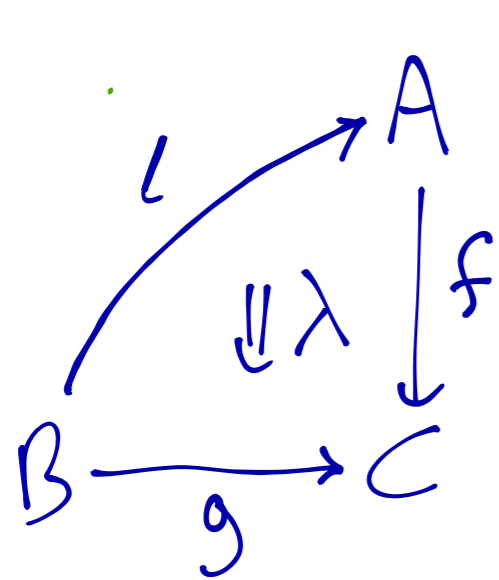
DEFN A MODULE $\text{Hom}_A(f, g) \rightarrow C \times B$ IS

-) LEFT REPRESENTABLE IF THERE EXISTS A FUNCTOR $L: B \rightarrow C$ AND AN EQUIVALENCE $\text{Hom}_C(L, C) \cong \text{Hom}_A(f, g)$ IN $\mathcal{K}/_{C \times B}$.

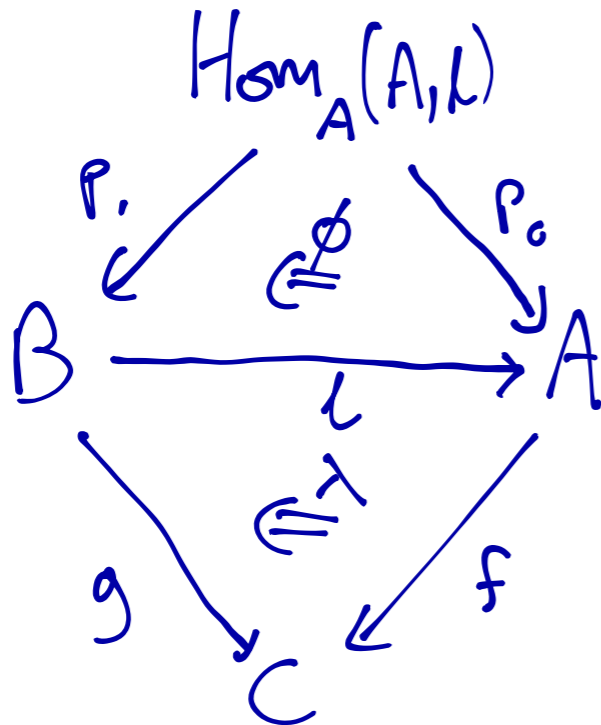
AN EQUIVALENCE
FIBRED OVER $C \times B$

-) RIGHT REPRESENTABLE IF THERE EXISTS A FUNCTOR $r: C \rightarrow B$ AND AN EQUIVALENCE $\text{Hom}_B(B, r) \cong \text{Hom}_A(f, g)$ IN $\mathcal{K}/_{C \times B}$

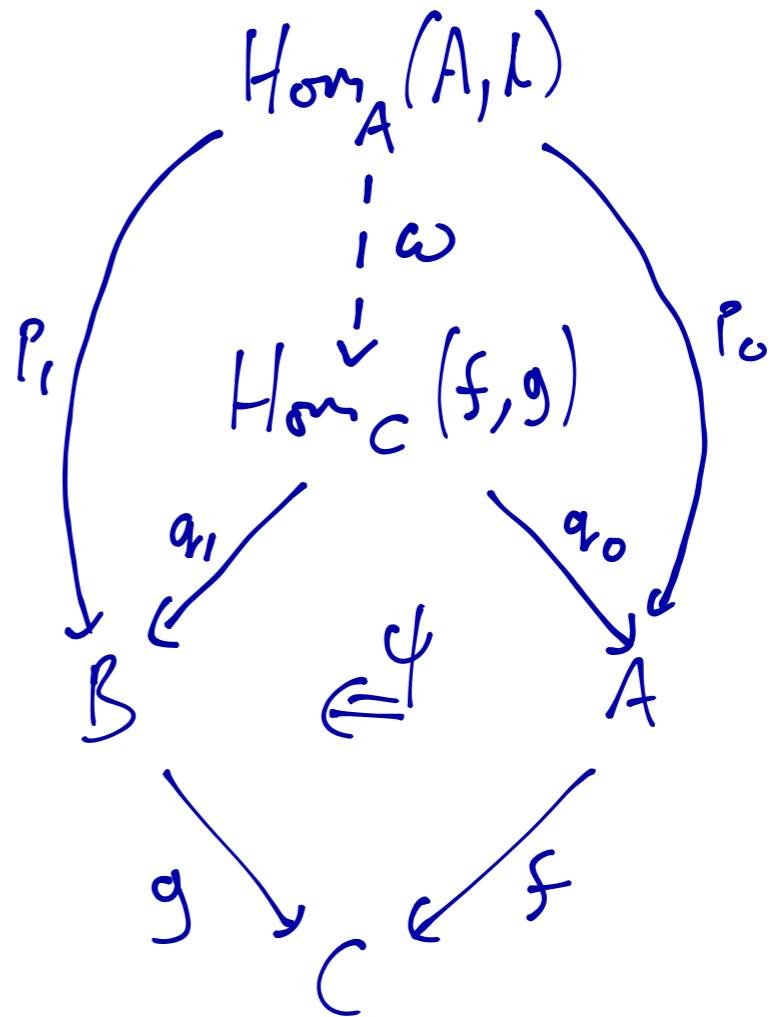
LIFTINGS MOM-WISE



LEMMA THE TRIANGLE ON THE LEFT IS AN ABSOLUTE RIGHT LIFTING IFF THE INDUCED FUNCTOR $\text{Hom}_A(A, L) \xrightarrow{\omega} \text{Hom}_C(f, g)$ IS AN EQUIVALENCE

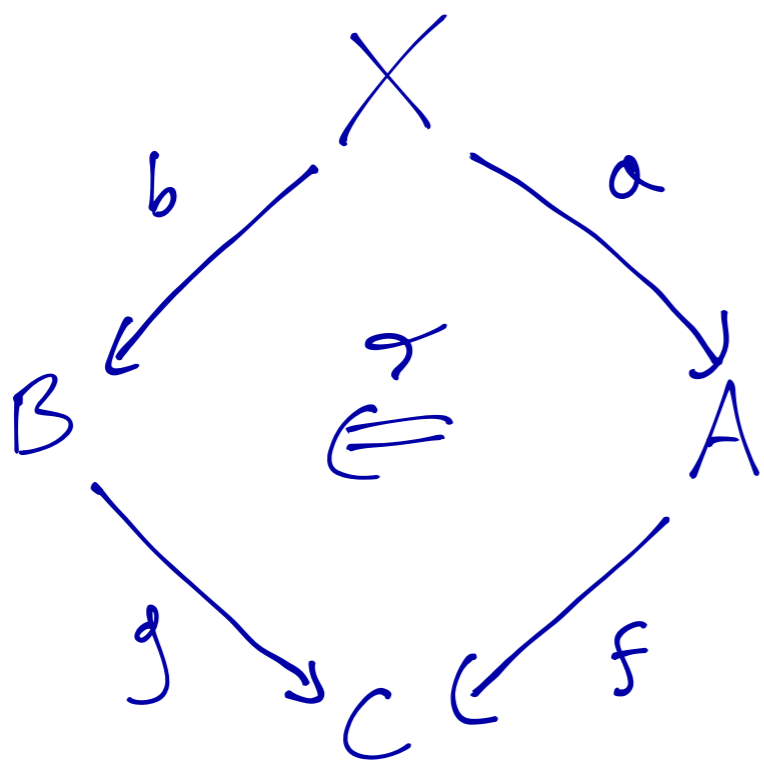


\rightsquigarrow
=



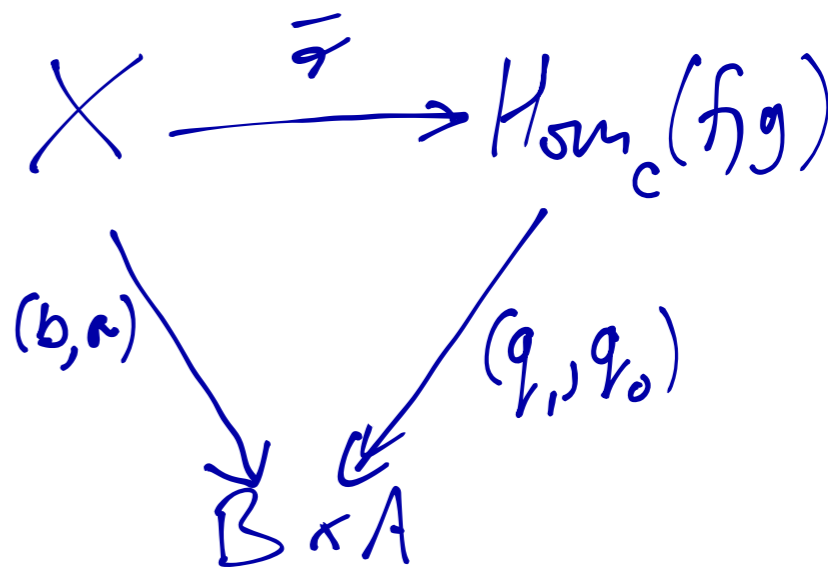
LIFTINGS MOM-WISE

KEY OBSERVATION IT IS A CONSEQUENCE OF THIS WEAK 2-UNIVERSAL PROPERTY OF $\text{Hom}_c(f, g)$ THAT THERE EXISTS A BIJECTION



