

SYNTHETIC ∞ -CATEGORY THEORY

INTRODUCTORY WORKSHOP :
HIGHER CATEGORIES & CATEGORIFICATION
MSRI , FEB 2020

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AIM: GIVE A PRECISE AND MODEL
INDEPENDENT ACCOUNT OF
 ∞ -CATEGORY THEORY.

RESOURCES

" ∞ -CATEGORY THEORY FROM SCRATCH"

https://journals.mq.edu.au/index.php/higher_structures/article/view/38

"ELEMENTS OF ∞ -CATEGORY THEORY"

<http://www.math.jhu.edu/~eriehl/elements.pdf>

$(\infty, 1)$ - CATEGORIES

SCHEMATIC

"DEFINITION"

AN $(\infty, 1)$ -CATEGORY IS
A CATEGORY WEAKLY
ENRICHED IN SPACES

∞ -GROUPOIDS

MODELS

QUASI-CATEGORIES
COMPLETE SEGAL SPACES
SEGAL CATEGORIES
MARKED QUASI-CATEGORIES

ALSO FIBRED
VARIANTS OF
THESE NOTIONS

∞ - CATEGORIES

AUXILIARY AIM: BUILD A FRAMEWORK THAT CAN BE EXTENDED TO GIVE FOUNDATIONS FOR THE CATEGORY THEORY OF (∞, n) - CATEGORIES

MODELS:

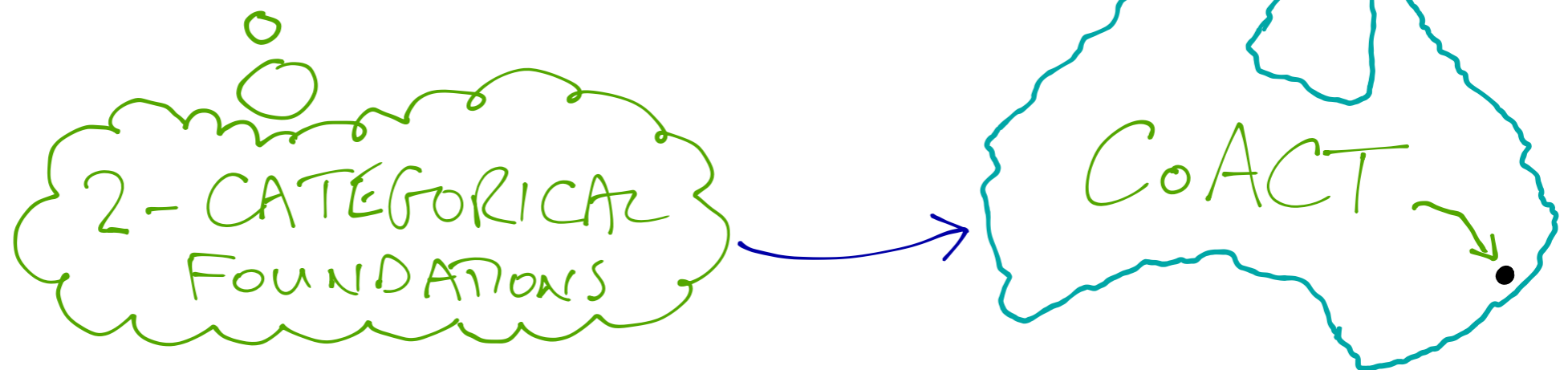
COMPLICIAL SETS
ITERATED SEGAL
 \mathbb{H}_n -SETS AND SPACES

CATEGORIES WEAKLY ENRICHED IN $(\infty, n-1)$ -CATEGORIES

WE WILL USE THE TERM ∞ -CATEGORY TO MEAN ANY STRUCTURE THAT INHABITS AN INSTANCE OF THE FRAMEWORK WE BUILD HERE.

BENEFITS

-) GIVE **SYNTHETIC** PROOFS OF ∞ -CATEGORICAL RESULTS, WHICH THEN APPLY IN ALL MODELS.
-) TRANSFER CATEGORICAL RESULTS DERIVED **ANALYTICALLY** IN ONE MODEL TO RELATED MODELS.
-) ADAPT AND APPLY INTUITIONS FROM "META-CATEGORY" THEORY.



PLAN

PART 1: AN AXIOMATIC FRAMEWORK FOR ∞ -CATEGORY THEORY.

PART 2: LIMITS AND ADJUNCTIONS.

PART 3: HOMOTOPY COHERENT MONADS AND BECK MONADICITY.

DIVING IN: 2-CATEGORIES

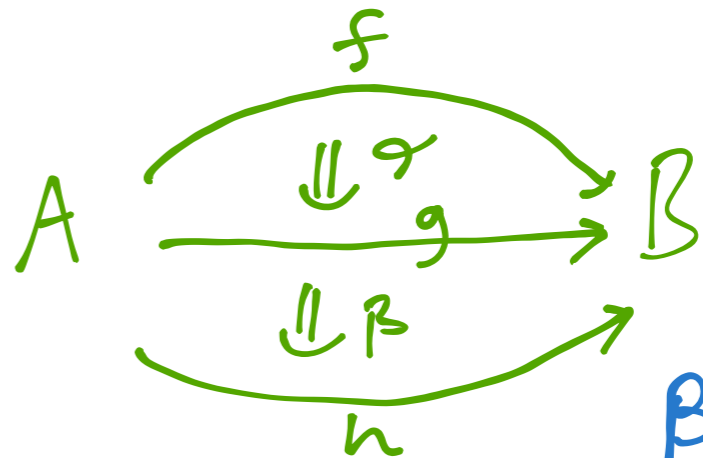
MODELED ON THE
TOTALITY OF CATEGORIES,
FUNCTORS + NATURAL
TRANSFORMATIONS

OF THE STRICT
VARIETY

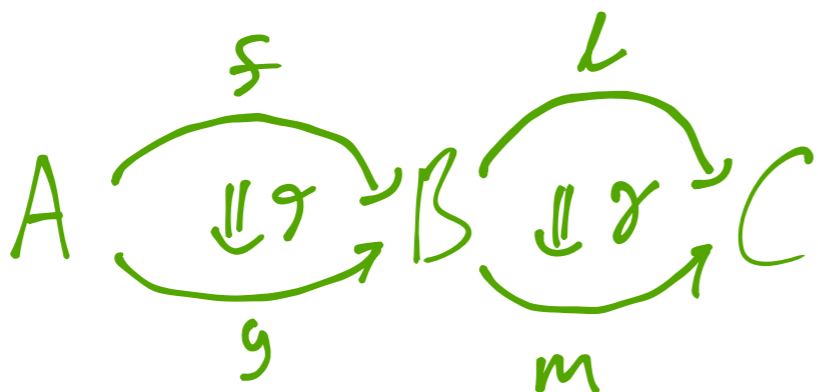
AKA
CATEGORIES
ENRICHED
IN $(\text{Cat}, \times, \mathbb{1})$

- OBJECTS A, B, C
- 1-CELLS $A \xrightarrow{f} B$
- 2-CELLS $A \begin{matrix} \xrightarrow{f} \\ \Downarrow \eta \\ \xrightarrow{g} \end{matrix} B$

2-CELLS COMPOSE
VERTICALLY



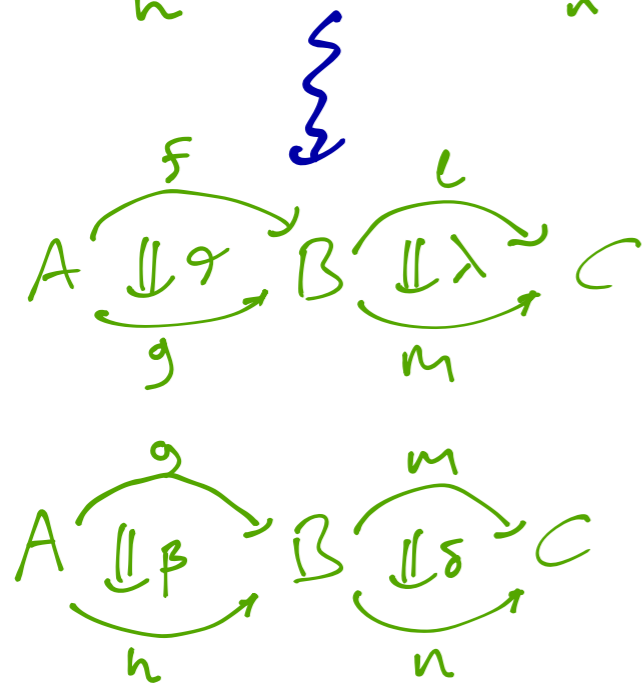
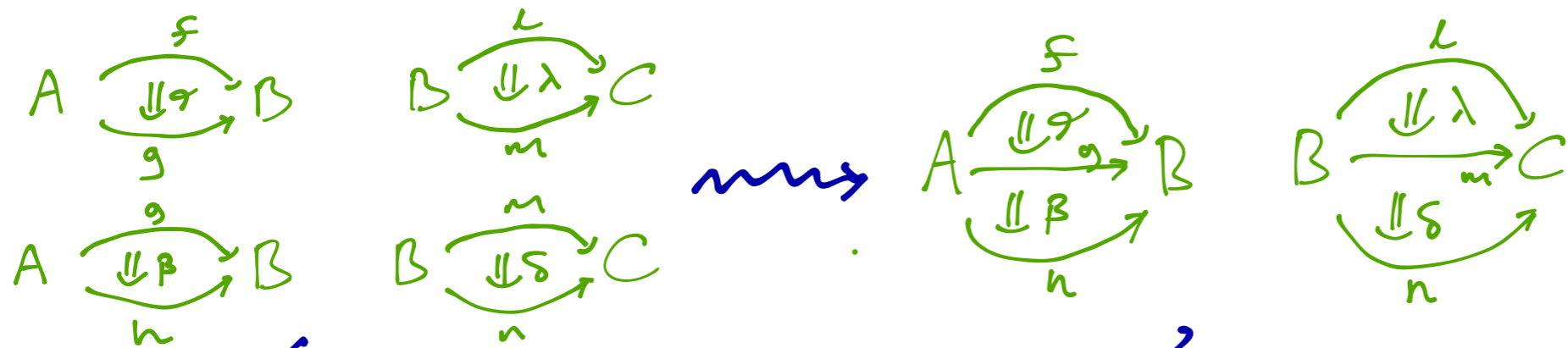
HORIZONTAL



$\gamma \circ \eta$

BOTH STRICTLY
ASSOCIATIVE

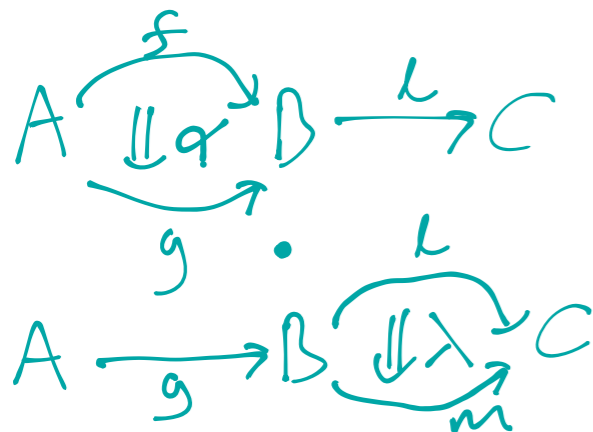
MIDDLE-4-INTERCHANGE



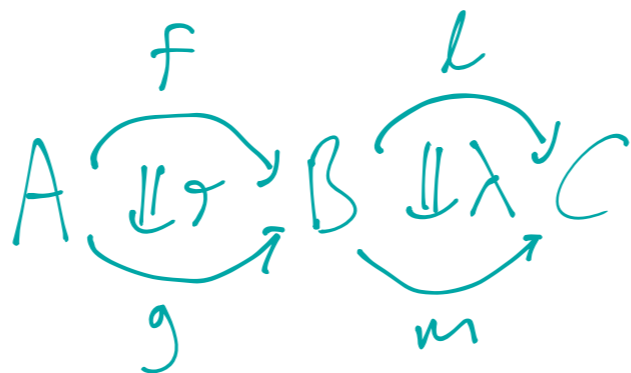
$(\delta \circ \beta) \cdot (\lambda \circ \gamma)$

WHISKERING

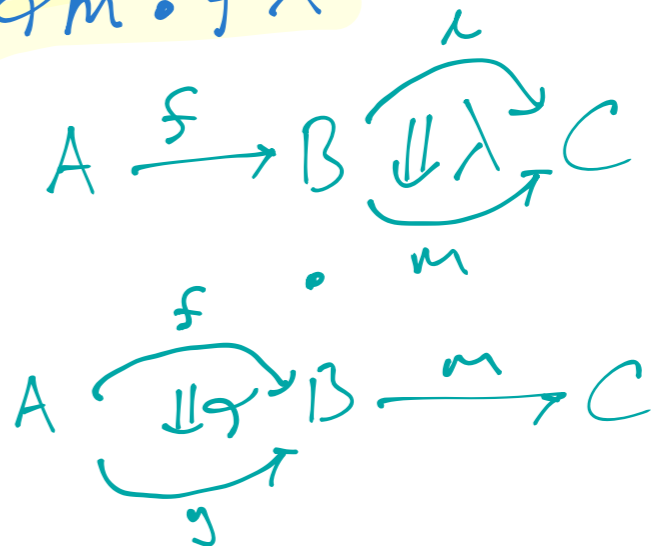
$\lambda \circ \gamma \cdot l \circ g = \lambda \circ \gamma = g \circ m \cdot f \cdot \lambda$



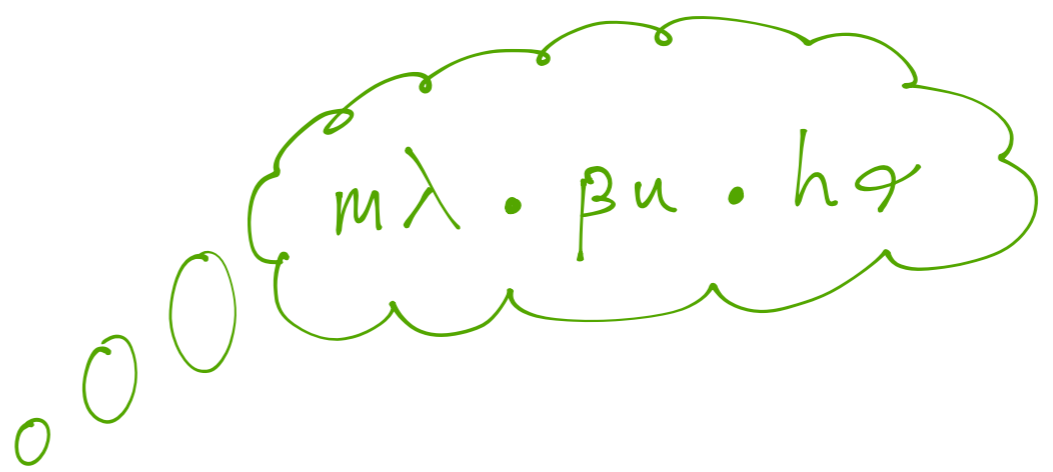
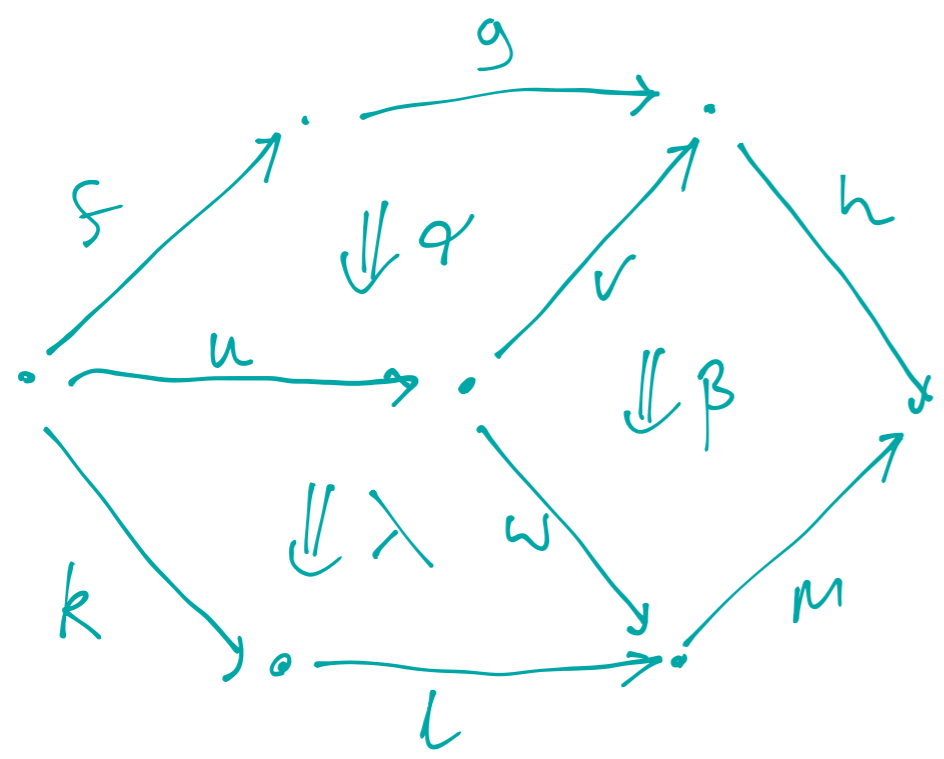
\rightsquigarrow



