MSRI : Introductory Workshop
Exercises for: Cobordism categories, classifying spaces and (invertible) TQFTs

1. Compute the unoriented cobordism groups in dimensions 0, 1, 2.

   Show that all elements in unoriented cobordism groups have order 2.

2. (a) Let Cob_2 be the 2-dimensional (discrete/homotopy) oriented cobordism category. For any monoid M, let $\mathcal{C}(M)$ denote the category with one object and morphisms set M. Show that
$$\Phi : Cob_2 \longrightarrow \mathcal{C}(\mathbb{Z})$$
   which sends a cobordism Σ from m to n circles to the number $n - m - \chi(\Sigma)$ defines a functor of symmetric monoidal categories; here $\chi(\Sigma)$ is the Euler characteristic of Σ.

   (b) Let Cob_2^∂ be the subcategory Cob_2 in which all connected components of the bordisms have outgoing boundary (so in particular there are no closed surfaces). Using Φ show that there is a functor $\Phi' : Con_2^\partial \to \mathcal{C}(\mathbb{N})$ with right inverse F and a natural transformation from the identity functor on Cob_2^∂ to $F \circ \Phi'$. Thus compute the homotopy type of the classifying space of Cob_2^∂?

3. Let X be a topological space (CW complex). Define $\mathcal{P}(X)$ to be the category with objects the set X^δ and morphisms between x_s and x_t the space of Moore paths from x_s to x_t.

   (a) Show $\mathcal{P}(X)$ is a category enriched in topological spaces..

   (b) Show that the associated homotopy category is $\pi_{\leqslant 1}X$.

   (c) What is the classifying space of $\mathcal{P}(X)$?

4. Let X be a space and consider the θ structure it defines.

   Determine the homotopy type of the classifying space of $Bord^X_{<0,1>}$.

   Determine the homotopy type of the classifying space of the subcategory $Cyl(X)$ of $Bord^X_{<1,2>}$ consisting only of cylindrical cobordisms.

5. Show that there are inclusions of cobordism categories $Bord_{<0,\ldots,d-1>} \subset Bord_{<0,\ldots,d>}$.

   Show that on classifying spaces these give a filtration of Ω^∞-spaces for Thom's space (up to an involution)
$$\Omega^\infty MTSO(0) \to \cdots \to \Omega^\infty\Sigma^d MTSO(d) \to \Omega^\infty MSO.$$

6. Show that $\oplus U_{d,n}$ defines a map $\Omega^\infty MTSO(d) \to \Omega^\infty\Sigma^\infty(BSO(d)_+)$.

7. Using the cobordism hypothesis compute the invertible 1-dimensional oriented TQFTs with value in $Vect_{\mathbb{C}}$.

References

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