2-CELLS

\rightsquigarrow
BIJECTION



ISO-CLASSES OF
1-CELLS OVER $B \times A$

THEOREM

A FUNCTOR $u: A \rightarrow B$ HAS A LEFT ADJOINT

IFF $\text{Hom}_B(B, u)$ IS LEFT REPRESENTABLE.

$$\text{Hom}_B(B, u) \xrightarrow{\sim} \text{Hom}_A(f, A)$$

$\searrow \quad \swarrow$
 $A \times B$

THIS IS THE USUAL
"HOM-WISE" CHARACTER
OF AN ADJUNCTION

THREE VIEWS OF ADJUNCTIONS

$$A \begin{array}{c} \xleftarrow{f} \\ \dashv \\ \xrightarrow{u} \end{array} B$$

$$\eta: \text{id}_B \Rightarrow uf$$

$$\varepsilon: fu \Rightarrow \text{id}_A$$

2-CATEGORICAL IN \mathcal{K}

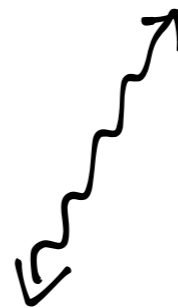
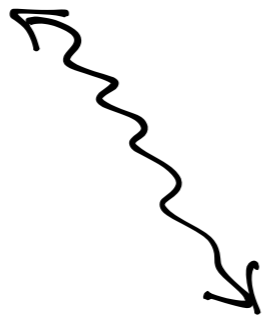


$$\text{Hom}_A(f, A) \rightarrow \text{Hom}_B(B, u)$$

$$\searrow \quad \swarrow \\ A \times B$$

"MODULES" + REPRESENTABILITY

HOMOTOPY
COHERENCE?
(PART 3)



$$A \xrightarrow{u} B$$

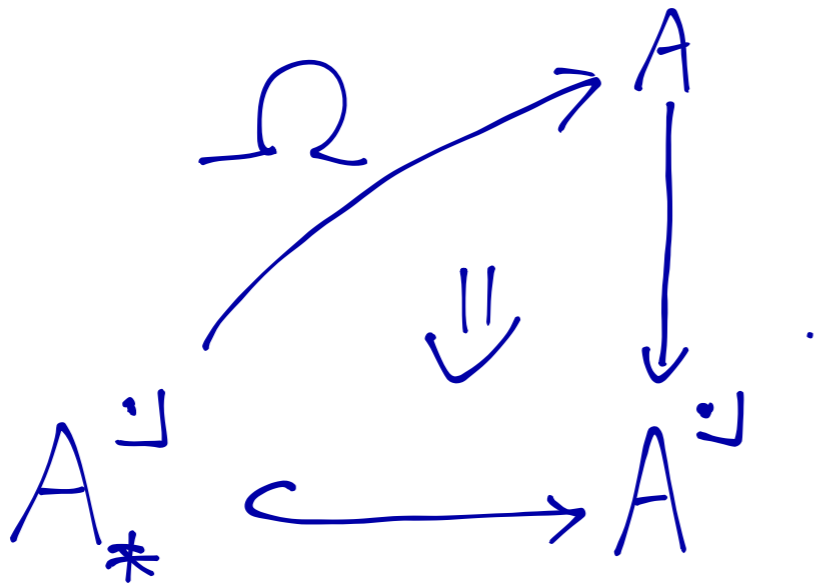
IS RIGHT ADJOINT IN \mathcal{K} IFF

$$\text{Fun}_{\mathcal{K}}(X, A) \xrightarrow{u_0} \text{Fun}_{\mathcal{K}}(X, B)$$

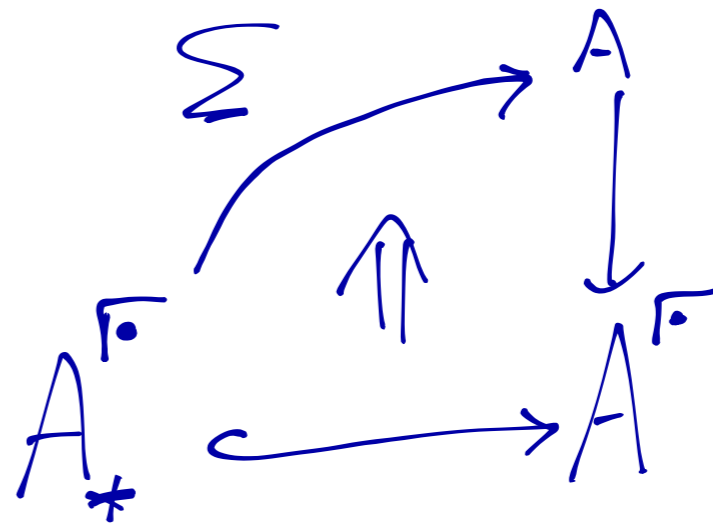
IS A RIGHT ADJOINT FUNCTOR OF QUASI-CATEGORIES FOR ALL $X \in \mathcal{K}$

RETURN TO LOOPS

WE SAY THAT A POINTED ∞ -CAT
A ADMITS LOOPS + SUSPENSIONS IF
THERE EXISTS:



ABSOLUTE
RIGHT LIFTING



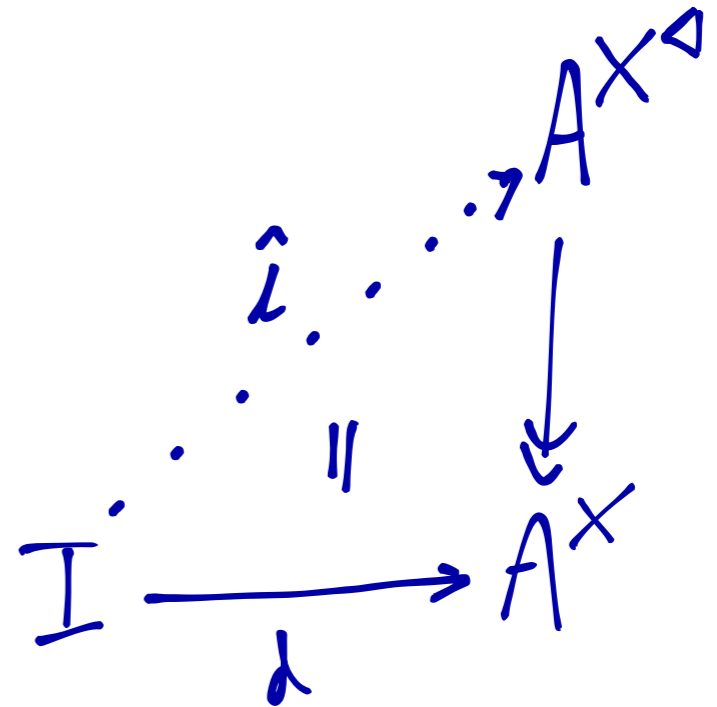
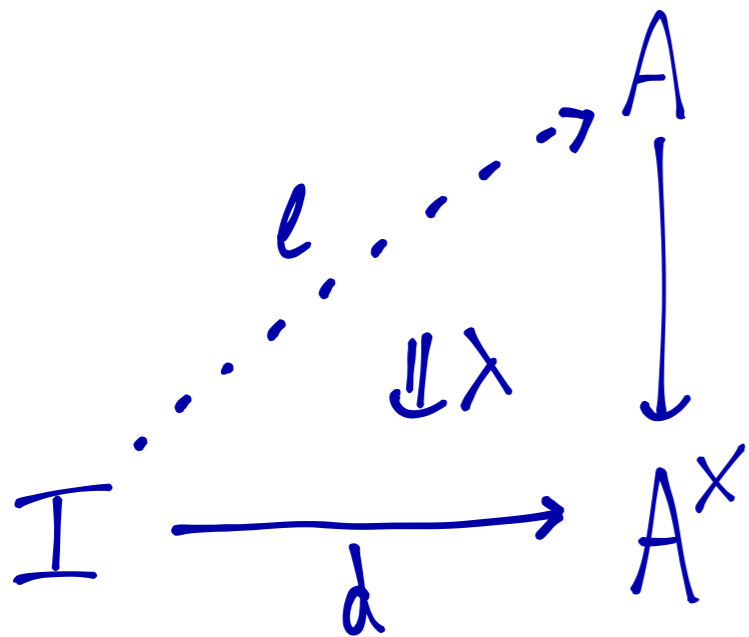
ABSOLUTE
LEFT LIFTING

(CO)LIMITS AS KAN

EXTENSIONS

× A DIAGRAM SHAPE (IN sSet)