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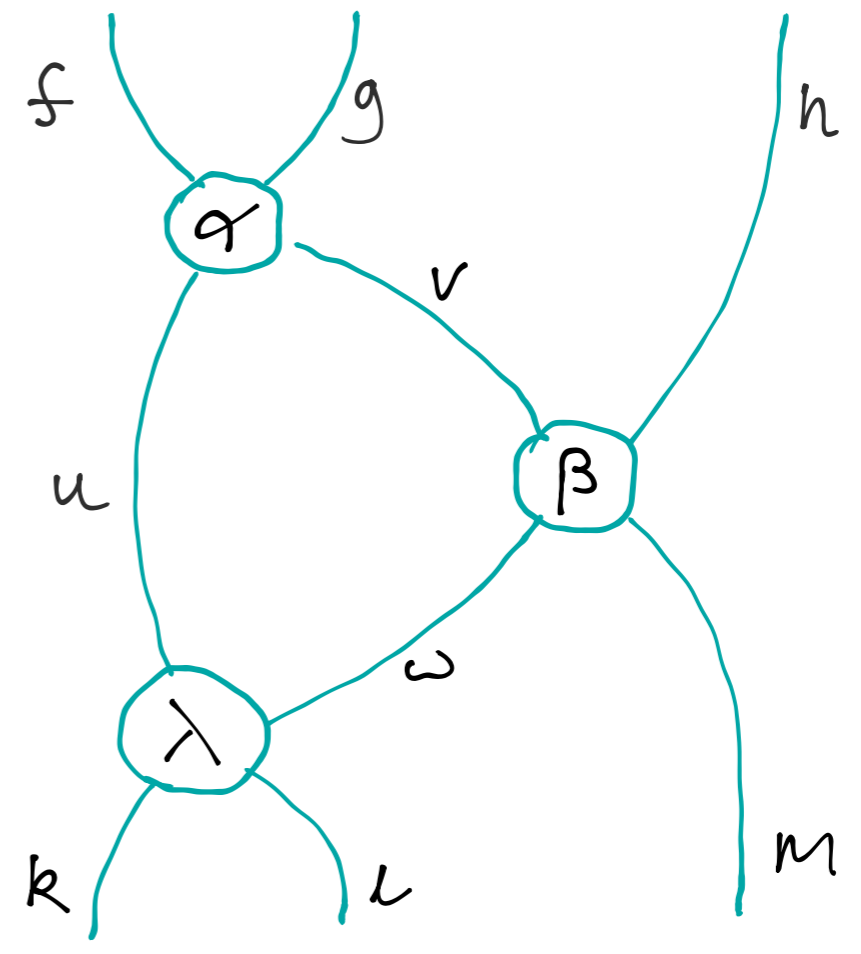
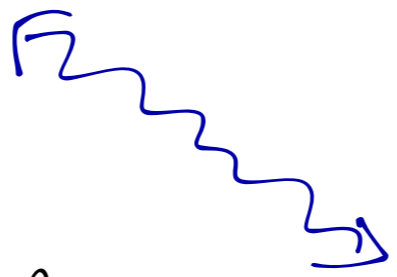


PASTING



STRING DIAGRAMS

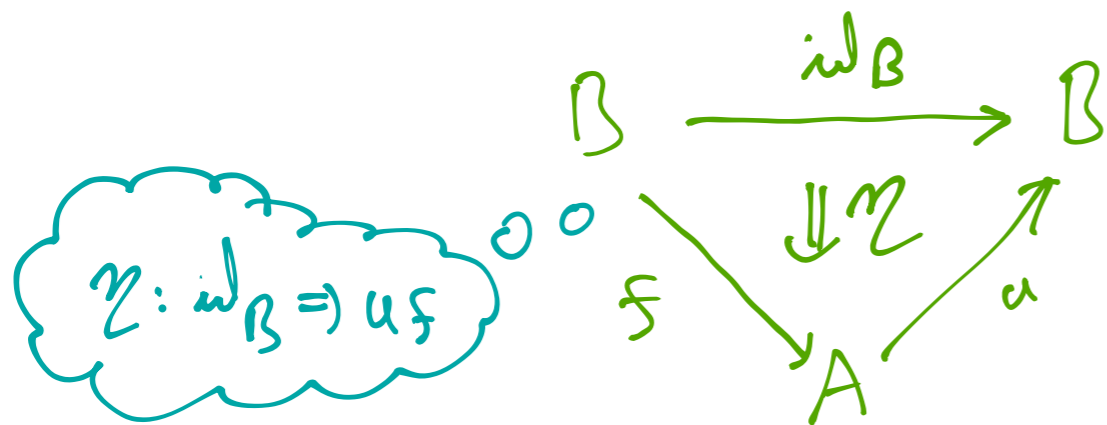
PLANAR
DUAL



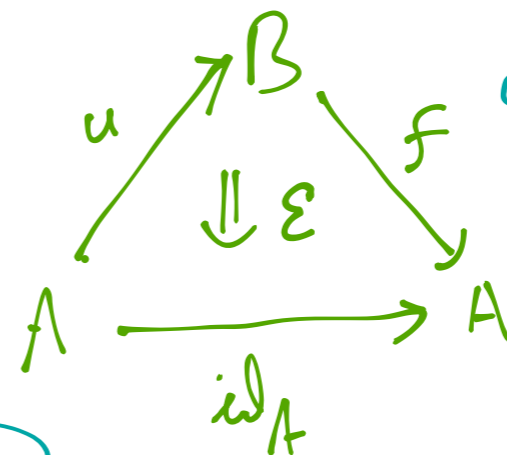
ADJUNCTIONS IN 2-CATEGORIES

DEFN AN ADJUNCTION IN A 2-CATEGORY COMPRISES

- A PAIR OF OBJECTS A AND B
- A PAIR OF 1-CELLS $A \xrightarrow{u} B$ AND $B \xrightarrow{f} A$
- A PAIR OF 2-CELLS:



$\eta: id_B \Rightarrow u \circ f$

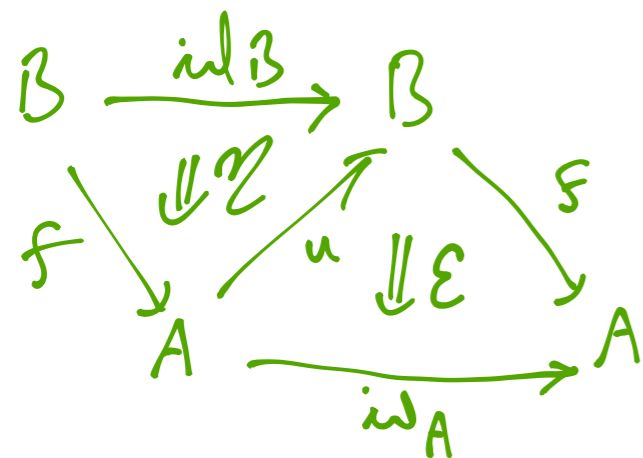


$\epsilon: f \circ u \Rightarrow id_A$

NOTATION
 $\epsilon: f \dashv u: \eta$

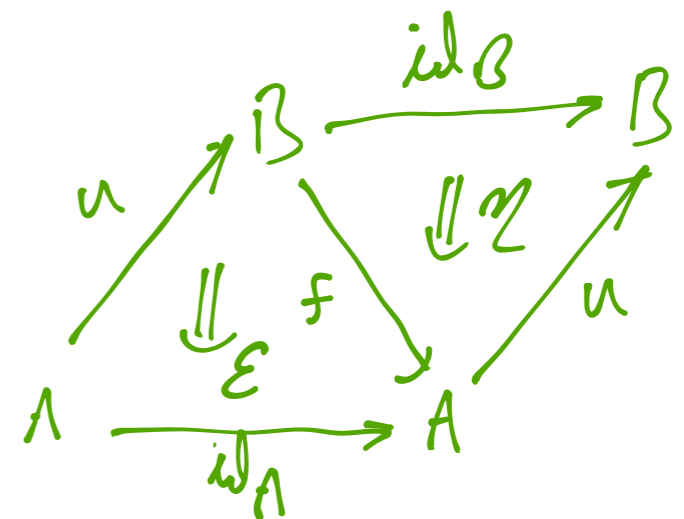
- TRIANGLE IDENTITIES

IDENTITY 2-CELLS



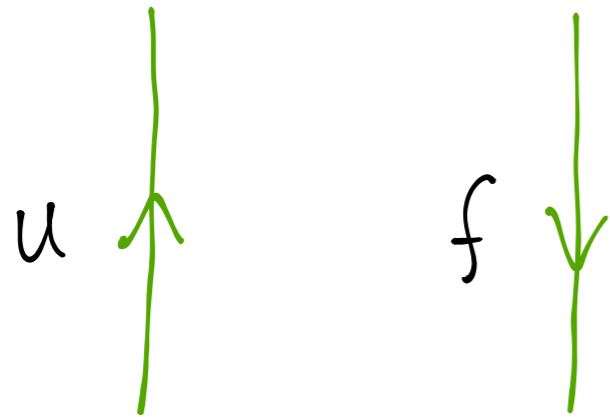
$= id_f$

$= id_u$

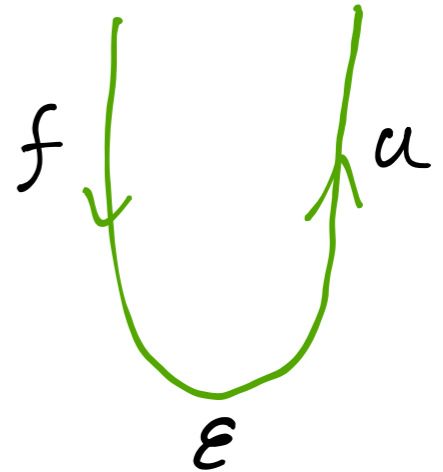
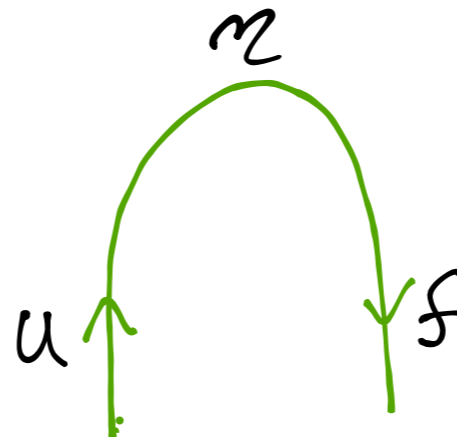


REPRISE WITH STRINGS

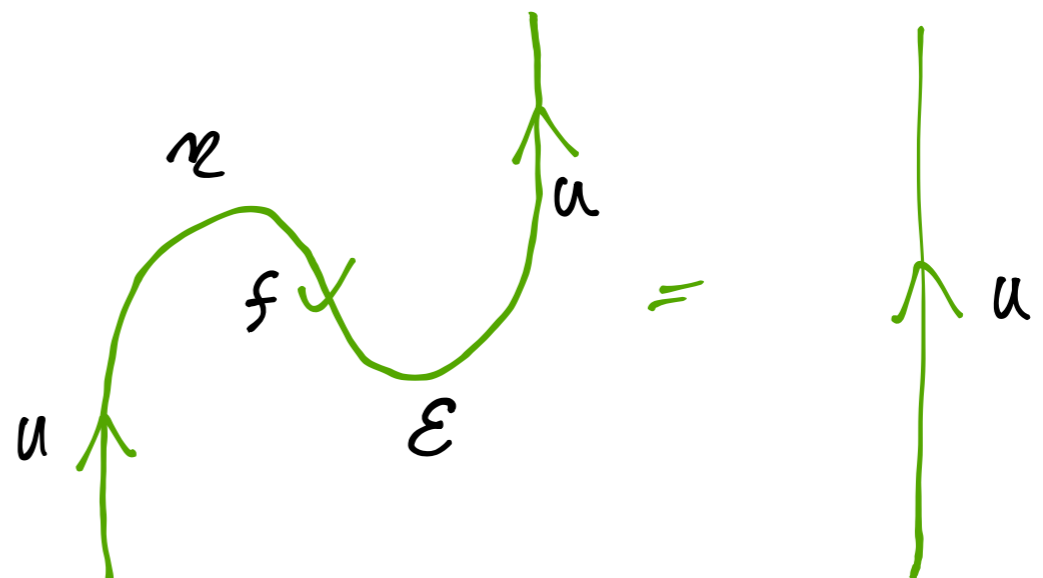
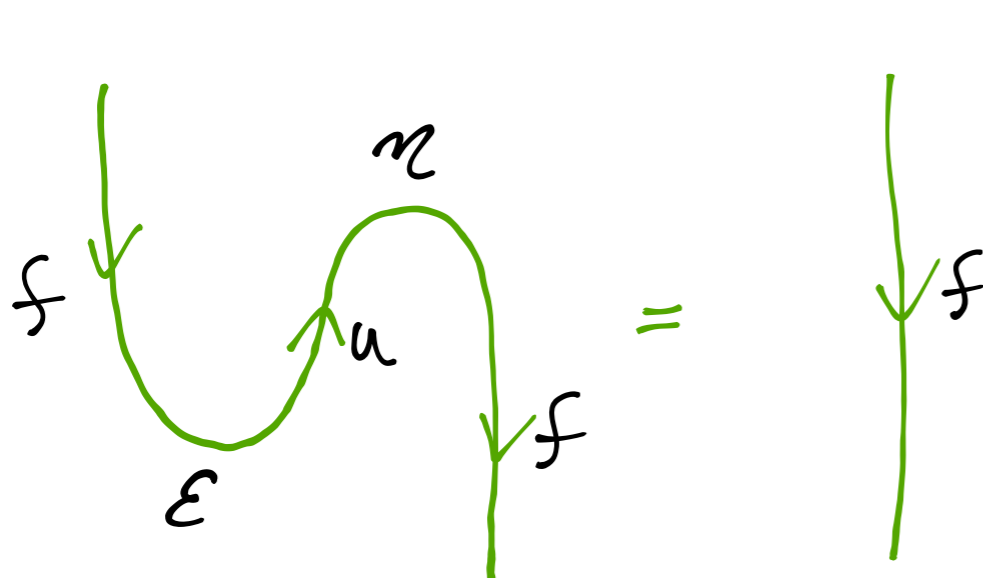
1-CELLS



2-CELLS



TRIANGLE IDENTITIES



SOME CONSEQUENCES

- COMPOSITION OF ADJUNCTIONS
- ADJOINTS ARE UNIQUE UP TO, AND STABLE UNDER, ISOMORPHISM.
- EQUIVALENCES MAY BE PROMOTED TO ADJOINT EQUIVALENCES.
- ADJUNCTIONS ARE STABLE UNDER EQUIVALENCES.

SLOGAN

CATEGORIES LIVE

INSIDE 2-CATEGORIES^{*}

- ⊛ IT IS MORE ACCURATE TO SAY THAT 2-CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP CATEGORY THEORY.

∞ -SLOGAN

∞ -CATEGORIES LIVE

INSIDE $(\infty, 2)$ -CATEGORIES^{*}

- * IT IS MORE ACCURATE TO SAY THAT $(\infty, 2)$ -CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP ∞ -CATEGORY THEORY.

$(\infty, 2)$ -CATEGORIES REALLY?!!

THE TRICK HERE IS TO:

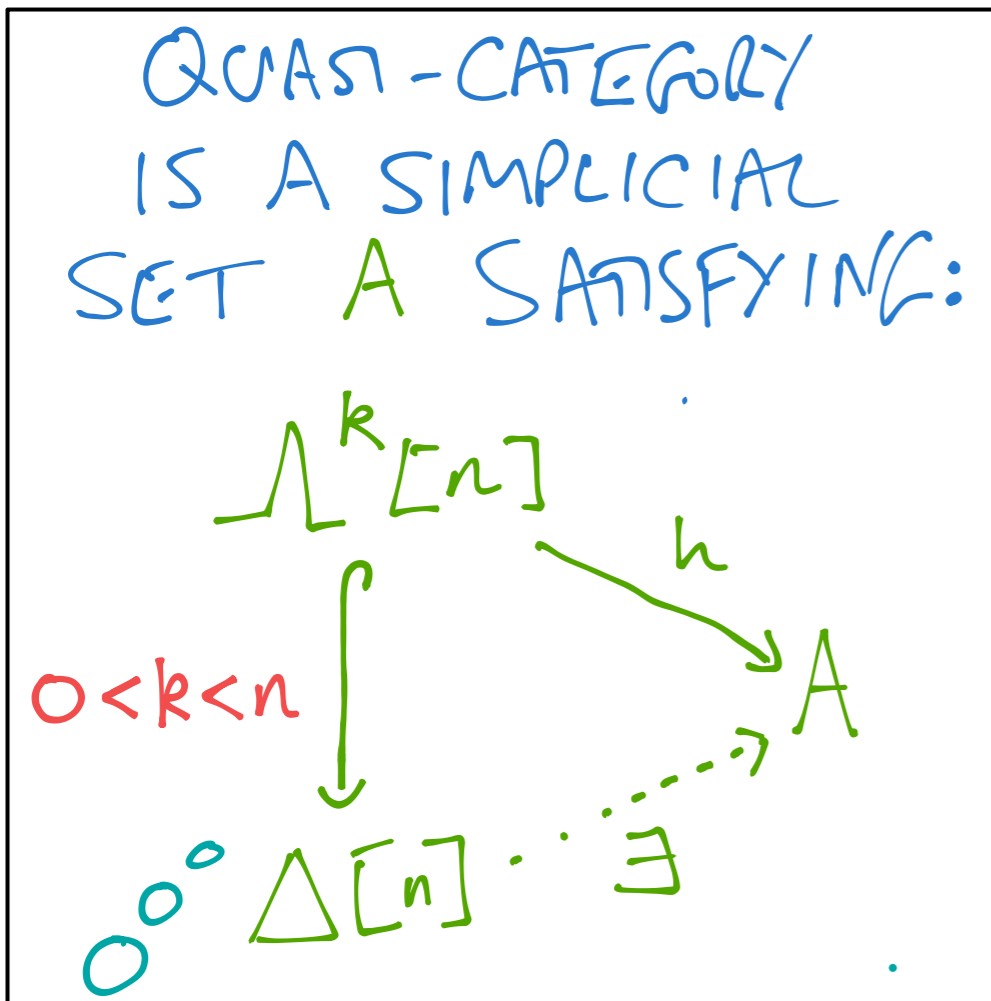
- PICK A MODEL OF $(\infty, 2)$ -CATEGORIES THAT IS FAMILIAR.
- SIMPLIFY MANY ARGUMENTS BY WORKING, WHERE POSSIBLE, IN THE HOMOTOPY 2-CATEGORY ASSOCIATED WITH OUR AMBIENT $(\infty, 2)$ -CATEGORY.