×[▷] OBTAINED BY ADDING INITIAL OBJECT

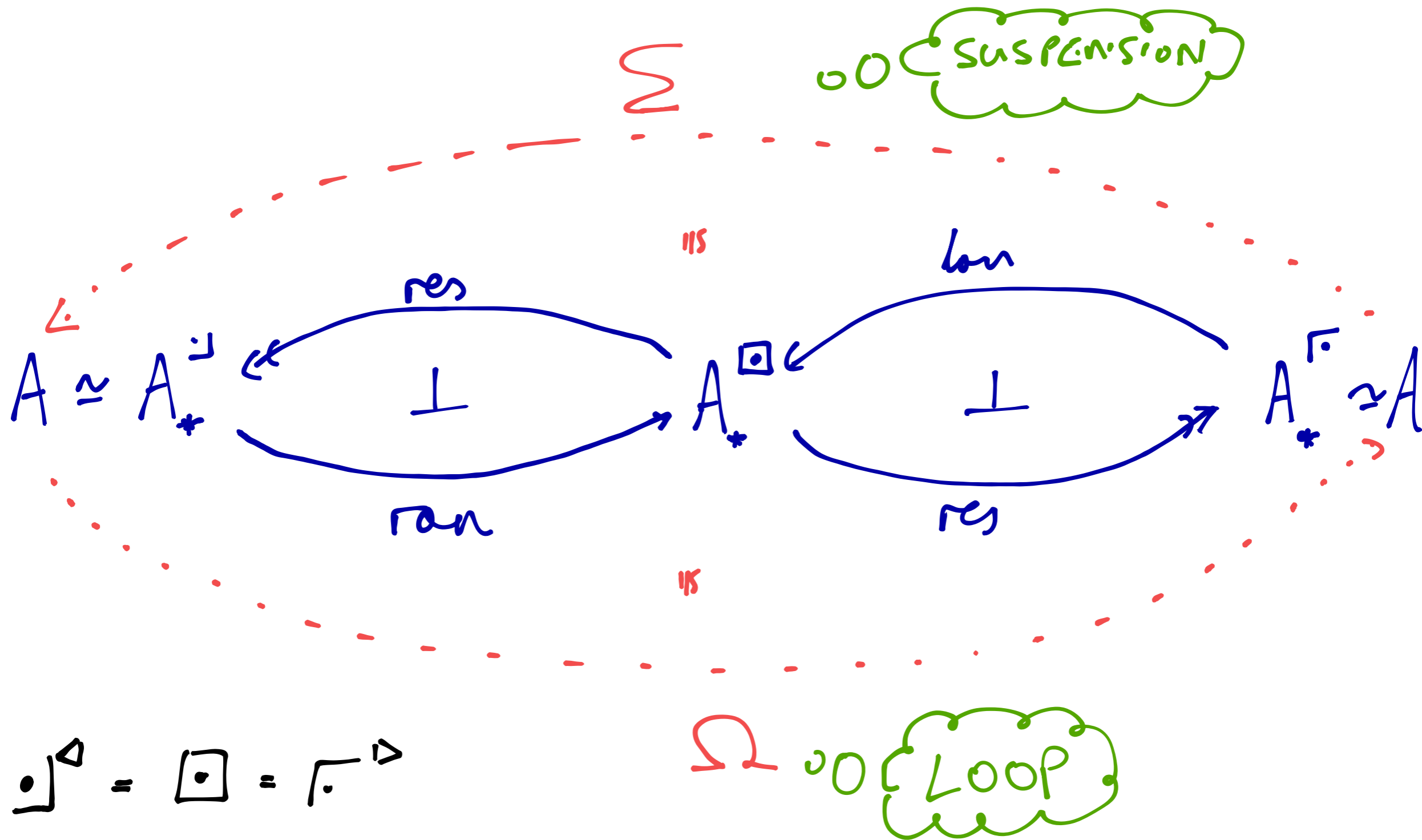


ADMITTS AN
ABSOLUTE RIGHT LIFTING



ADMITTS AN ABSOLUTE
RIGHT LIFT WHOSS 2-CELL
IS THE IDENTITY.

LOOP - SUSPENSION ADJUNCTION



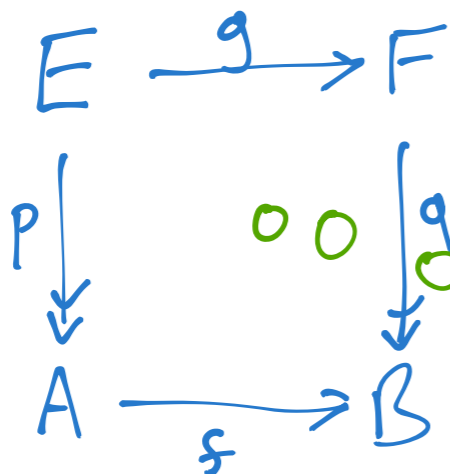
NEW ∞ -COSMOS FROM OLD.

IMPORTANT: MANY (2-) CATEGORICALLY INSPIRED CONSTRUCTIONS APPLY TO ∞ -COSMOS GIVING NEW STRUCTURES WHICH ARE AGAIN ∞ -COSMOS.

SLICES \mathcal{K}/A

OBJECTS ARE ISOFIBRATIONS $p: E \rightarrow A$

COSMOS OF ISOFIBRATIONS \mathcal{K}^2



OBJECTS ARE ISOFIBRATIONS

ARROWS ARE COMMUTATIVE SQUARES

(CO) COMPLETE ∞ -CATS

$\text{Lex}_{\omega}(\mathcal{K})$

OBJECTS: THOSE OF \mathcal{K} ADMITTING SOME CLASS OF LIMITS

ARROWS: FUNCTORS IN \mathcal{K} THAT PRESERVE THOSE LIMITS

POINTED ∞ -CATS

\mathcal{K}_*

POINTED OBJECTS OF \mathcal{K} + POINT PRESERVING FUNCTORS

ASIDE: COMPARING $h(K/B)$ AND hK/B

THE SLICE CONSTRUCTIONS ARE CALLED THE INTERNAL AND EXTERNAL SLICE RESPECTIVELY.

BUT WE DO HAVE A CANONICAL COMPARISON

UGH! THESE ARE NOT, IN GENERAL, EQUIVALENT 2-CATS

$$h(K/B)$$



$$hK/B$$

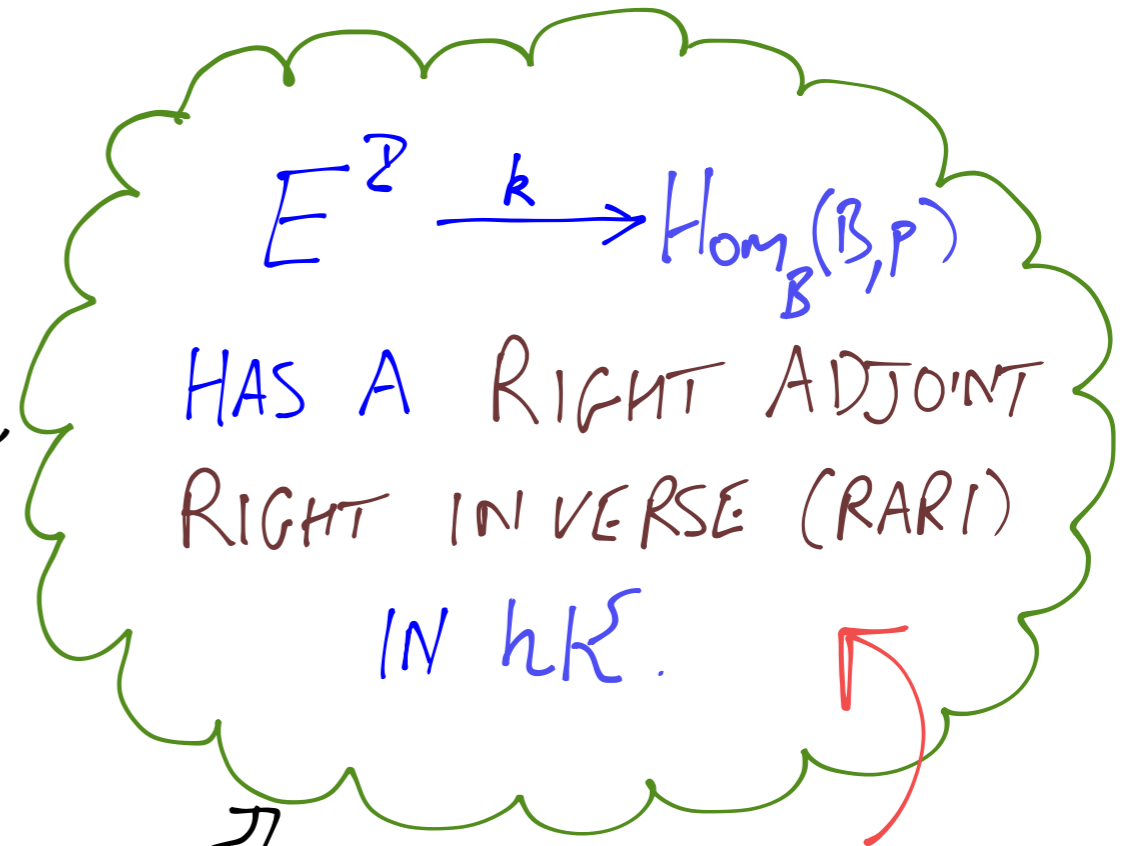
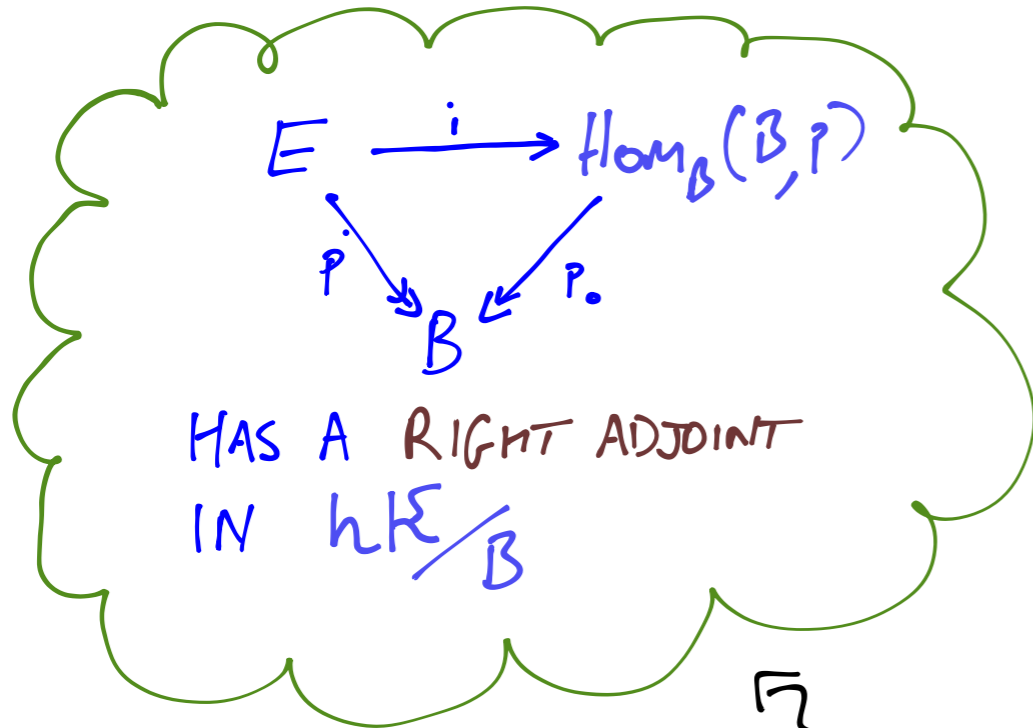
WHICH IS 2-SMOTHERING

IN PARTICULAR ADJUNCTIONS LIFT ALONG THIS COMPARISON.

OBSCURITY CORNER: IN THE 2-DERIVATOR WORLD THIS IS A KEY AXIOM.

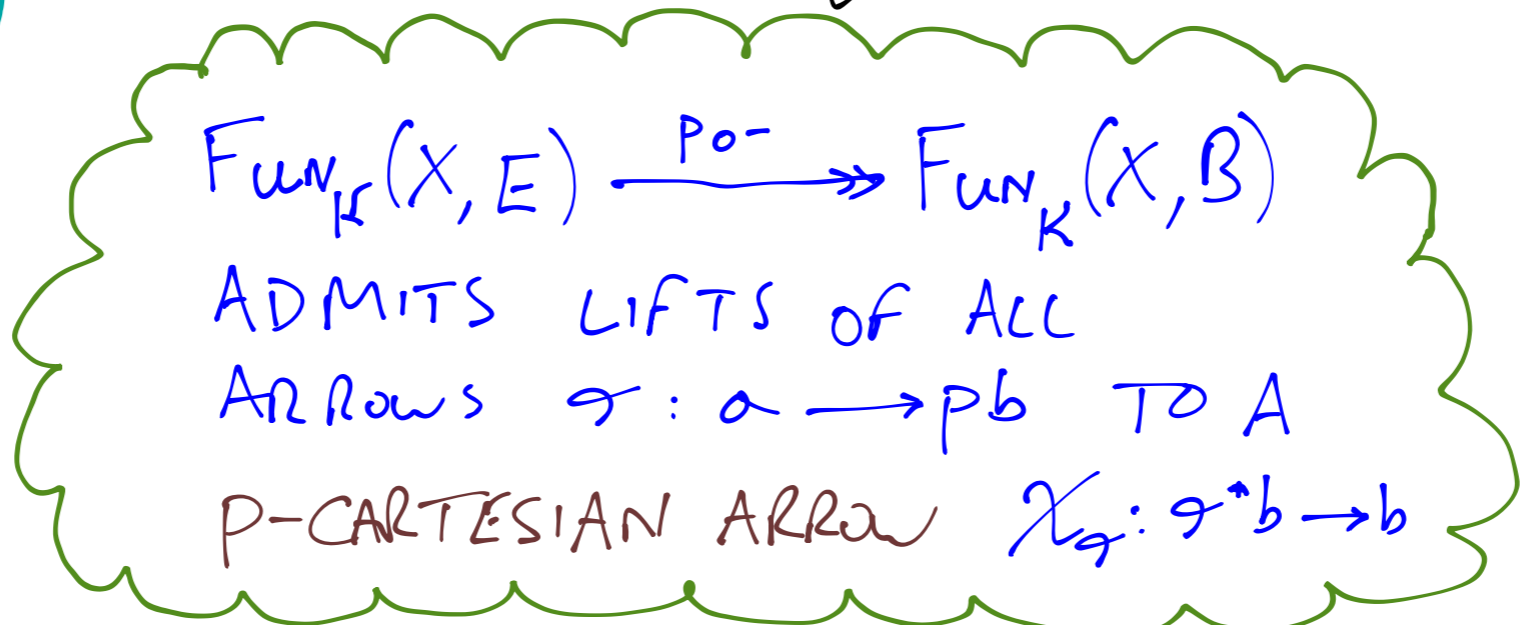
THREE VIEWS OF GROTHENDIECK FIBRATIONS

$p: E \rightarrow B$ AN ISOFIBRATION IN \mathcal{K} .



"CHEVALLEY
CRITERION"

OFTEN CALLED
CARTESIAN
FIBRATIONS IN
THE ∞ -CAT
LITERATURE



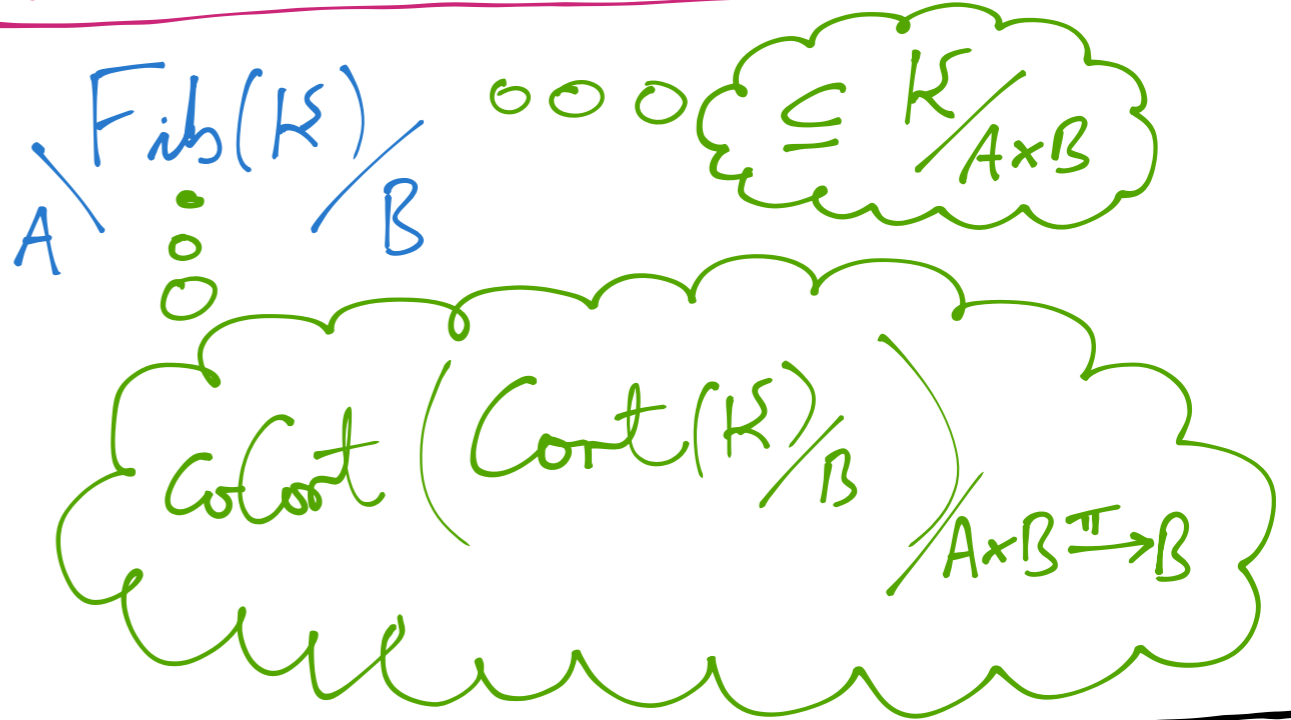
MORE ∞ -COSMOI

CARTESIAN FIBRATIONS

$\text{Cort}(K) / A \circ \circ \circ \subseteq K / A$ OVER A FIXED BASE

CARTESIAN FIBRATIONS WITH BASE A + CARTESIAN FUNCTORS BETWEEN THEM

2-SIDED CARTESIAN FIBRATIONS



ALL CARTESIAN FIBRATIONS

$\text{Cort}(K) \circ \circ \circ \subseteq K^2$



CARTESIAN FUNCTOR: PRESERVES CARTESIAN 1-ARROWS

CARTESIAN FIBRATION

GROUPOIDAL ∞ -CATS

$\text{Gpd}(K) \circ \circ \circ \subseteq K$

$A \in \text{Gpd}(K) \text{ IFF } \forall X \in K$


$\text{Fun}_K(X, A)$

IS A KAN COMPLEX.