COMMITTING TO AN $(\infty, 2)$ -CATEGORICAL MODEL

AN ∞ -COSMOS IS, IN ESSENCE,
A CATEGORY OF FIBRANT
OBJECTS ENRICHED IN THE
JOYAL MODEL STRUCTURE

QUASI-CATEGORIES

RECALL:

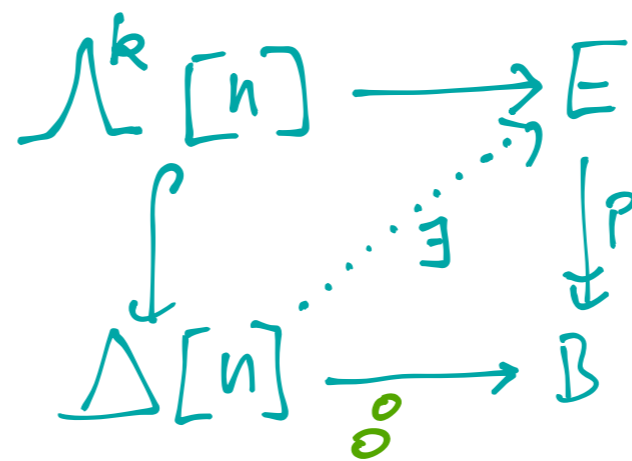


INNER HORN

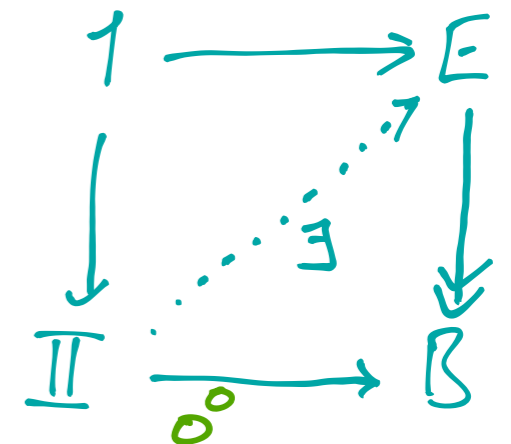
JOYAL MODEL STRUCTURE

ON SSet HAS

-) FIBRANT OBJECTS THE QUASI-CATEGORIES
-) FIBRATIONS, CALLED ISOFIBRATIONS



LIFTING OF COMPOSITES



LIFTING OF ISOMORPHISMS
 $II := \{ \bullet \cong \bullet \}$

-) COFIBRATIONS \equiv MONOS

∞-COSMOS IN ONE PAGE...

A SIMPLICIAL
CATEGORY K

- EQUIPPED WITH
CLASSES OF
- ISOFIBRATIONS
 - WEAK EQUIVALENCES

PRODUCTS AND
PULLBACKS OF
ISOFIBRATIONS

COFIBRANT
REPLACEMENT
 $A_c \xrightarrow{\sim} A$

ALL OBJECTS
FIBRANT $!A \rightarrow 1$

FIBRATION NOTIONS
STABLE UNDER
PULLBACK

ALL LIMITS
SIMPLICIALLY
ENRICHED

COTENSORS SATISFYING "SM7"
WRT JOYAL MODEL STRUCTURE

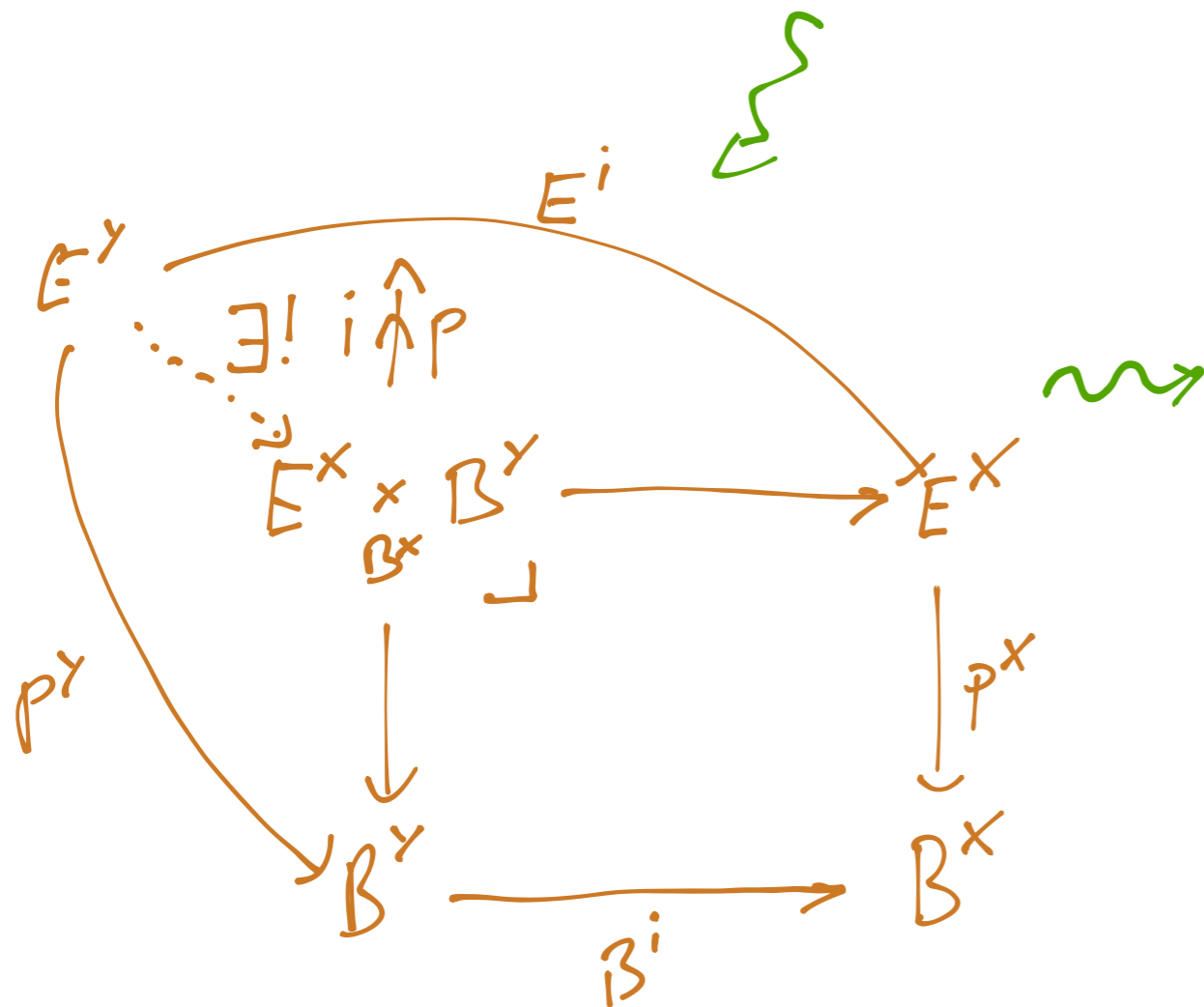
FUNCTOR SPACE

AXIOM "JM7"

$$\text{Fun}_{\mathcal{K}}(A, B)^X \cong \text{Fun}_{\mathcal{K}}(A, B^X)$$

COTENSOR

INCLUSION $i: X \hookrightarrow Y \in \underline{\text{Set}}$
 ISOFIBRATION $p: E \rightarrow B \in \mathcal{K}$



"JM7"

$i \hat{\wedge} p$ IS AN ISOFIBRATION AND IT IS A TRIVIAL FIBRATION IF i IS A WEAK EQUIVALENCE IN THE JOYAL MODEL STRUCTURE OR p IS A TRIVIAL FIBRATION IN \mathcal{K} .

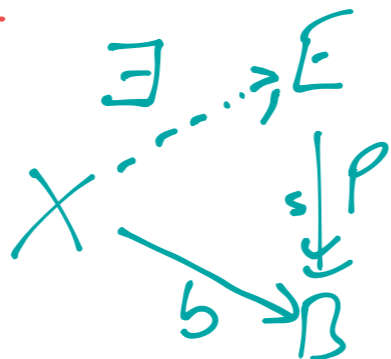
∞ -COSMOS EXAMPLES

$(\infty, 1)$ - MODELS

- QUASI-CATEGORIES
- SEGAL CATEGORIES
- COMPLETE SEGAL SPACES
- MARKED QUASI-CATS

MOST COMMON
EXAMPLES HAVE
ALL OBJECTS

COFIBRANT



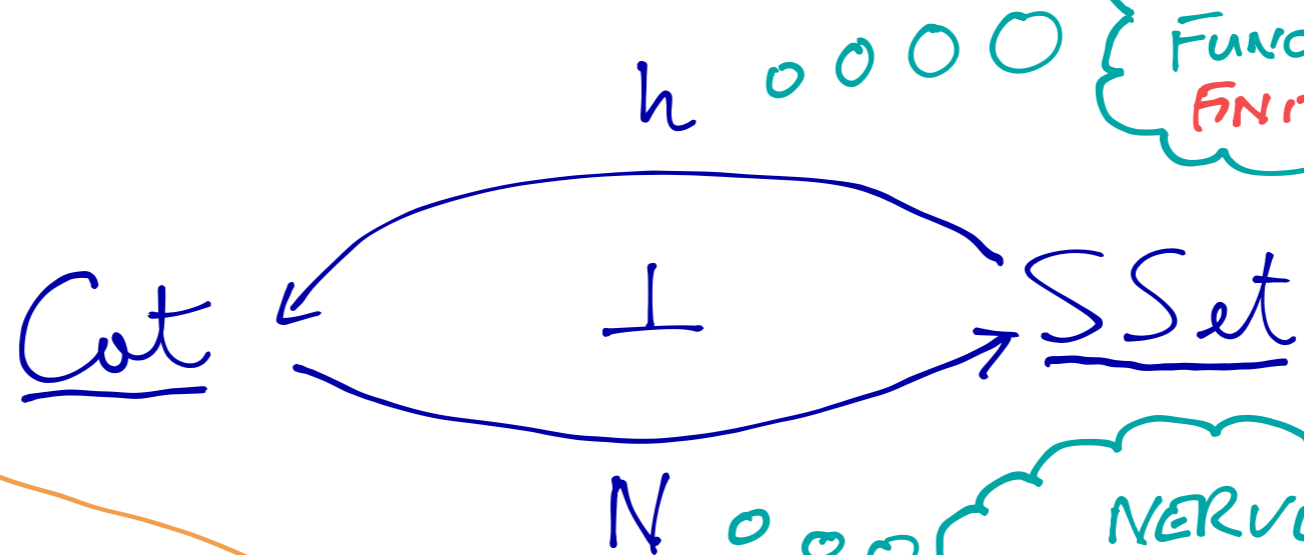
(∞, n) - MODELS

- SEGAL (∞, n) -SPACES
- \mathbb{H}_n -SETS + SPACES
- n -COMPLICIAL SETS

DERIVED EXAMPLES

- SLICES
- ∞ -CATEGORIES FIBRED OVER A BASE.
- (CO)COMPLETE ∞ -CATS

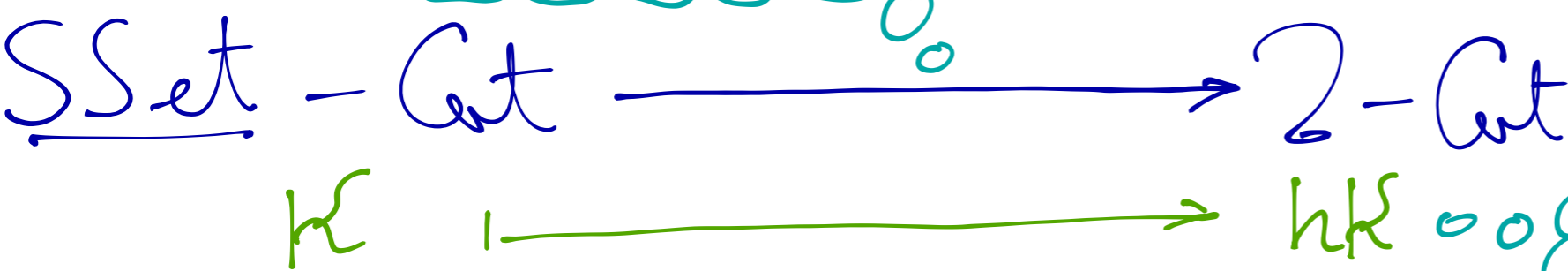
HOMOTOPY 2-CATEGORIES



HOMOTOPY CATEGORY
FUNCTOR - PRESERVES
FINITE PRODUCTS

NERVE
"ALL CATEGORIES
ARE SIMPLICIAL SETS"

APPLY h TO
HOM-SPACES



THE HOMOTOPY
2-CATEGORY OF K

HOMOTOPY 2-CATEGORIES OF

∞ -COSMOS

WHEN A IS COFIBRANT
IN \mathcal{K} THEN ANY HOMSPACE
 $\text{Fun}_{\mathcal{K}}(A, B)$ IS A QUASI-CAT



WE SHALL RESTRICT
THE HTY 2-CAT hK
OF AN ∞ -COSMOS
TO COFIBRANT OBJECTS

EXPLICIT DESCRIPTION
OF hK

- 0-CELLS THE COFIBRANT OBJECTS IN \mathcal{K}
- 1-CELLS ARROWS BETWEEN COF OBJECTS IN \mathcal{K}
- 2-CELLS HOMOTOPY CLASSES OF 1-SIMPLICES, THAT IS OF $(\infty-)$ NATURAL TRANSFORMATIONS, IN $\text{Fun}_{\mathcal{K}}(A, B)$.

∞ -CATEGORIES

$(\infty-)$ FUNCTORS

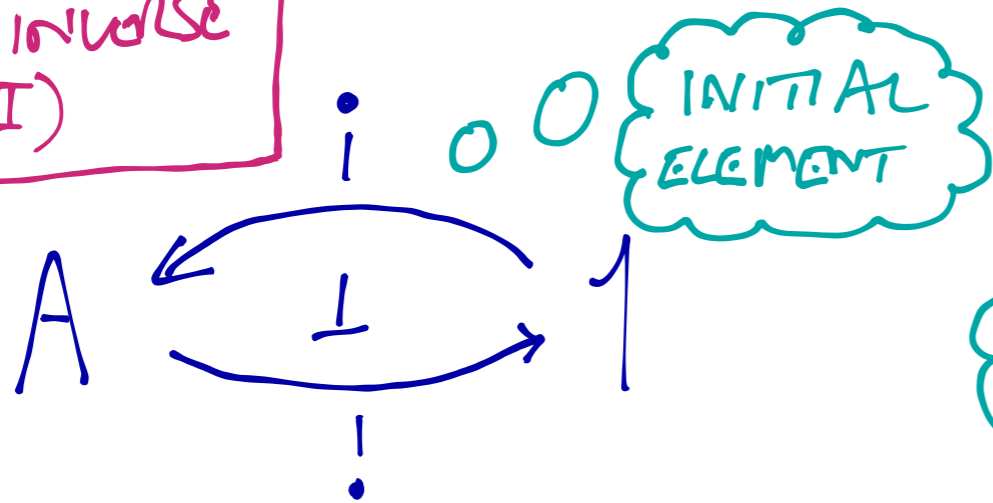
A WILD CONJECTURE!

2-CATEGORICAL ADJUNCTIONS
IN THE HOMOTOPY 2-CATEGORY
 hK OF AN ∞ -COSMOS K ARE
THE CORRECT NOTION WITH RESPECT
TO ∞ -CATEGORY THEORY.

FIRST STEPS

... TOWARDS ∞ -CATEGORY THEORY.

LEFT ADJOINT
RIGHT INVOLVE
(LARI)



RIGHT ADJOINT
RIGHT INVOLVES
(RARI)

MINIMAL DATA PRESENTING A TERMINAL ELEMENT

(i) AN ELEMENT $t : 1 \rightarrow A$

(ii) A 2-CELL $A \xrightarrow{id_A} A$

NATURAL FAMILY
OF MAPS FROM THE
ELEMENTS OF A TO t

(iii) $\varepsilon t = id_t$

WHOSE COMPONENT
AT t IS THE IDENTITY

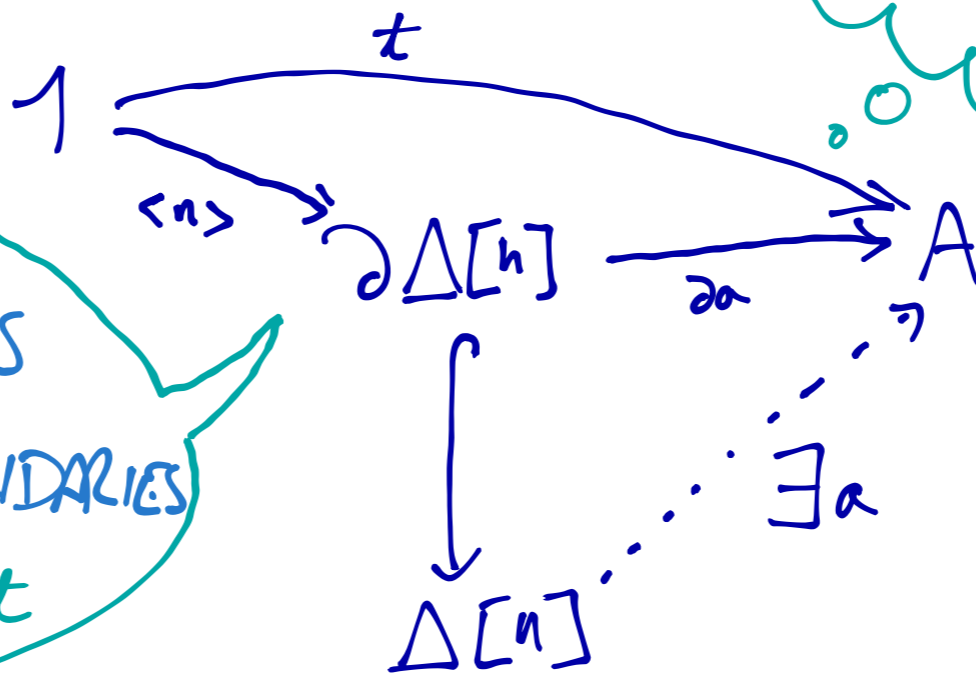
EXERCISE

SUPPOSE THAT A IS A QUASI-CATEGORY;
AN OBJECT IN THE ∞ -COSMOS $QCat$.
LET t BE A VERTEX OF A , THEN

$t : 1 \rightarrow A$ IS A TERMINAL ELEMENT



THIS IS JOYAL'S
TERMINAL OBJECT
NOTION



A ADMITS FILLERS
FOR SIMPLEX BOUNDARIES
WITH LAST VERTEX t

IMMEDIATE CONSEQUENCES

- TERMINAL ELEMENTS PRESERVED BY RIGHT ADJOINTS.

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B AND FUNCTORS $f: B \rightarrow A$ WE HAVE

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \searrow \downarrow \exists! & \nearrow \tau \\ & 1 & \end{array} \text{ IN } \mathcal{K}.$$

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B THE GENERALISED ELEMENT $B \xrightarrow{!} 1 \xrightarrow{\tau} A$ IS TERMINAL IN THE QUASI-CAT $\text{Fun}_{\mathcal{K}}(B, A)$.

LIMITS AND COLIMITS

$$\begin{array}{ccc} \underline{\text{SSet}}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \times & & A \longmapsto A^x \end{array}$$

$$\begin{array}{ccc} \mathcal{K}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \cup & & A \longmapsto A^u \end{array}$$

COTENORS

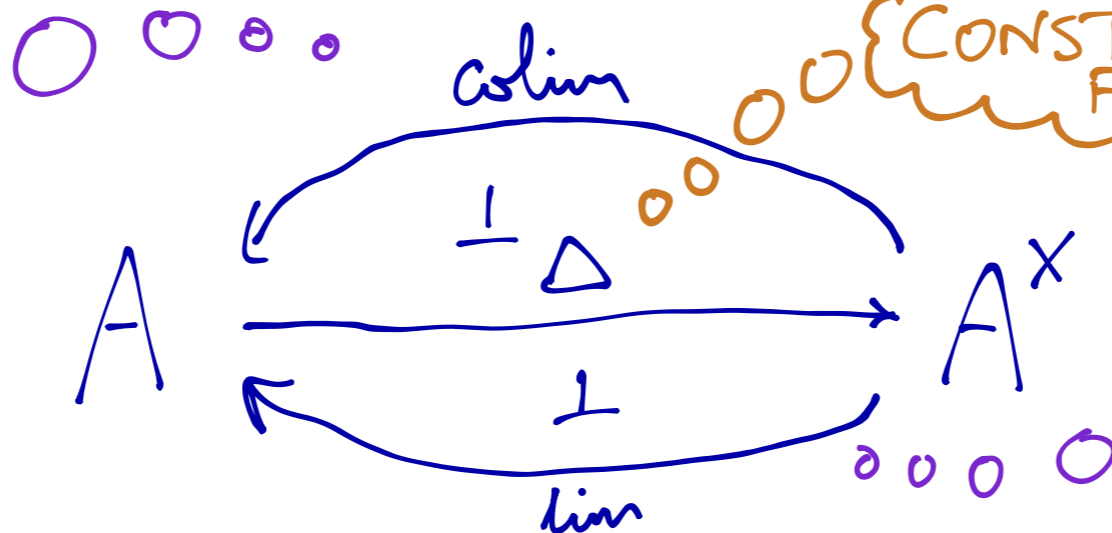
(CO)LIMITS OF SIMPLICIAL SET SHAPED DIAGRAMS

CLOSURE

(WHEN \mathcal{K} CLOSING IS CLOSED)

LIMITS OF DIAGRAMS WHOSE SHAPES ARE ∞ -CATEGORIES IN \mathcal{K}

COLIMITS OF DIAGRAMS OF SHAPE X



CONSTANT DIAGRAM FUNCTOR

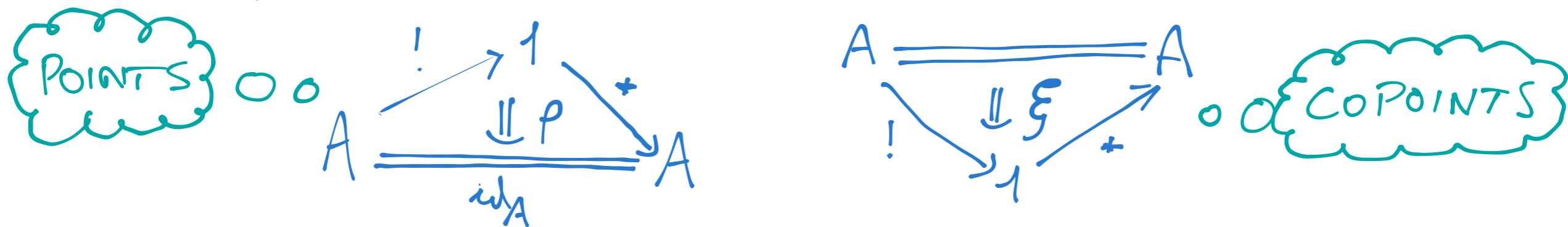
LIMITS OF DIAGRAMS OF SHAPE X

FAMILIES OF DIAGRAMS

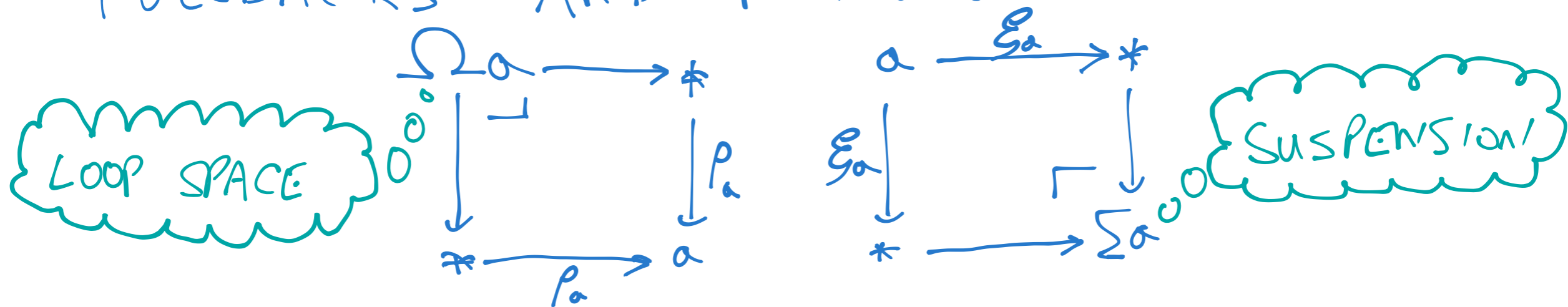
WE MAY WISH TO ASK THAT AN ∞ -CATEGORY HAS SOME, BUT NOT ALL, LIMITS OF A GIVEN SHAPE.

EXAMPLE

- POINTED ∞ -CATEGORY HAS AN ELEMENT $*$: $1 \rightarrow A$ WHICH IS BOTH INITIAL + TERMINAL

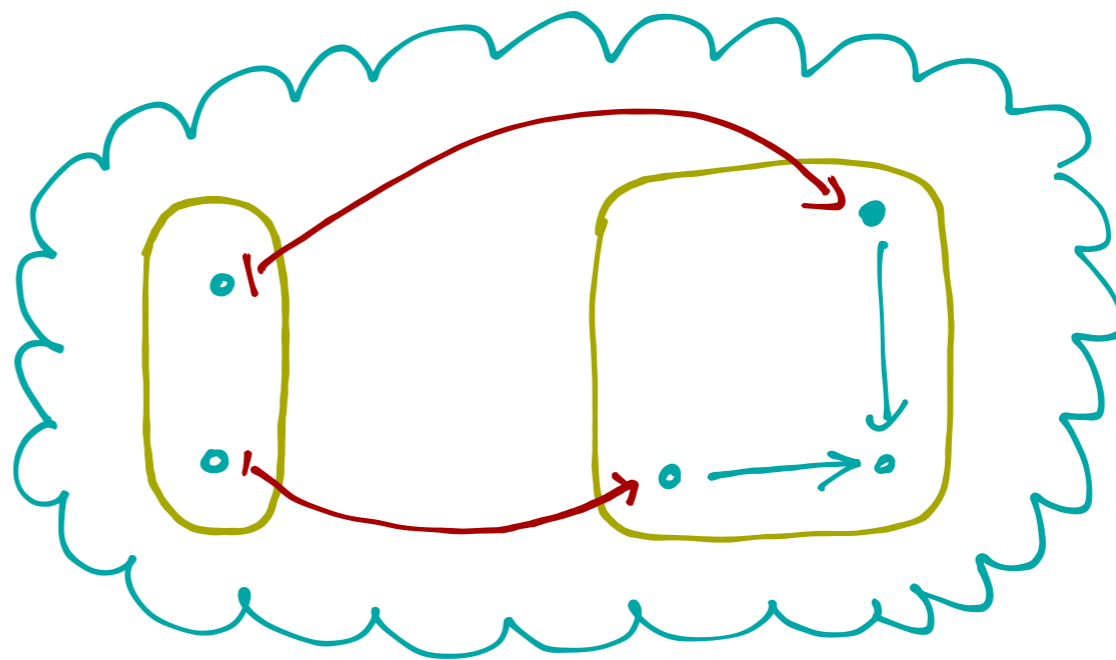


- IT IS COMMON ONLY TO ASK FOR CERTAIN PULLBACKS AND PUSHOUTS



LOOP SPACES

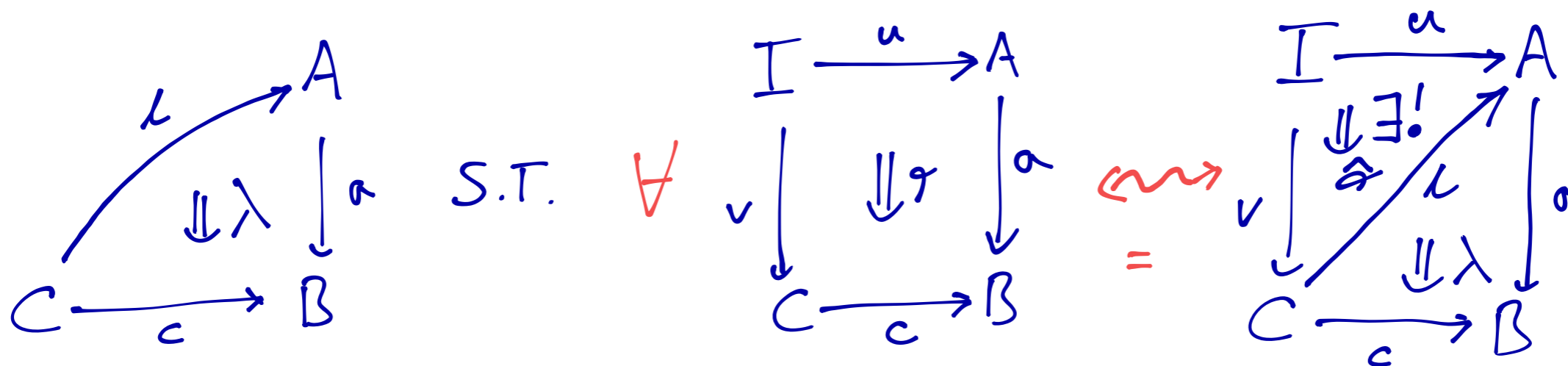
$$\begin{array}{ccc}
 A^{\square} & \xrightarrow{f} & A^{\square} \\
 \downarrow & & \downarrow \circ \circ \circ \\
 1 & \xrightarrow{(*,*)} & A \times A
 \end{array}$$



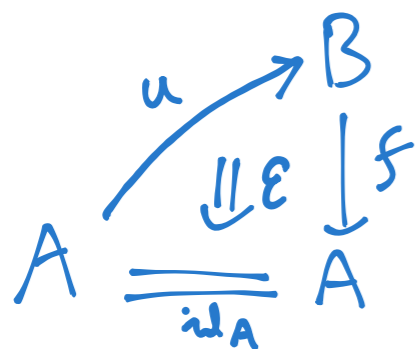
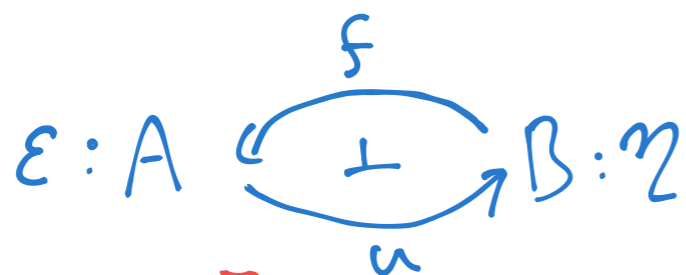
- LOOP SPACES ARE PULLBACKS,
SO THEIR UNIVERSAL PROPERTY GIVES
IN THE DIAGRAM ∞ -CATEGORY $A^{\square} \dots$
- \dots BUT WE ONLY WANT TO ASK FOR
LIMITS OF DIAGRAMS IN A^{\square}
- $A^{\square} \xrightarrow{f} A^{\square}$ SPECIFIES A FAMILY OF DIAGRAMS
IN A^{\square} WHOSE LIMITS ARE LOOP SPACES.

LIMITS OF FAMILIES

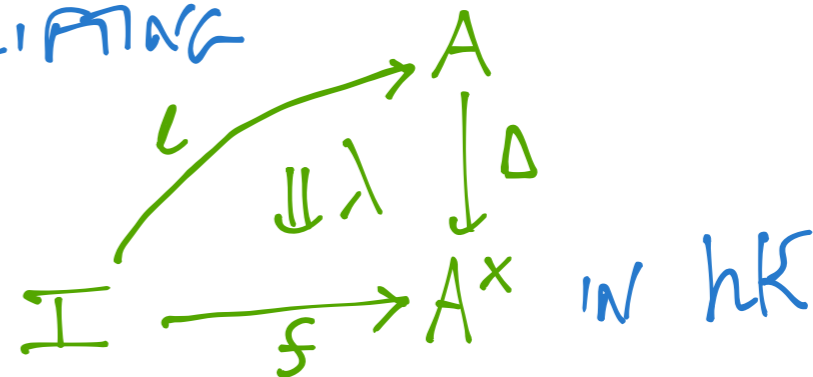
ABSOLUTE RIGHT LIFTING (IN A 2-CATEGORY)



EXAMPLE



DEFN A FAMILY OF DIAGRAMS $f: I \rightarrow A^x$ ADMITS A LIMIT IF THERE IS AN ABSOLUTE RIGHT LIFTING



PRESERVATION

RIGHT ADJOINTS PRESERVE LIMITS OF FAMILIES

$$\begin{array}{ccc}
 & f & \\
 A & \xleftarrow{\quad} & B \\
 & \perp & \\
 & u & \\
 & \xrightarrow{\quad} &
 \end{array}$$

GIVEN AN ADJUNCTION

$$\begin{array}{ccc}
 \text{IF} & & \\
 & \curvearrowright & A \\
 & \Downarrow \lambda & \downarrow \Delta \\
 I & \xrightarrow{d} & A^x
 \end{array}$$

IS AN ABSOLUTE
RIGHT LIFTING

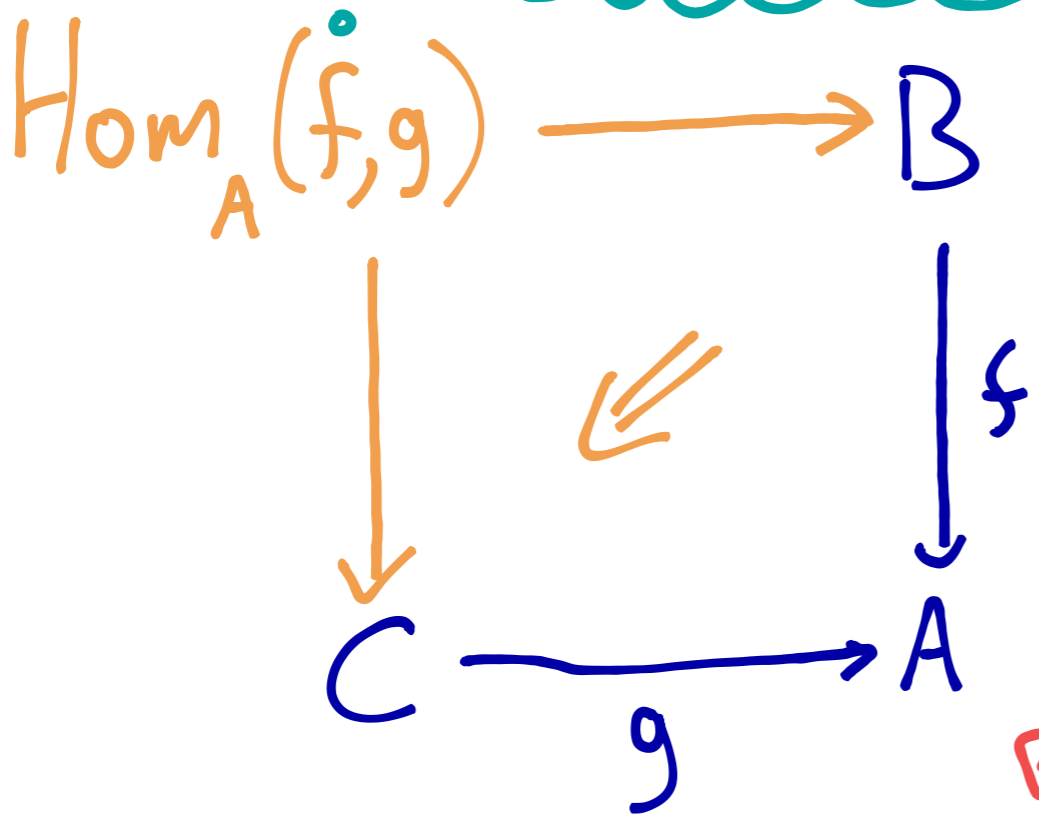
\Rightarrow

$$\begin{array}{ccccc}
 & \curvearrowright & A & \xrightarrow{u} & B \\
 & \Downarrow \lambda & \downarrow \Delta & & \downarrow \Delta \\
 I & \xrightarrow{d} & A^x & \xrightarrow{u^x} & B^x
 \end{array}$$