MODULES

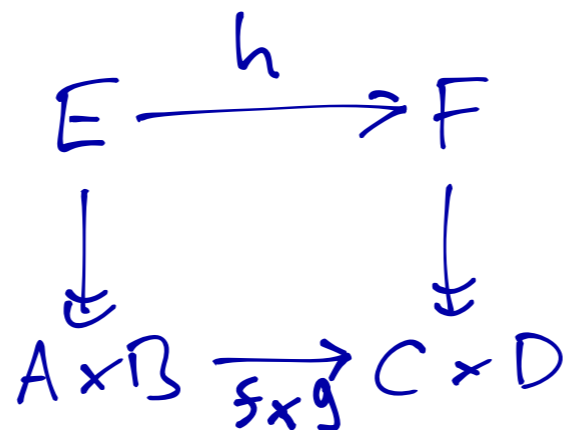
AKA PROFUNCTORS

$$A \backslash \text{Mod}(K) / B \quad \bullet \bullet = \text{Gpd} \left(A \backslash \text{Fib}(K) / B \right)$$

GIVEN $A \xrightarrow{f} C \xleftarrow{g} B$
 $\text{Hom}_C(g, f)$ IS A MODULE


ALL MODULES

$\text{Mod}(K) \circlearrowright$ + MAPS

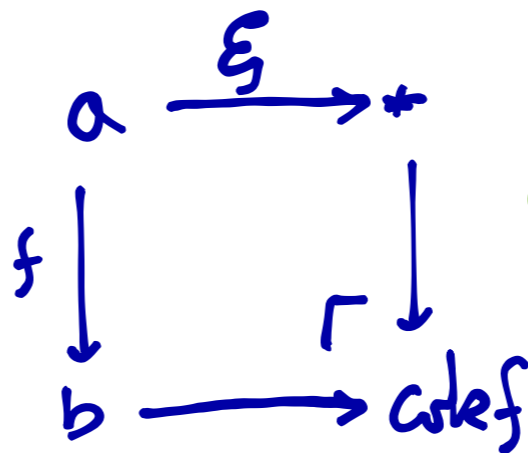
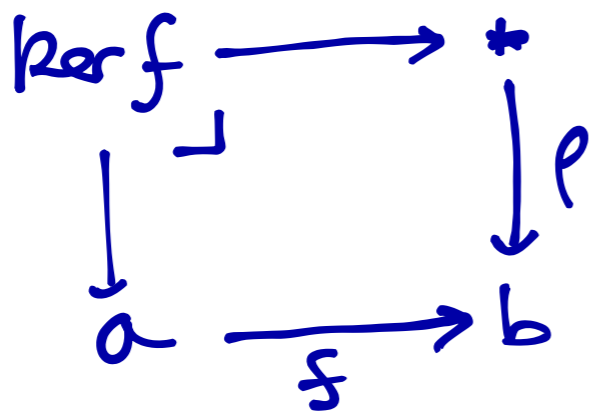


THIS IS AGAIN!
AN ∞ -COSMOS.

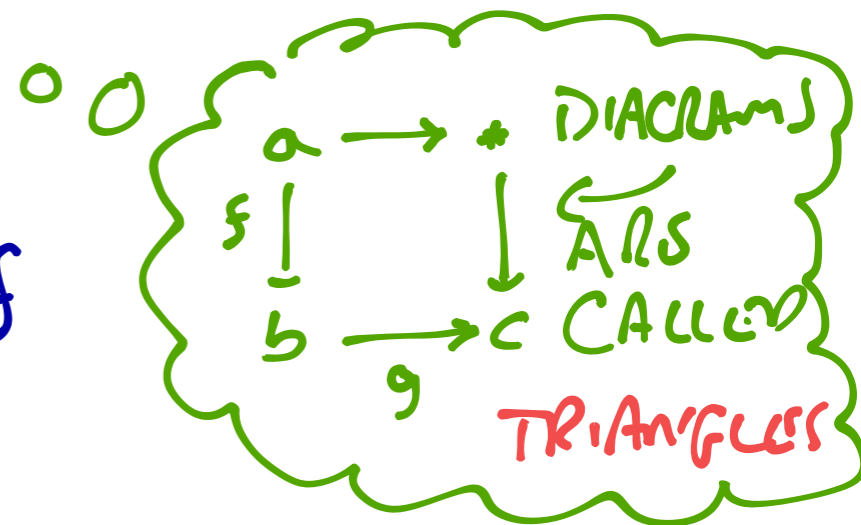
STABLE ∞ -CATEGORIES

AN ∞ -CATEGORY \mathcal{A} IS SAID TO BE STABLE IF

- \mathcal{A} HAS A ZERO OBJECT $*$: $1 \rightarrow \mathcal{A}$
- EVERY X -INDEXED FAMILY $f: a \rightarrow b$ OF ARROWS IN \mathcal{A} ADMITS KERNELS + COKERNELS

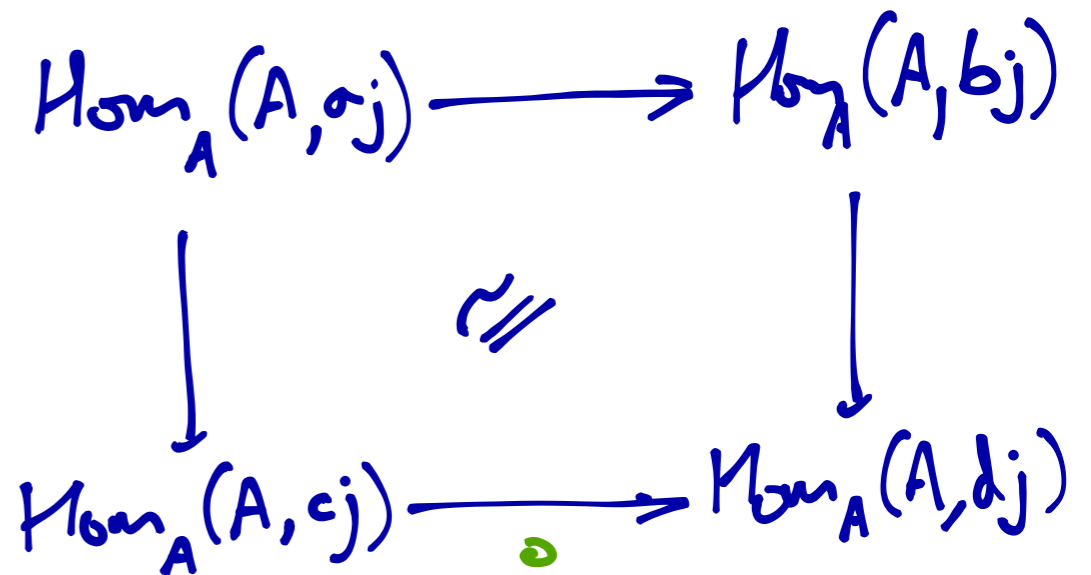
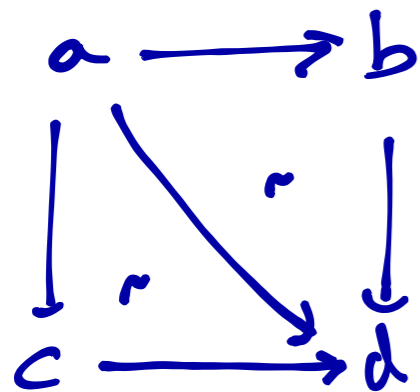


$$f: X \rightarrow A^2$$



- A TRIANGLE IS A PUSHOUT (COKERNEL) IFF IT IS A PULLBACK (KERNEL).

A HOM-WISE CHARACTERISATION OF PULLBACKS



I-INDEXED COMMUTATIVE SQUARES IN A

ISO-SQUARES IN
2-CAT $\text{Mod}_K(J, A)$

THIS SQUARE IS AN I-INDEXED FAMILY OF PULLBACKS \Leftrightarrow

THIS ISO-SQUARE IS A WEAK ISO-COMMA FOR ALL $j: J \rightarrow I$.

APPLICATION: THE COMPOSITION + CANCELLATION LAWS HOLD FOR PULLBACKS IN ANY \mathcal{A} -CAT.

ELEMENTARY RESULTS ...

- COMPOSITION + CANCELLATION OF PULLBACK / PULLBACK SQUARES \Rightarrow STABLE ∞ -CATS ADMIT ALL PULLBACKS / PUSHOUTS:

$$\begin{array}{ccccc}
 \ker kg & \longrightarrow & a & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow f & \lrcorner & \downarrow \\
 c & \xrightarrow{g} & b & \xrightarrow{k} & \text{coker } f
 \end{array}$$

- THE LOOP-SUSPENSION ADJUNCTION OF A STABLE ∞ -CAT IS AN EQUIVALENCE.
- A POINTED ∞ -CATEGORY IS STABLE IFF IT ADMITS KERNELS + ITS LOOP FUNCTOR IS AN EQUIVALENCE.

STABILISATION

THE SUBCATEGORY OF

-) POINTED ∞ -CATEGORIES ADMITTING KERNELS
-) ∞ -FUNCTORS THAT PRESERVE POINTS AND KERNELS.