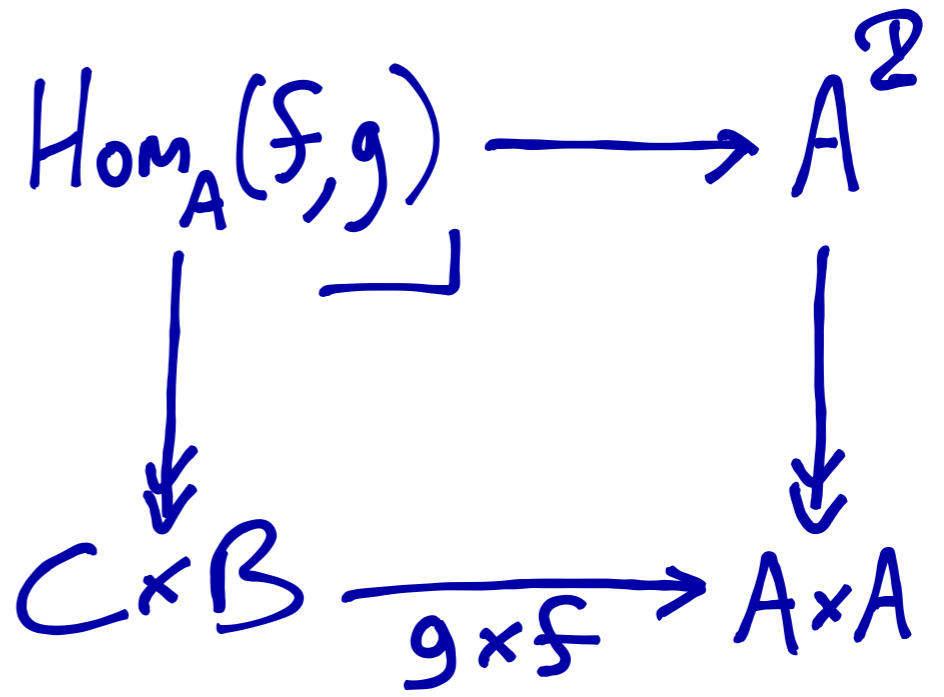
IS ALSO AN ABSOLUTE
RIGHT LIFTING.

Hom - SPACES

THE HOM SPACE FROM f TO g



A PARAMETERISED OBJECT IN THE ∞ -CATEGORY A



WEAK COMMA OBJECTS

HOM-SPACES POSSESS A **WEAK** 2-UNIVERSAL PROPERTY IN THE HOMOTOPY 2-CATEGORY.

SPECIFICALLY FOR EACH X THE INDUCED FUNCTOR

$$hK(X, \text{Hom}_A(f, g)) \longrightarrow hK(X, f) \downarrow hK(X, g)$$

IS

- SURJECTIVE ON OBJECTS,

- FULL, AND

- CONSERVATIVE.

AKA
SMOTHERING

WE SAY THAT $\text{Hom}_A(f, g)$ IS THE **WEAK COMMA** OF THE FUNCTORS f AND g IN hK .

HOM-WISE CHARACTER OF ADJUNCTIONS

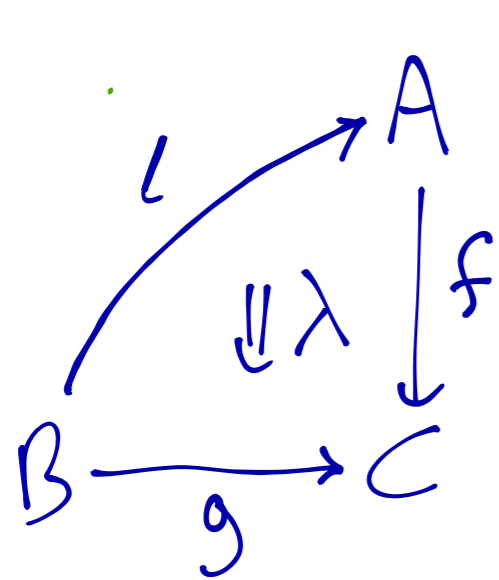
DEFN A MODULE $\text{Hom}_A(f, g) \rightarrow C \times B$ IS

-) LEFT REPRESENTABLE IF THERE EXISTS A FUNCTOR $L: B \rightarrow C$ AND AN EQUIVALENCE $\text{Hom}_C(L, C) \cong \text{Hom}_A(f, g)$ IN $\mathcal{K}/_{C \times B}$.

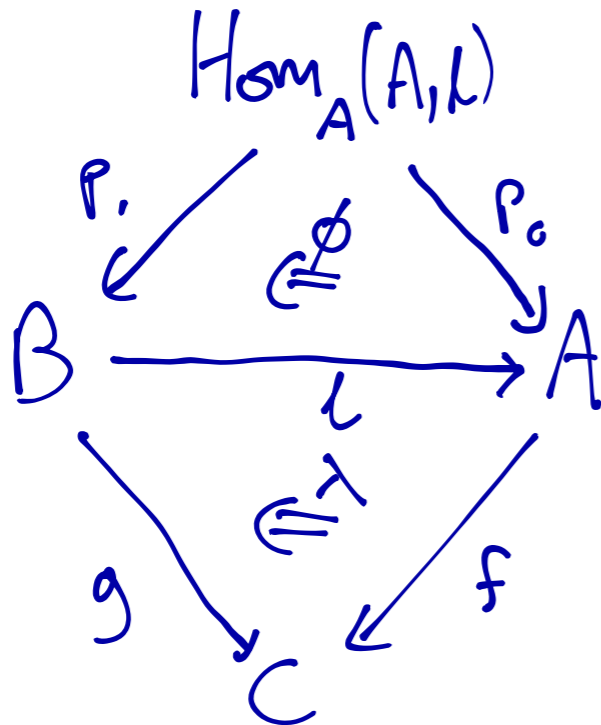
AN EQUIVALENCE
FIBRED OVER $C \times B$

-) RIGHT REPRESENTABLE IF THERE EXISTS A FUNCTOR $r: C \rightarrow B$ AND AN EQUIVALENCE $\text{Hom}_B(B, r) \cong \text{Hom}_A(f, g)$ IN $\mathcal{K}/_{C \times B}$

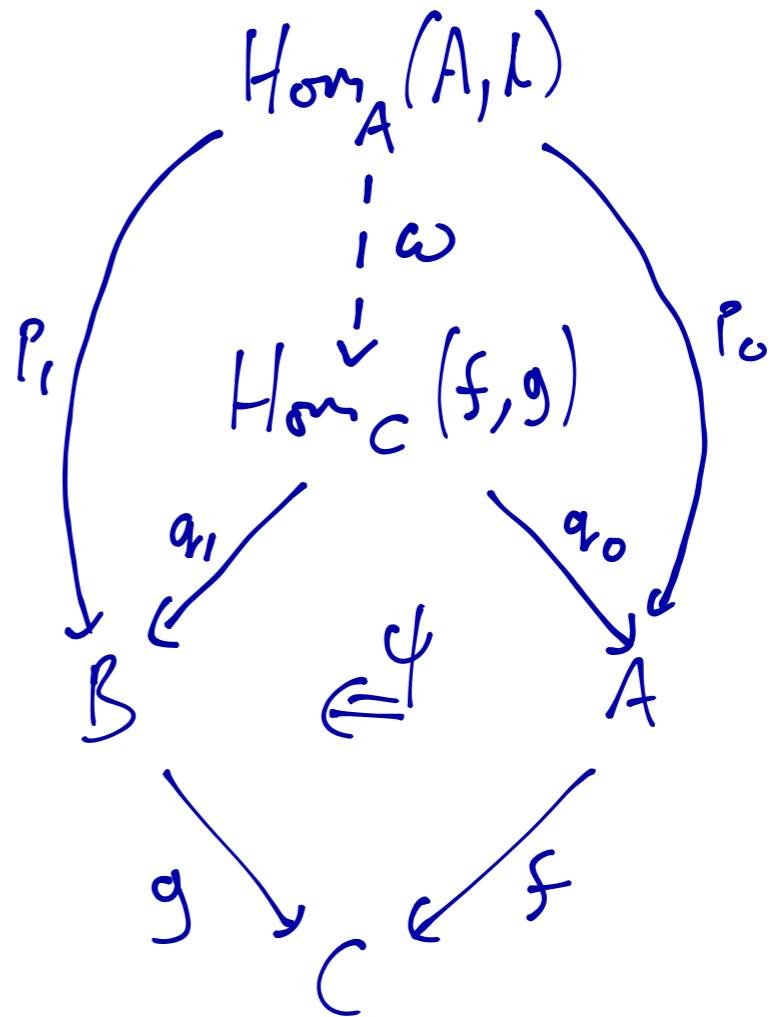
LIFTINGS MOM-WISE



LEMMA THE TRIANGLE ON THE LEFT IS AN ABSOLUTE RIGHT LIFTING IFF THE INDUCED FUNCTOR $\text{Hom}_A(A, L) \xrightarrow{\omega} \text{Hom}_C(f, g)$ IS AN EQUIVANCE

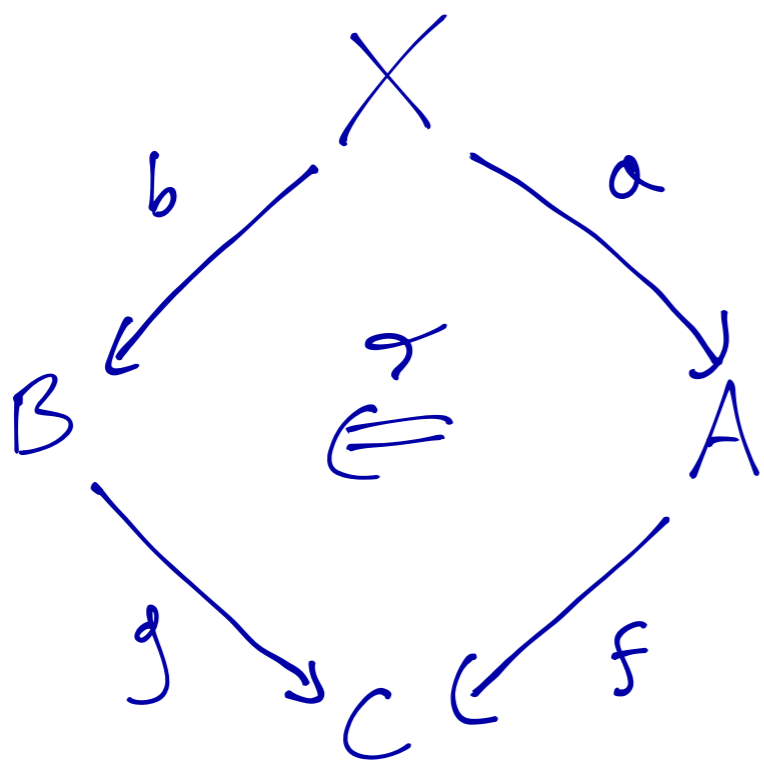


\rightsquigarrow
=



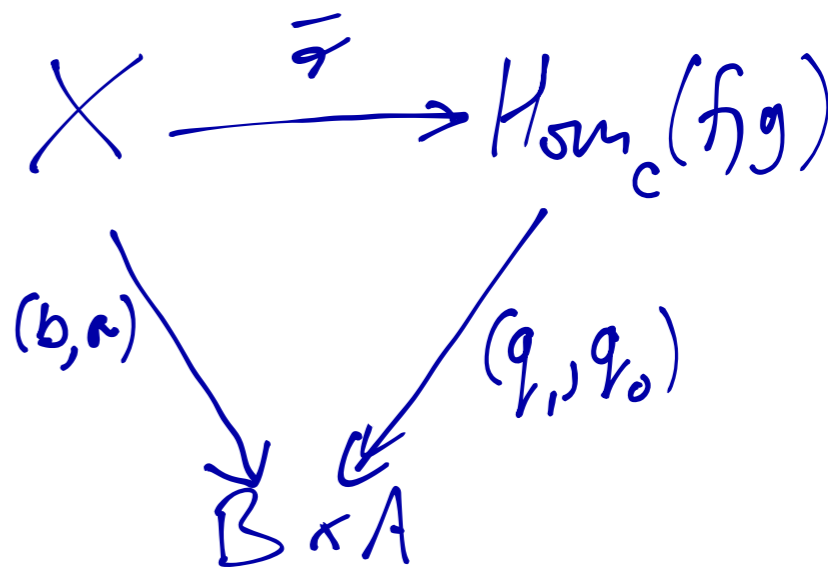
LIFTINGS MOM-WISE

KEY OBSERVATION IT IS A CONSEQUENCE OF THIS WEAK 2-UNIVERSAL PROPERTY OF $\text{Hom}_c(f, g)$ THAT THERE EXISTS A BIJECTION



2-CELLS

BIJECTION



ISO-CLASSES OF 1-CELLS OVER $B \times A$

THEOREM

A FUNCTOR $u: A \rightarrow B$ HAS A LEFT ADJOINT

IFF $\text{Hom}_B(B, u)$ IS LEFT REPRESENTABLE.

$$\text{Hom}_B(B, u) \xrightarrow{\sim} \text{Hom}_A(f, A)$$

$\searrow \quad \swarrow$
 $A \times B$

THIS IS THE USUAL
"HOM-WISE" CHARACTER
OF AN ADJUNCTION

THREE VIEWS OF ADJUNCTIONS

$$A \begin{array}{c} \xleftarrow{f} \\ \dashv \\ \xrightarrow{u} \end{array} B$$

$$\eta: \text{id}_B \Rightarrow uf$$

$$\varepsilon: fu \Rightarrow \text{id}_A$$

2-CATEGORICAL IN \mathcal{K}

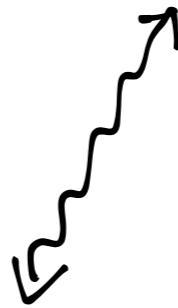
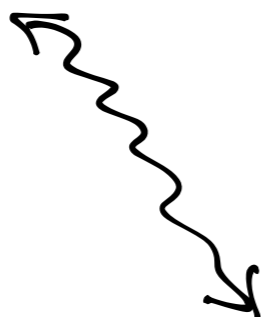


$$\text{Hom}_A(f, A) \rightarrow \text{Hom}_B(B, u)$$

$$\searrow \quad \swarrow \\ A \times B$$

"MODULES" + REPRESENTABILITY

HOMOTOPY COHERENCE?
(PART 3)



$$A \xrightarrow{u} B$$

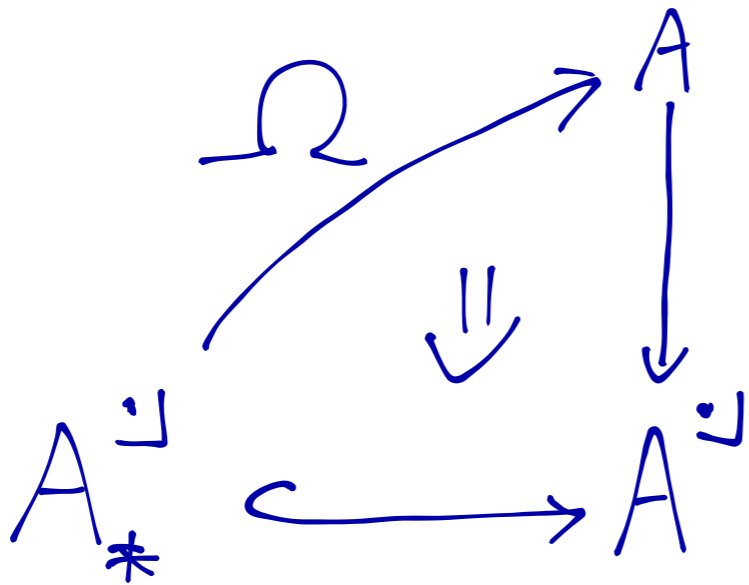
IS RIGHT ADJOINT IN \mathcal{K} IFF

$$\text{Fun}_{\mathcal{K}}(X, A) \xrightarrow{u_0} \text{Fun}_{\mathcal{K}}(X, B)$$