IS A **SUB- ∞ -COSMOS** SO IT ADMITS A HOMOTOPY LIMIT OF EACH CHAIN:

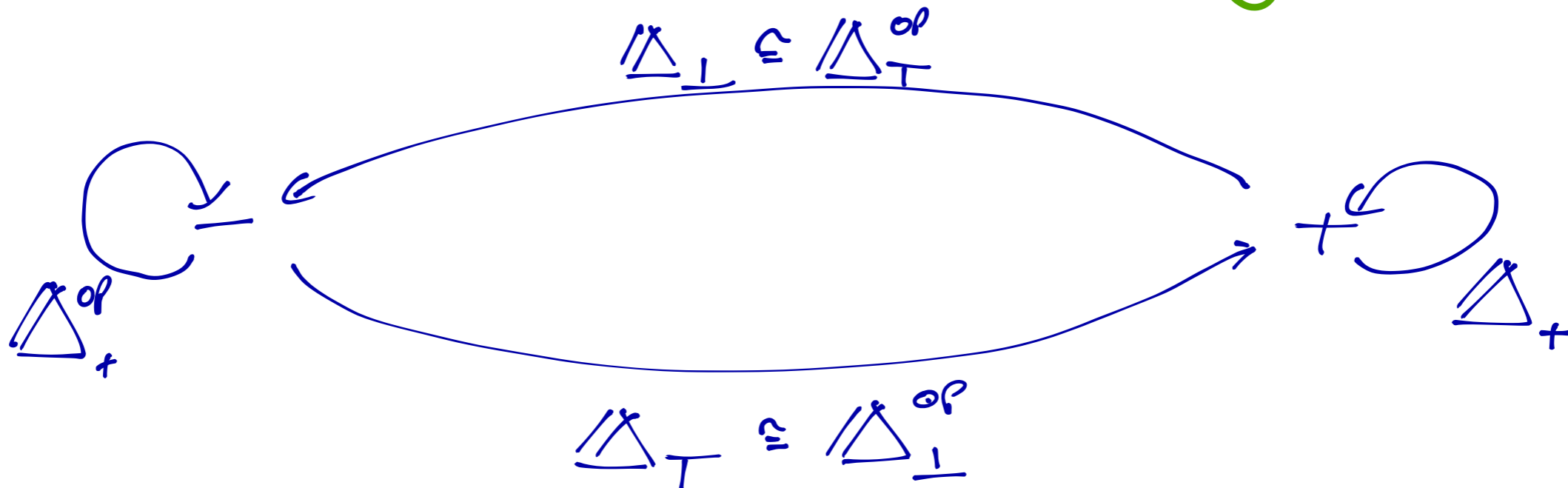
$$\dots \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A$$

PROP LOOPS FUNCTOR ON THIS LIMITING ∞ -CATEGORY IS AN EQUIVALENCE.

\Rightarrow THIS ∞ -CATEGORY $\text{STAB}(A)$ IS STABLE.

SPECTRA
LIVE
HERE!

THE 2-CATEGORY Adj



$$[0] : - \longrightarrow + \quad \text{circled } \underline{u}$$

$$[0] : + \longrightarrow - \quad \text{circled } \underline{f}$$

$$\underline{f} \underline{u} = [0] : - \longrightarrow - \quad \text{id}_- = [-1] : - \longrightarrow - \quad [-1] \rightarrow [0] \in \Delta_+$$

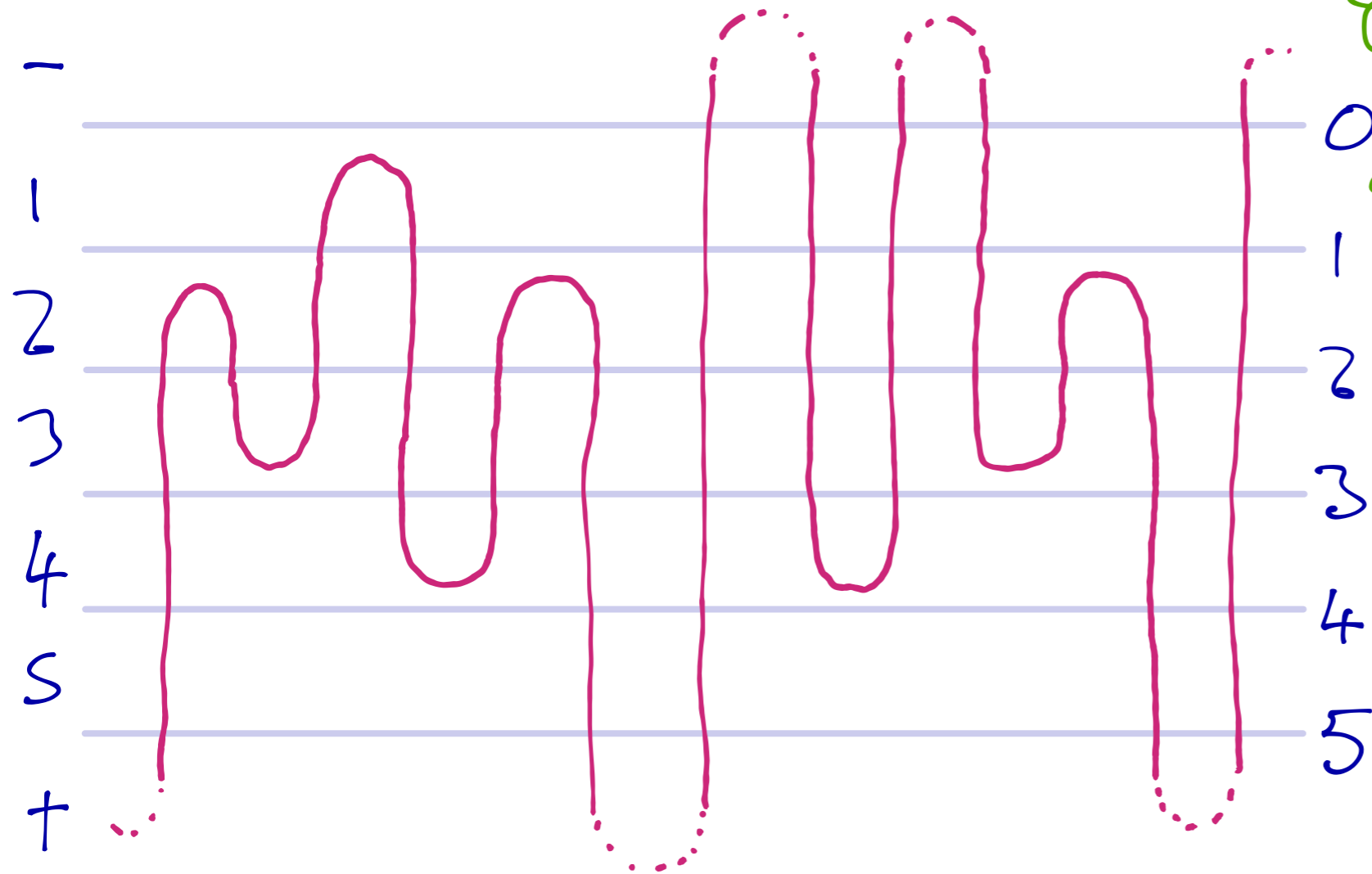
$\xi : \underline{f} \underline{u} \Rightarrow \text{id}_- \in \text{Adj}(-, -) = \Delta_+^{\text{op}}$

$$\underline{u} \underline{f} = [0] : + \longrightarrow + \quad \text{id}_+ = [-1] : + \longrightarrow + \quad [-1] \rightarrow [0] \in \Delta_+$$

$\eta : \text{id}_+ \Rightarrow \underline{u} \underline{f} \in \text{Adj}(+, +) = \Delta_+$