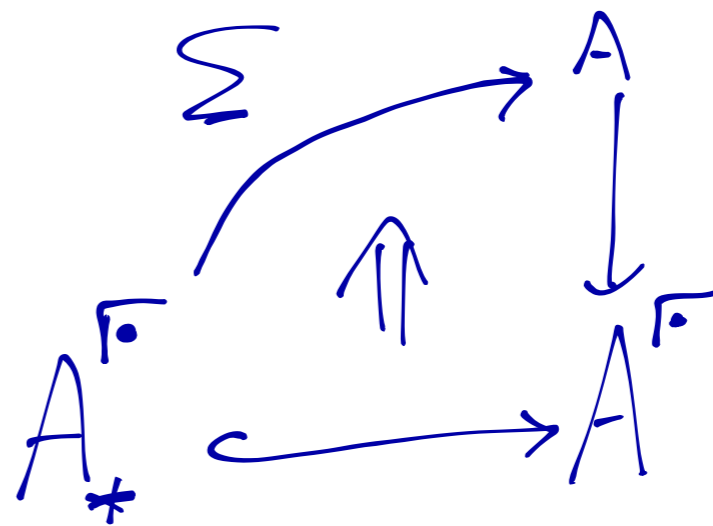
IS A RIGHT ADJOINT FUNCTOR OF QUASI-CATEGORIES FOR ALL $X \in \mathcal{K}$

RETURN TO LOOPS

WE SAY THAT A POINTED ∞ -CAT
A ADMITS LOOPS + SUSPENSIONS IF
THERE EXISTS:



ABSOLUTE
RIGHT LIFTING

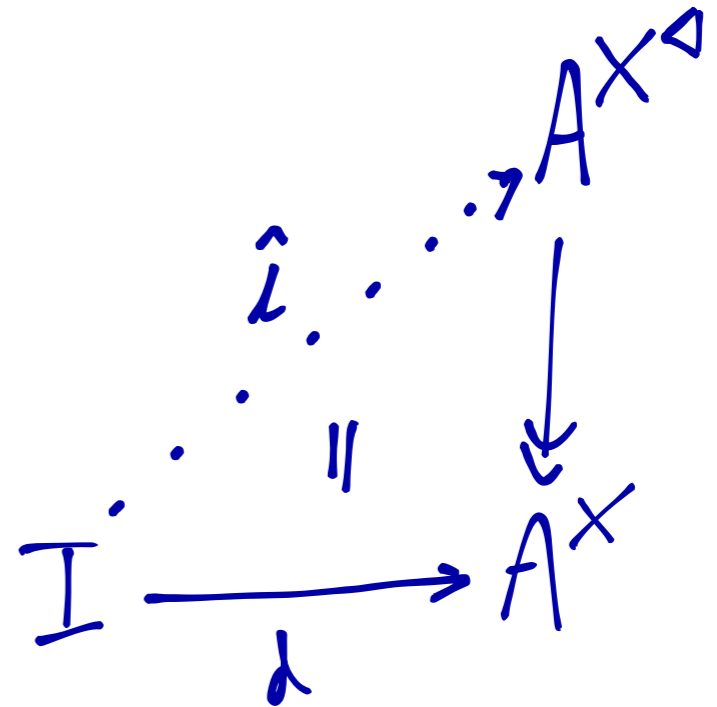
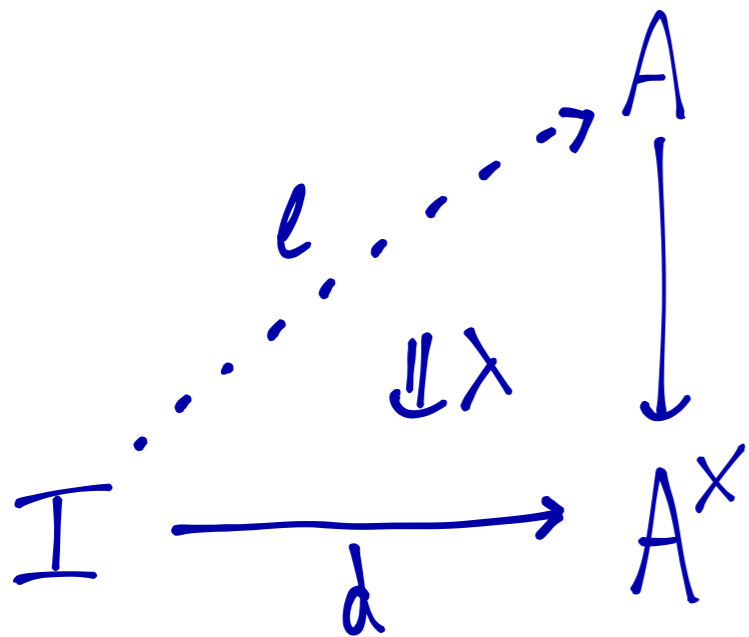


ABSOLUTE
LEFT LIFTING

(CO)LIMITS AS KAN EXTENSIONS

× A DIAGRAM SHAPE (IN sSet)

×[▷] OBTAINED BY ADDING INITIAL OBJECT

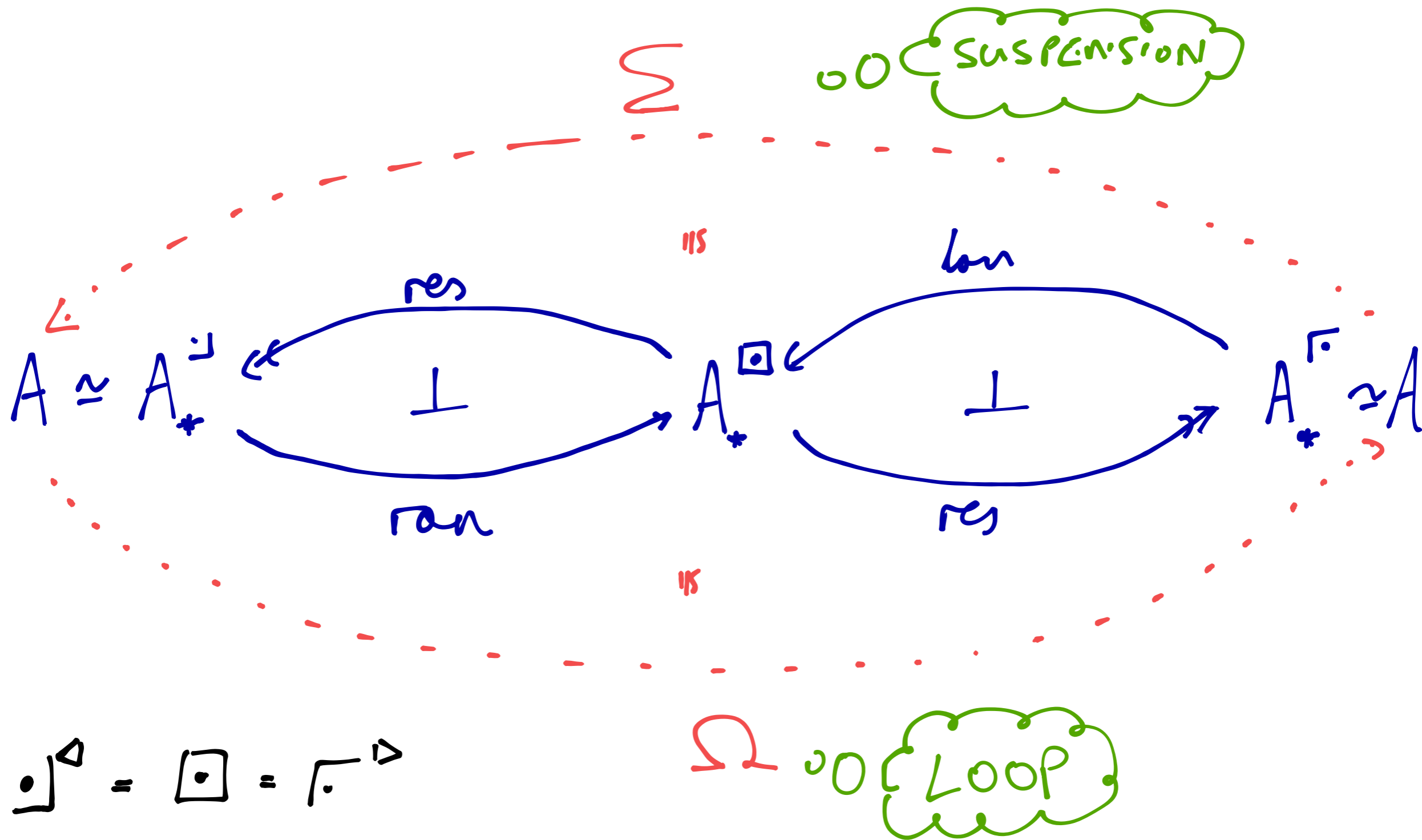


ADMITTS AN
ABSOLUTE RIGHT LIFTING



ADMITTS AN ABSOLUTE
RIGHT LIFT WHOSS 2-CELL
IS THE IDENTITY.

LOOP - SUSPENSION ADJUNCTION



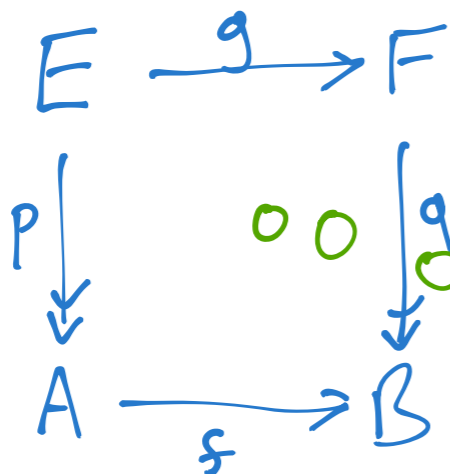
NEW ∞ -COSMOS FROM OLD.

IMPORTANT: MANY (2-) CATEGORICALLY INSPIRED CONSTRUCTIONS APPLY TO ∞ -COSMOS GIVING NEW STRUCTURES WHICH ARE AGAIN ∞ -COSMOS.

SLICES \mathcal{K}/A

OBJECTS ARE ISOFIBRATIONS $p: E \rightarrow A$

COSMOS OF ISOFIBRATIONS \mathcal{K}^2



OBJECTS ARE ISOFIBRATIONS

ARROWS ARE COMMUTATIVE SQUARES

(CO) COMPLETE ∞ -CATS

$\text{Lex}_{\omega}(\mathcal{K})$

OBJECTS: THOSE OF \mathcal{K} ADMITTING SOME CLASS OF LIMITS

ARROWS: FUNCTORS IN \mathcal{K} THAT PRESERVE THOSE LIMITS

POINTED ∞ -CATS

\mathcal{K}_*

POINTED OBJECTS OF \mathcal{K} + POINT PRESERVING FUNCTORS

ASIDE: COMPARING $h(K/B)$ AND hK/B

THE SLICE CONSTRUCTIONS ARE CALLED THE INTERNAL AND EXTERNAL SLICE RESPECTIVELY.

BUT WE DO HAVE A CANONICAL COMPARISON

UGH! THESE ARE NOT, IN GENERAL, EQUIVALENT 2-CATS

$$h(K/B) \xrightarrow{\circ\circ} hK/B \quad \text{WHICH IS 2-SMOTHERING}$$

IN PARTICULAR ADJUNCTIONS LIFT ALONG THIS COMPARISON.

OBSCURITY CORNER: IN THE 2-DERIVATOR WORLD THIS IS A KEY AXIOM.

THREE VIEWS OF GROTHENDIECK FIBRATIONS

$p: E \rightarrow B$ AN ISOFIBRATION IN \mathcal{K} .

$$\begin{array}{ccc}
 E & \xrightarrow{i} & \text{Hom}_B(B, p) \\
 \downarrow p & & \downarrow p_0 \\
 B & & B
 \end{array}$$

HAS A RIGHT ADJOINT
IN $\mathcal{K}E/B$

\Leftrightarrow

$$E^2 \xrightarrow{k} \text{Hom}_B(B, p)$$

HAS A RIGHT ADJOINT
RIGHT INVERSE (RARI)
IN $\mathcal{K}E$.

"CHEVALLEY
CRITERION"

OFTEN CALLED
CARTESIAN
FIBRATIONS IN
THE ∞ -CAT
LITERATURE

$$\text{Fun}_{\mathcal{K}}(X, E) \xrightarrow{p_0} \text{Fun}_{\mathcal{K}}(X, B)$$

ADMITS LIFTS OF ALL
ARROWS $g: a \rightarrow pb$ TO A
P-CARTESIAN ARROW $\chi_g: g^*b \rightarrow b$

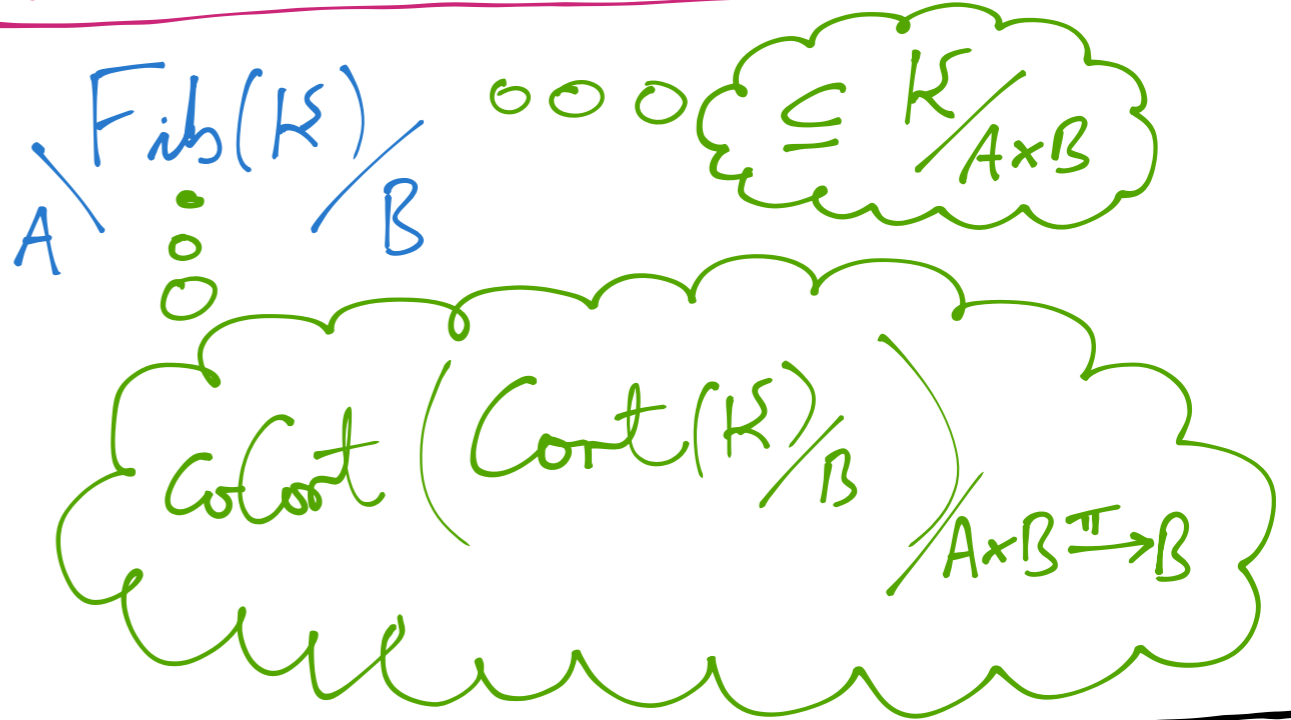
MORE ∞ -COSMOI

CARTESIAN FIBRATIONS

$\text{Cort}(K) / A \circ \circ \circ \subseteq K / A$ OVER A FIXED BASE

CARTESIAN FIBRATIONS WITH BASE A + CARTESIAN FUNCTORS BETWEEN THEM

2-SIDED CARTESIAN FIBRATIONS



ALL CARTESIAN FIBRATIONS

$\text{Cort}(K) \circ \circ \circ \subseteq K^2$



CARTESIAN FUNCTOR: PRESERVES CARTESIAN 1-ARROWS

CARTESIAN FIBRATION

GROUPOIDAL ∞ -CATS

$\text{Gpd}(K) \circ \circ \circ \subseteq K$

$A \in \text{Gpd}(K) \text{ IFF } \forall X \in K$


$\text{Fun}_K(X, A)$

IS A KAN COMPLEX.

MODULES

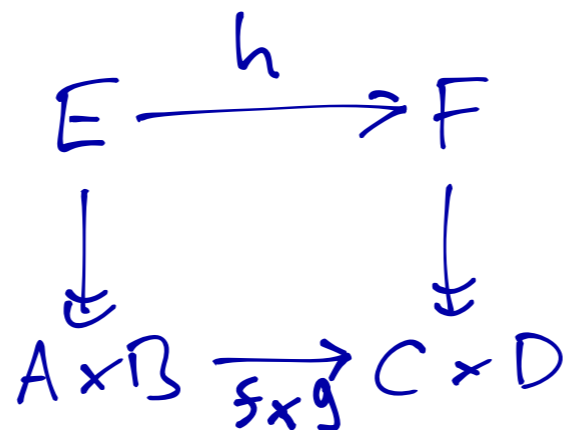
AKA PROFUNCTORS

$$A \backslash \text{Mod}(K) / B \quad \bullet \bullet = \text{Gpd} \left(A \backslash \text{Fib}(K) / B \right)$$

GIVEN $A \xrightarrow{f} C \xleftarrow{g} B$
 $\text{Hom}_C(g, f)$ IS A MODULE


ALL MODULES

$\text{Mod}(K) \circledast$ + MAPS

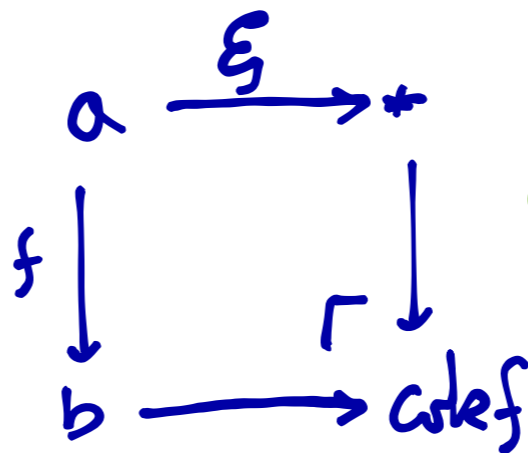
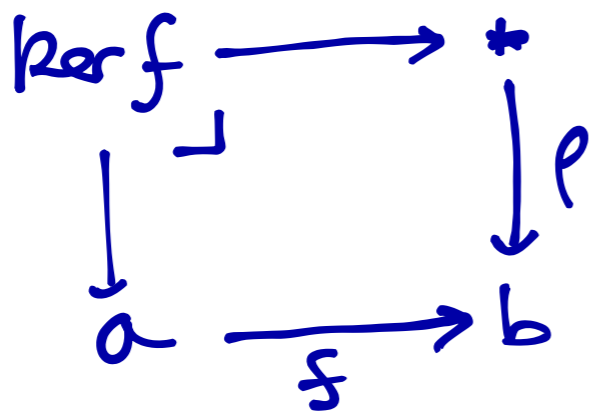


THIS IS AGAIN!
AN ∞ -COSMOS.