THE SIMPLICIAL CATEGORY Adj

n-ARROWS



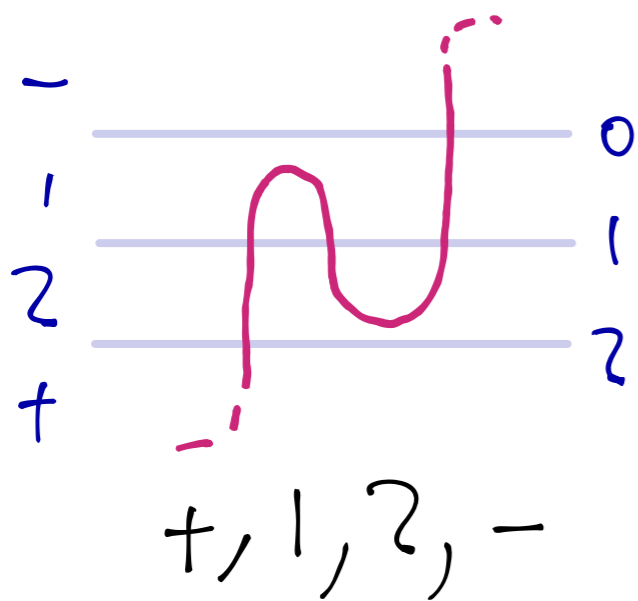
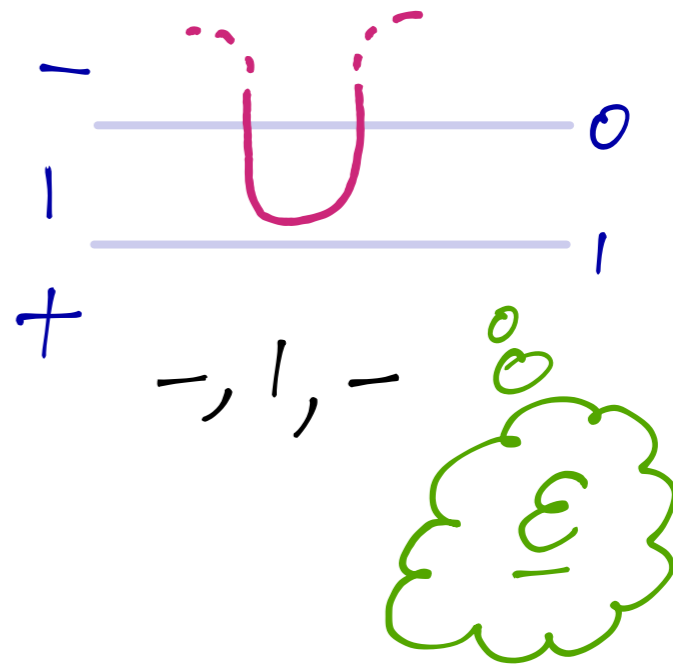
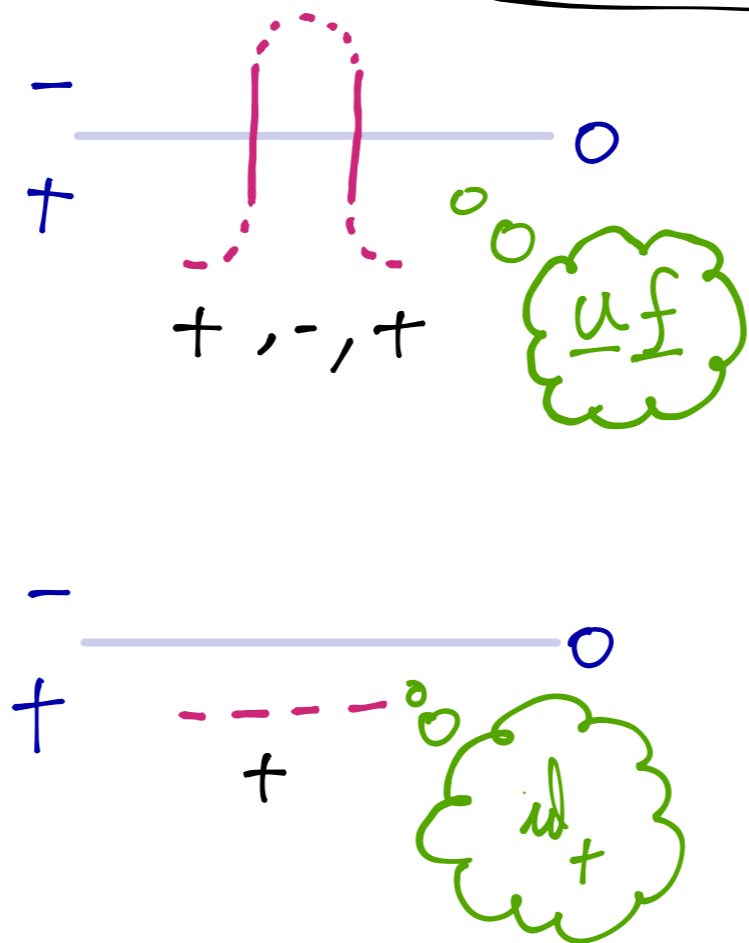
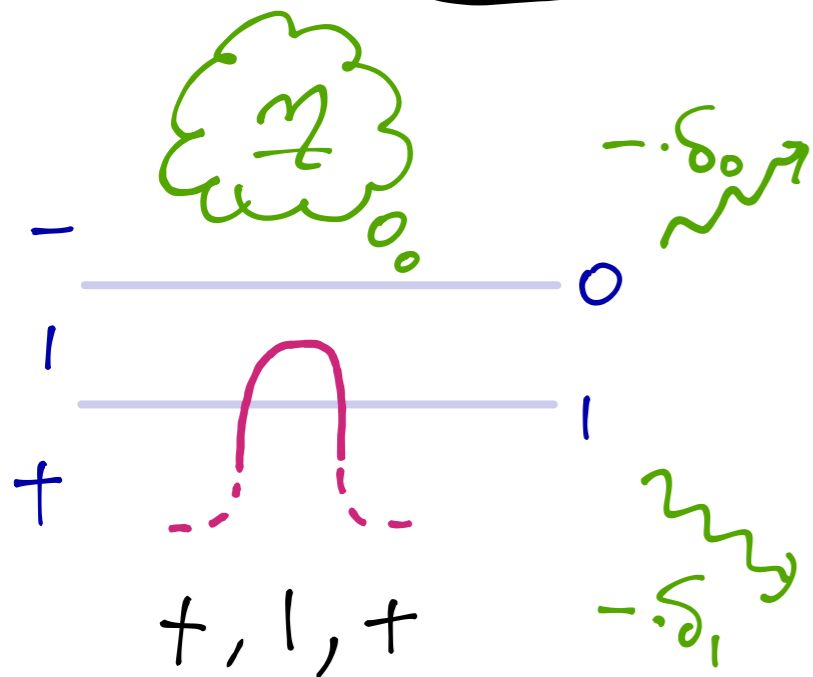
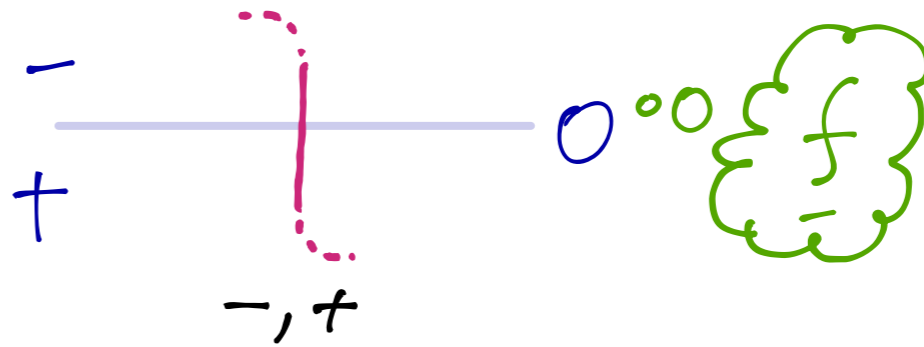
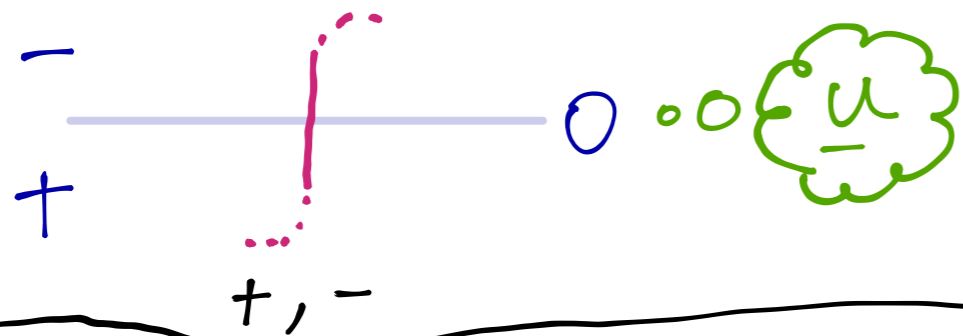
STRICTLY UNDUCTING
SQUIGGLE \equiv TREMELO

THIS DIAGRAM
DEPICTS A
5-ARROW

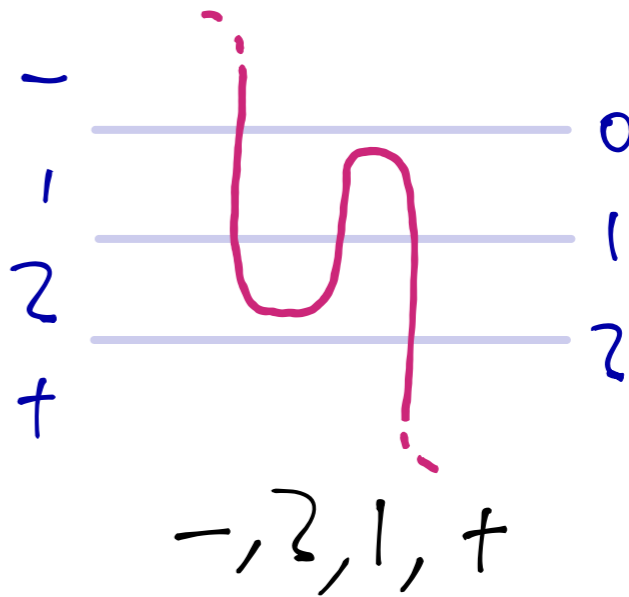
$\oplus, 2, 3, 1, 4, 2, +, -, 4, -, 3, 2, +, \ominus$