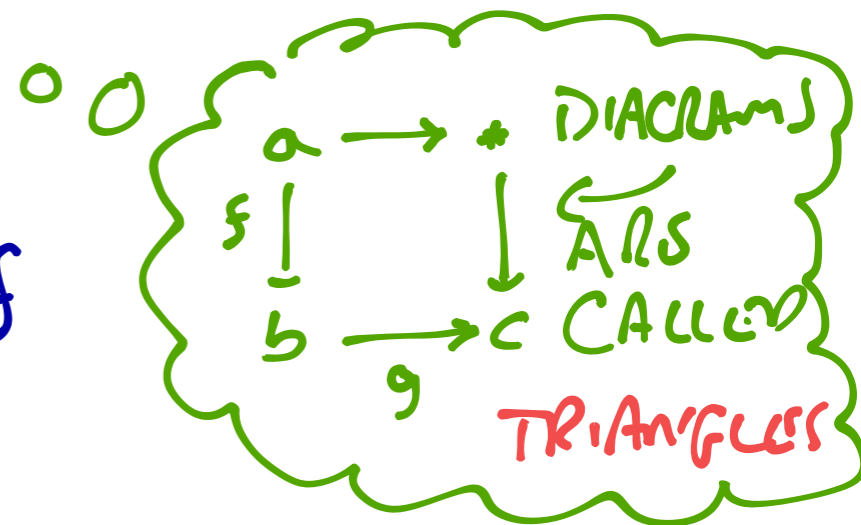
STABLE ∞ -CATEGORIES

AN ∞ -CATEGORY \mathcal{A} IS SAID TO BE STABLE IF

- \mathcal{A} HAS A ZERO OBJECT $*$: $1 \rightarrow \mathcal{A}$
- EVERY X -INDEXED FAMILY $f: a \rightarrow b$ OF ARROWS IN \mathcal{A} ADMITS KERNELS + COKERNELS

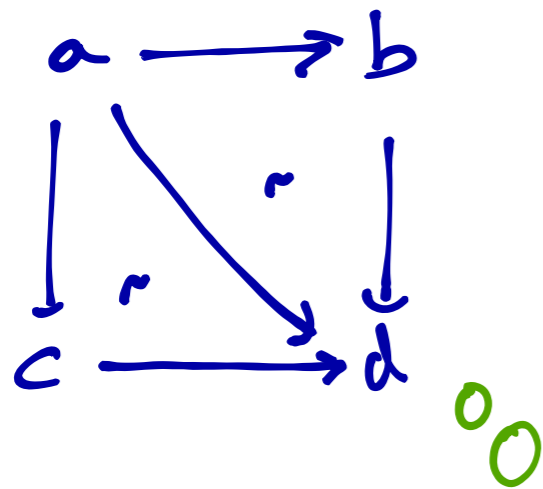


$$f: X \rightarrow A^2$$



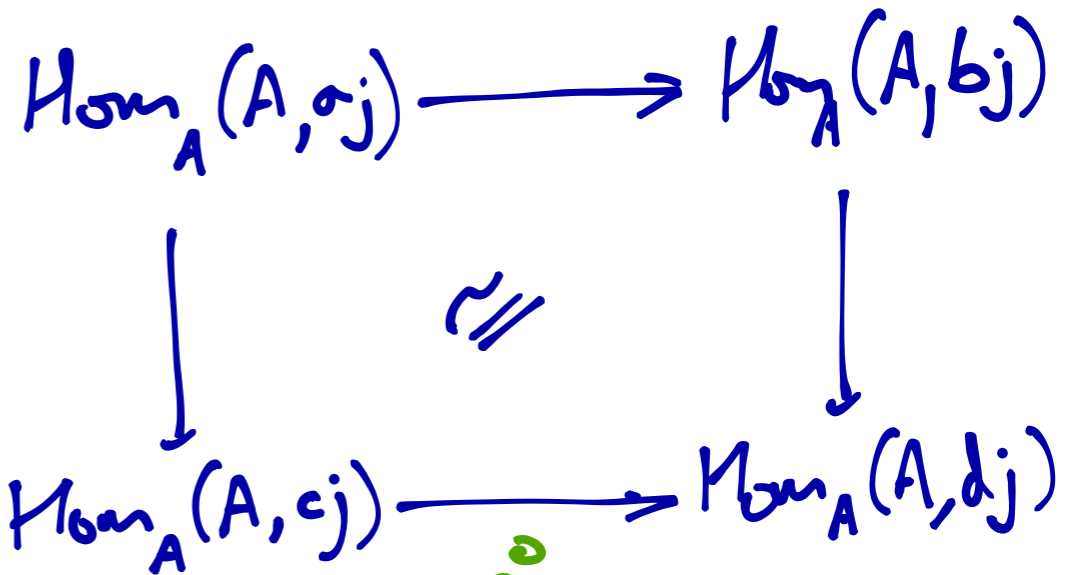
- A TRIANGLE IS A PUSHOUT (COKERNEL) IFF IT IS A PULLBACK (KERNEL).

A HOM-WISE CHARACTERISATION OF PULLBACKS



I-INDEXED COMMUTATIVE SQUARES IN A

THIS SQUARE IS AN I-INDEXED FAMILY OF PULLBACKS \Leftrightarrow



ISO-SQUARES IN 2-CAT $\text{Mod}_K(J, A)$

THIS ISO-SQUARE IS A WEAK ISO-COMMA FOR ALL $j: J \rightarrow I$.

APPLICATION: THE COMPOSITION + CANCELLATION LAWS HOLD FOR PULLBACKS IN ANY \mathcal{A} -CAT.

ELEMENTARY RESULTS ...

- COMPOSITION + CANCELLATION OF PULLBACK / PULLBACK SQUARES \Rightarrow STABLE ∞ -CATS ADMIT ALL PULLBACKS / PUSHOUTS:

$$\begin{array}{ccccc}
 \ker kg & \longrightarrow & a & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow f & \lrcorner & \downarrow \\
 c & \xrightarrow{g} & b & \xrightarrow{k} & \text{coker } f
 \end{array}$$

- THE LOOP-SUSPENSION ADJUNCTION OF A STABLE ∞ -CAT IS AN EQUIVALENCE.
- A POINTED ∞ -CATEGORY IS STABLE IFF IT ADMITS KERNELS + ITS LOOP FUNCTOR IS AN EQUIVALENCE.

STABILISATION

THE SUBCATEGORY OF

-) POINTED ∞ -CATEGORIES ADMITTING KERNELS
-) ∞ -FUNCTORS THAT PRESERVE POINTS AND KERNELS.

IS A **SUB- ∞ -COSMOS** SO IT ADMITS A HOMOTOPY LIMIT OF EACH CHAIN:

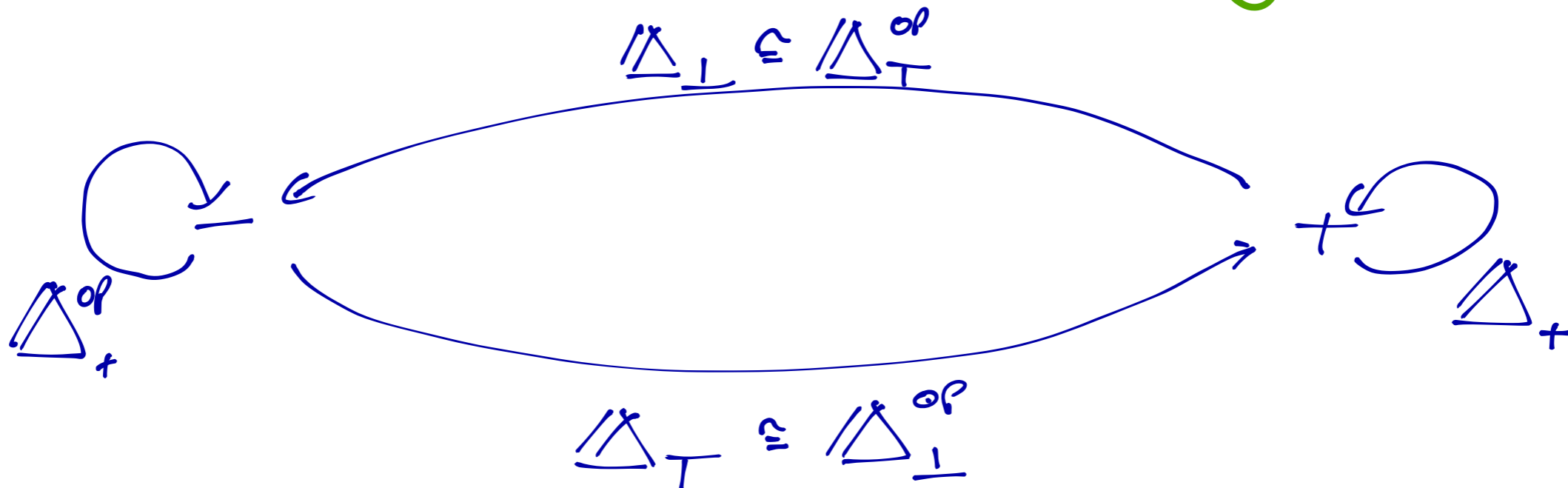
$$\dots \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A$$

PROP LOOPS FUNCTOR ON THIS LIMITING ∞ -CATEGORY IS AN EQUIVALENCE.

\Rightarrow THIS ∞ -CATEGORY $\text{STAB}(A)$ IS STABLE.

SPECTRA
LIVE
HERE!

THE 2-CATEGORY Adj



$$[0] : - \longrightarrow + \quad \circ \quad \underbrace{\quad}_{\underline{u}}$$

$$[0] : + \longrightarrow - \quad \circ \quad \underbrace{\quad}_{\underline{f}}$$

$$\underline{f} \underline{u} = [0] : - \longrightarrow - \quad \text{id}_- = [-1] : - \longrightarrow - \quad [-1] \rightarrow [0] \in \Delta_+$$

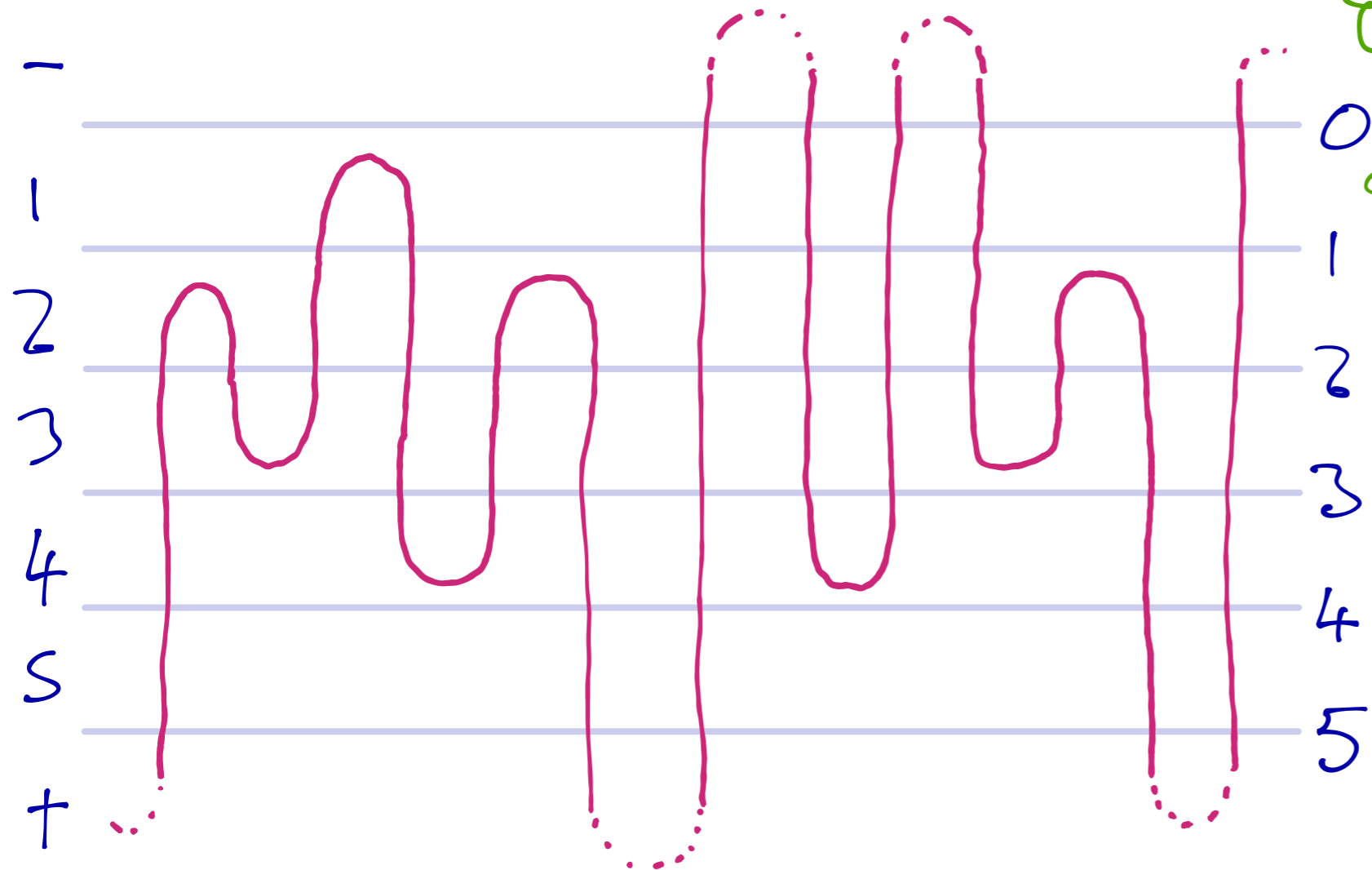
$$\xi : \underline{f} \underline{u} \Rightarrow \text{id}_- \in \text{Adj}(-, -) = \Delta_+^{\text{op}}$$

$$\underline{u} \underline{f} = [0] : + \longrightarrow + \quad \text{id}_+ = [-1] : + \longrightarrow + \quad [-1] \rightarrow [0] \in \Delta_+$$

$$\eta : \text{id}_+ \Rightarrow \underline{u} \underline{f} \in \text{Adj}(+, +) = \Delta_+$$

THE SIMPLICIAL CATEGORY Adj

n-ARROWS



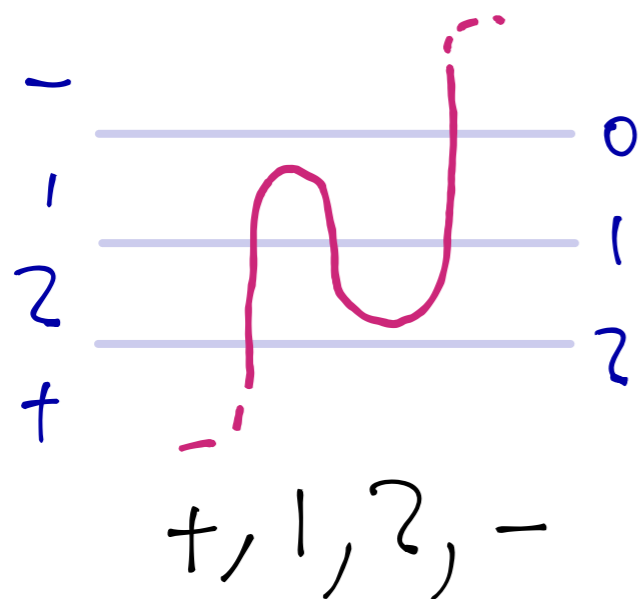
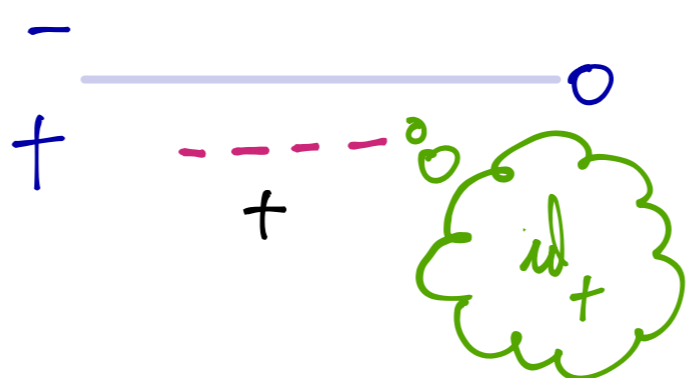
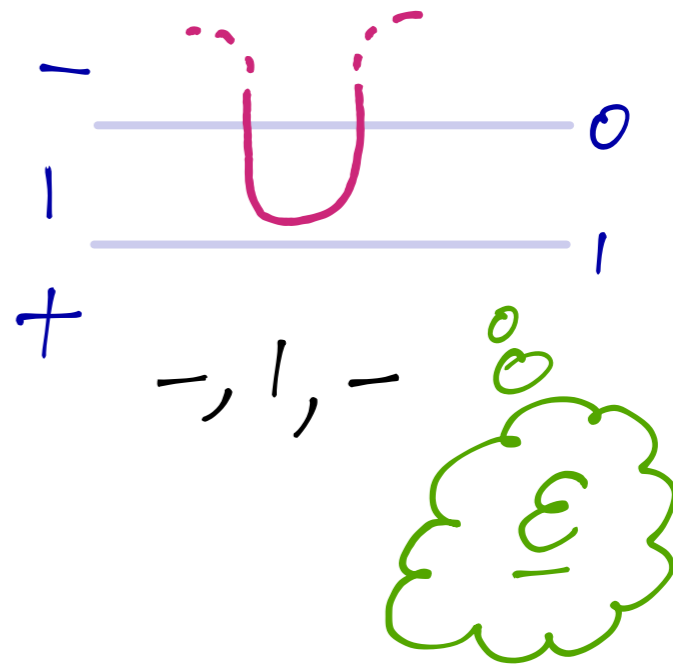
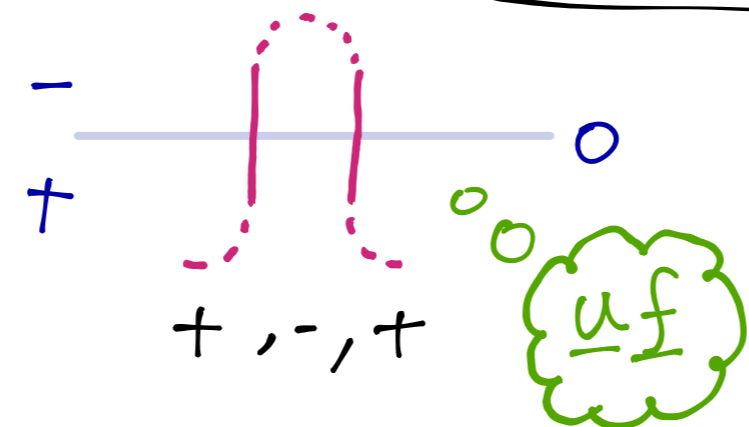
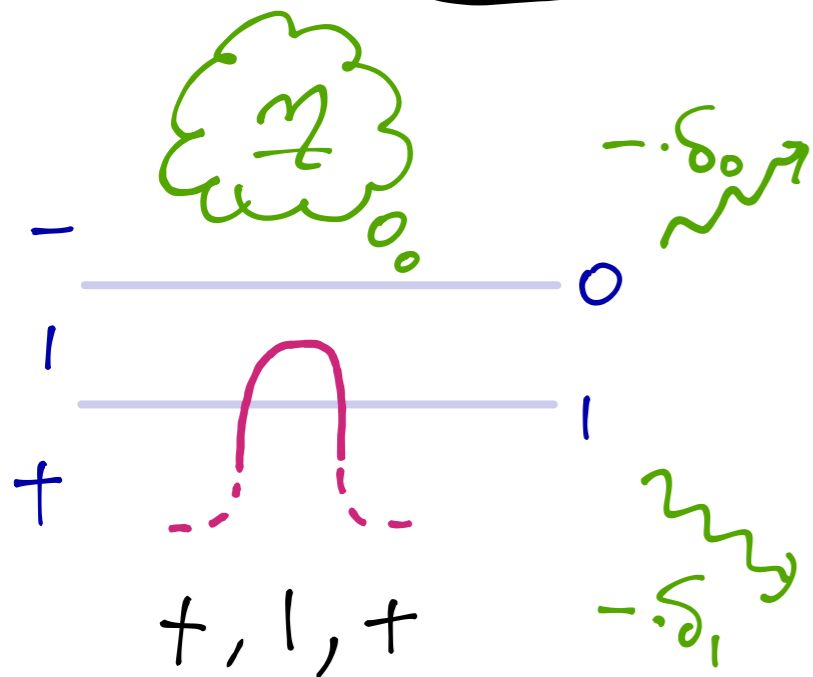
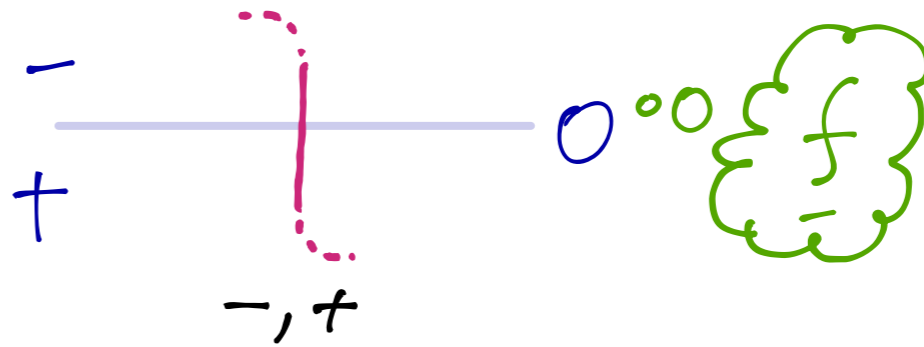
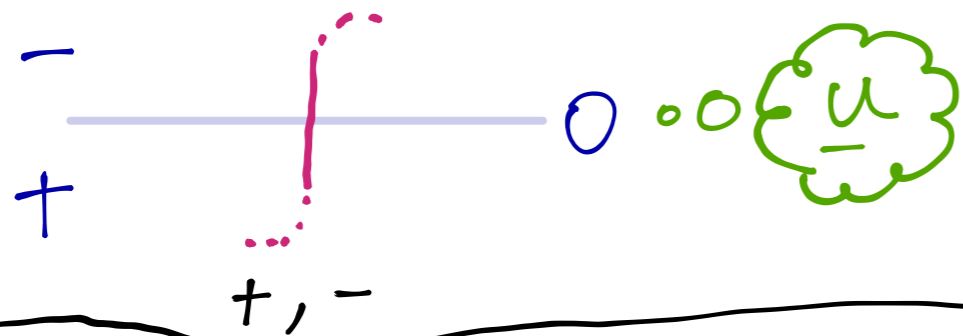
STRICTLY UNDUCTING
SQUIGGLE \equiv TREMELO

THIS DIAGRAM
DEPICTS A
5-ARROW

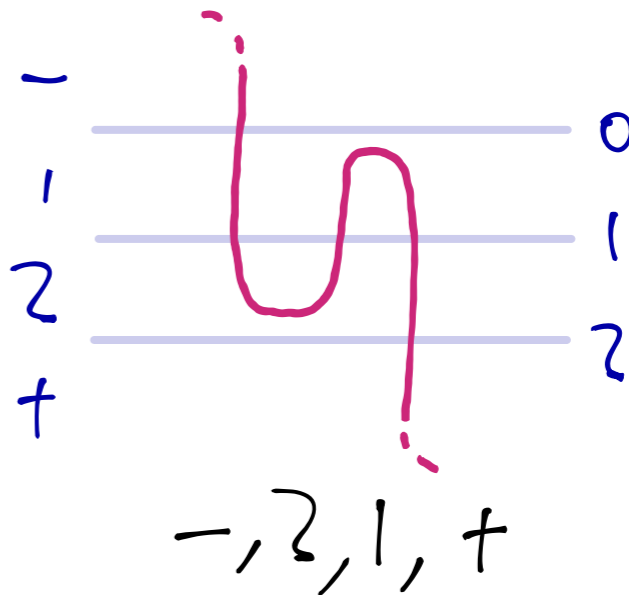
$\oplus, 2, 3, 1, 4, 2, +, -, 4, -, 3, 2, +, \ominus$

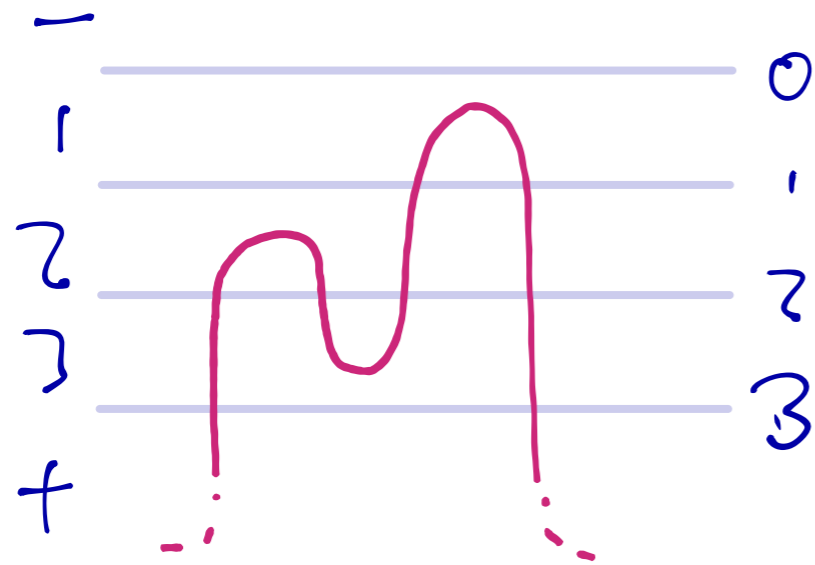
←
CODOMAIN

→
DOMAIN

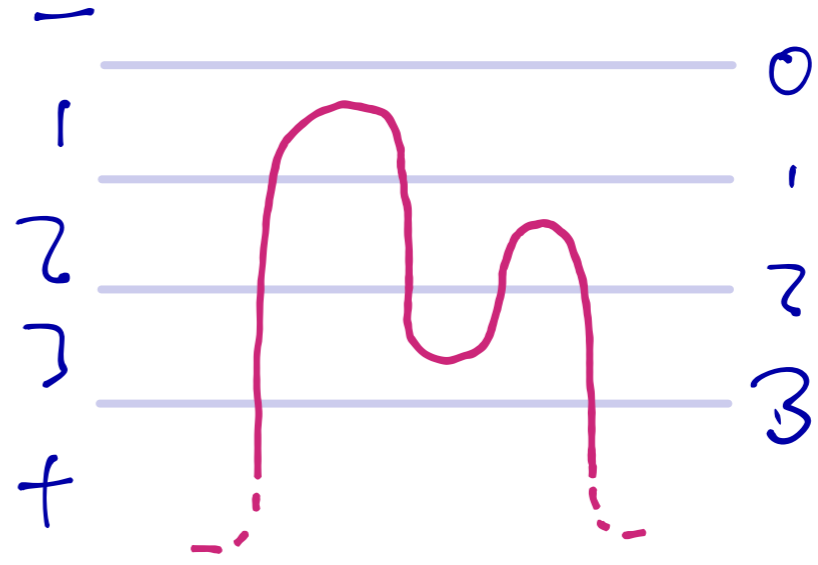


WITNESSES
OF THE
TRIANGLE
IDENTITIES





+ 2, 3, 1, +



+ 1, 3, 2, +

SOME HIGHER
 COHERENCES BETWEEN
 TRIANGLE IDENTITIES

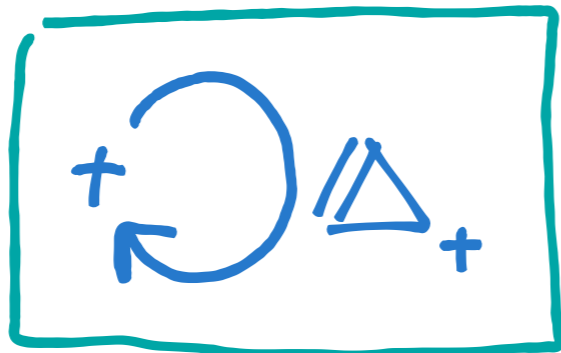
ADJUNCTIONS ARE COHERENT

THEOREM: AN ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$ GIVES RISE TO A SIMPLICIAL FUNCTOR $\mathcal{A} : \mathcal{A}dj \rightarrow \mathcal{K}$ WHICH CARRIES THE CANONICAL ADJUNCTION $\underline{\mathcal{F}} \dashv \underline{\mathcal{U}}$ IN $\mathcal{A}dj$ TO THE ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$. THE SPACE OF FUNCTORS THAT EXTEND $\mathcal{F} \dashv \mathcal{U}$ IN THIS WAY IS CONTRACTIBLE.

OBSCURITY CORNER: THIS IS ANOTHER IMPORTANT INTERNALISATION AXIOM IN THE WORLD OF 2-DERIVATORS.

HOMOTOPY COHERENT MONADS

$\mathcal{M}nd$

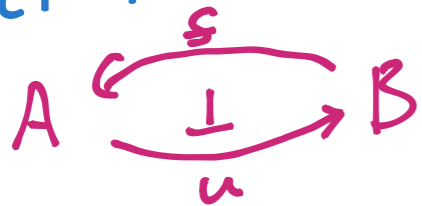


THE "WALKING"
HOMOTOPY COHERENT
MONAD.

FULL SUBCATEGORY
OF Adj SPANNING
THE OBJECT +

A HOMOTOPY COHERENT MONAD
IN AN ∞ -COSMOS \mathcal{K} IS SIMPLY
A SIMPLICIAL FUNCTOR

THE HOMOTOPY COHERENCE
RESULT FOR AN ADJUNCTION



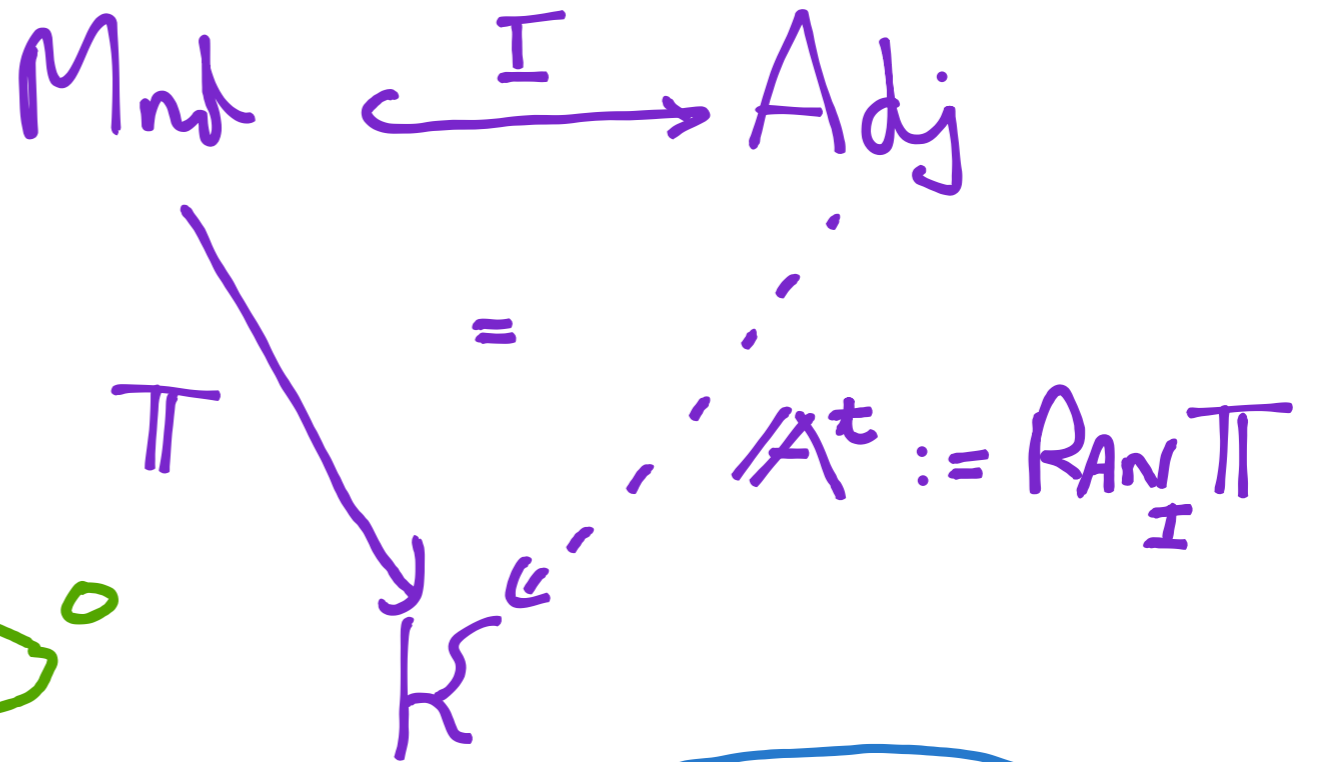
IN \mathcal{K} IMPLIES THAT
SUCH INDUCES AN MC
MONAD ON B AND AN
MC COMONAD ON A .

$$\Pi : \mathcal{M}nd \longrightarrow \mathcal{K}$$

$$\begin{array}{ccc}
 A \in \mathcal{K} & + & A \xrightarrow{t} A^{\Delta_+} \\
 A \xrightarrow{t} A^{\Delta_+} & & \begin{array}{ccc}
 A & \xrightarrow{t} & A^{\Delta_+} \\
 \downarrow t & & \downarrow A^{\oplus} \\
 A^{\Delta_+} & \xrightarrow{t^{\Delta_+}} & A^{\Delta_+ \times \Delta_+}
 \end{array}
 \end{array}$$

EILENBERG - MOORE OBJECTS

∞ -COSMOS ADMIT
ALL FLEXIBLY
WEIGHTED LIMITS



THIS KAN EXTENSION
EXISTS BECAUSE
Adj IS A COMPUTAD \Rightarrow
THE WEIGHT USED TO
COMPUTE IT IS FLEXIBLE

FLEXIBLE
 \equiv
COFIBRANT

THE OBJECT $A^\pi := A^t(-)$ IS THE ∞ -CAT
OF EILENBERG - MOORE ALGEBRAS.