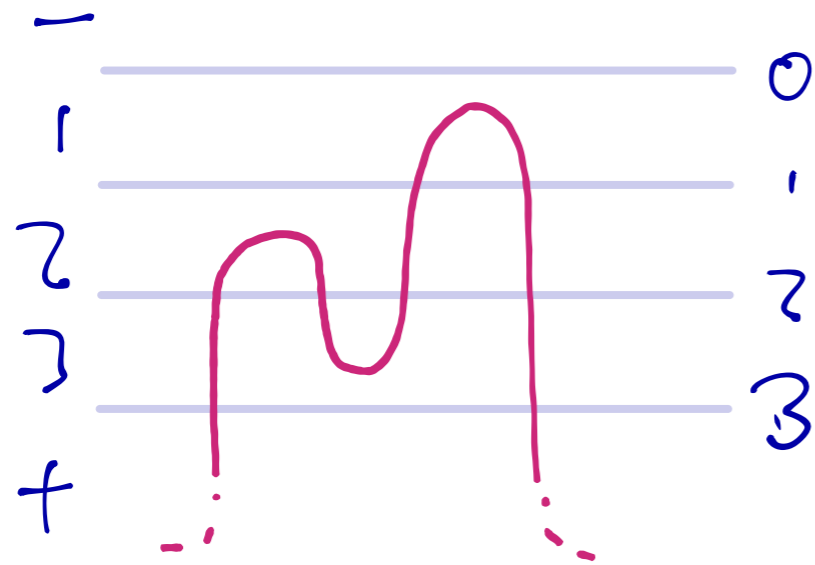
←
CODOMAIN

→
DOMAIN

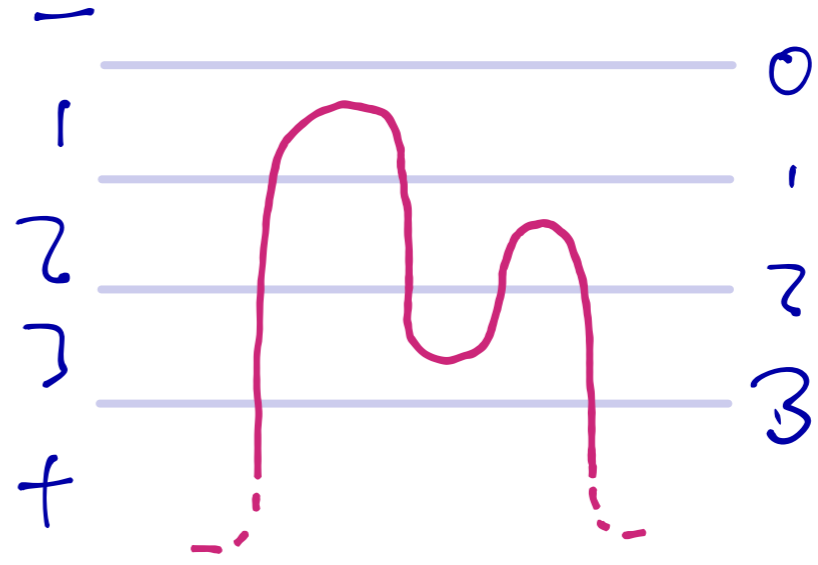


WITNESSES
OF THE
TRIANGLE
IDENTITIES





+ , 2 , 3 , 1 , +



+ , 1 , 3 , 2 , +

SOME HIGHER
COHERENCES BETWEEN
TRIANGLE IDENTITIES

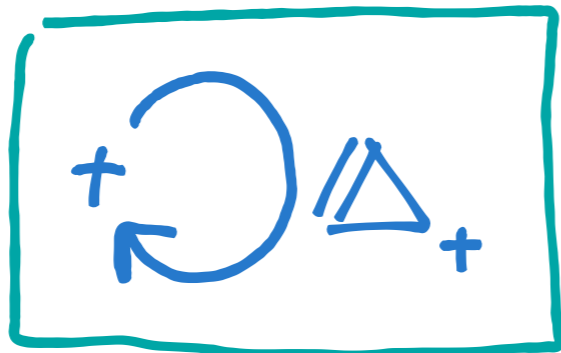
ADJUNCTIONS ARE COHERENT

THEOREM: AN ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$ GIVES RISE TO A SIMPLICIAL FUNCTOR $\mathcal{A} : \mathcal{A}dj \rightarrow \mathcal{K}$ WHICH CARRIES THE CANONICAL ADJUNCTION $\underline{\mathcal{F}} \dashv \underline{\mathcal{U}}$ IN $\mathcal{A}dj$ TO THE ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$. THE SPACE OF FUNCTORS THAT EXTEND $\mathcal{F} \dashv \mathcal{U}$ IN THIS WAY IS CONTRACTIBLE.

OBSCURITY CORNER: THIS IS ANOTHER IMPORTANT INTERNALISATION AXIOM IN THE WORLD OF 2-DERIVATORS.

HOMOTOPY COHERENT MONADS

$\mathcal{M}nd$

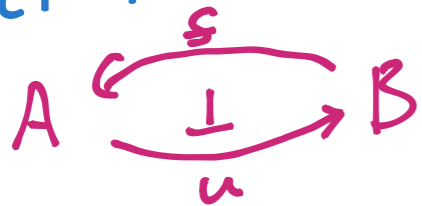


THE "WALKING"
HOMOTOPY COHERENT
MONAD.

FULL SUBCATEGORY
OF Adj SPANNING
THE OBJECT $+$

A HOMOTOPY COHERENT MONAD
IN AN ∞ -COSMOS \mathcal{K} IS SIMPLY
A SIMPLICIAL FUNCTOR

THE HOMOTOPY COHERENCE
RESULT FOR AN ADJUNCTION



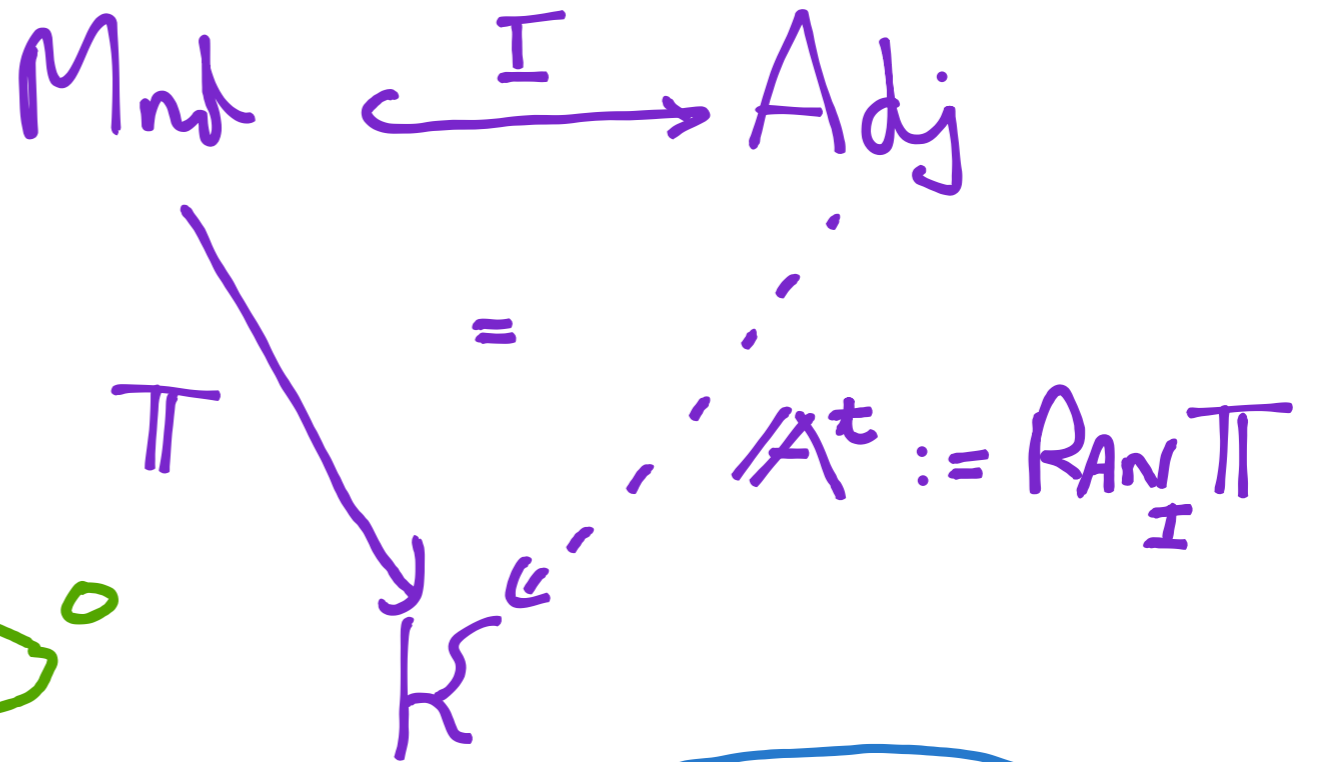
IN \mathcal{K} IMPLIES THAT
SUCH INDUCES AN MC
MONAD ON B AND AN
MC COMONAD ON A .

$$\Pi : \mathcal{M}nd \longrightarrow \mathcal{K}$$

$$\begin{array}{ccc}
 A \in \mathcal{K} & + & A \xrightarrow{t} A^{\Delta_+} \\
 A \xrightarrow{t} A^{\Delta_+} & & \begin{array}{ccc}
 A & \xrightarrow{t} & A^{\Delta_+} \\
 \downarrow t & & \downarrow A^{\oplus} \\
 A^{\Delta_+} & \xrightarrow{t^{\Delta_+}} & A^{\Delta_+ \times \Delta_+}
 \end{array}
 \end{array}$$

EILENBERG - MOORE OBJECTS

∞ -COSMOS ADMIT
ALL FLEXIBLY
WEIGHTED LIMITS



THIS KAN EXTENSION
EXISTS BECAUSE
Adj IS A COMPUTAD \Rightarrow
THE WEIGHT USED TO
COMPUTE IT IS FLEXIBLE

FLEXIBLE
 \equiv
COFIBRANT

THE OBJECT $\mathbf{A}^\pi := \mathbf{A}^t(-)$ IS THE ∞ -CAT
OF EILENBERG - MOORE ALGEBRAS.