An Introduction to Categorification of Quantum Groups & Link Invariants III

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We saw: Following the CKM approach to $\mathfrak{sl}_n$-knot invariants requires two ingredients:

- $\dot{U}_q(\mathfrak{gl}_m)$
- Quantum Weyl group action

Today: We categorify these ingredients in order to categorify $\mathfrak{sl}_n$ link polynomials?

What does categorification mean?

$\mathfrak{sl}_n$-polynomial (Laurent polynom)

$\varphi_{\mathfrak{sl}_n,V}(K) \in \mathbb{Z}[q,q^{-1}] \xrightarrow{\text{categorify}} ?$

$\dot{U}_q(\mathfrak{gl}_m)$ linear category $\xrightarrow{\text{categorify}} ?$

idea: decategorification is a specified process of destroying info $(n-1)\text{-cat} \longleftarrow n\text{-cat}$

categorification is the process of inverting such a destruction for a given (n-1)-cat of interest.

Ex. "Additive" categorification

- Start w/ $\mathcal{C}$ an additive category ($\oplus$, 0)
- Define (split) Grothendieck group $K_0(\mathcal{C})$ to be free abelian gp. on iso classes $[X]$ of objects in $\mathcal{C}$ modulo relation $[X \oplus Y] = [X] + [Y]$
- If $\mathcal{C}$ is monoidal (i.e., $\otimes$-prod associative) $K_0(\mathcal{C})$ is a ring w/ $[X \otimes Y] = [X] \cdot [Y]$
- If $\mathcal{C}$ is graded (i.e. has an invertible auto-equiv $\langle 1 \rangle : \mathcal{C} \to \mathcal{C}$ s. t. $\forall X$, $X\langle t \rangle$ $t \in \mathbb{Z}$

$\Rightarrow K_0(\mathcal{C})$ is a $\mathbb{Z}[q, q^{-1}]$-modl $[X\langle t \rangle] = q^t [X]$

Ex. finite dim'l vector spaces $\underline{\text{Fin Vect}}$

Iso class of object $[\mathbb{k}^n]$ $\qquad$ $\mathbb{k}$ ground field

$\mathbb{k}^n = \mathbb{k} \oplus \dots \oplus \mathbb{k}$

$[\mathbb{k}^n] = n[\mathbb{k}]$

$K_0(\text{Fin Vect}) \cong \mathbb{Z}$

$V \mapsto \dim(V) \cdot [\mathbb{k}]$

$\underline{\text{Gr Vect}}$ objects $V = \bigoplus_{s \in \mathbb{Z}} V_s$

morphs: ~~homomorphisms~~ maps

$$K_0(\mathrm{GrVect}) \cong \mathbb{Z}[q,q^{-1}]$$

$$V \longmapsto \mathrm{grdim}(V) = \sum_{s\in\mathbb{Z}} z^s \dim V_s \; [\mathbb{K}]$$

note: $[V] \in \mathbb{N}[q,q^{-1}]$

Kom(GrVect):

objects: $V^n \xrightarrow{d} V^{n-1} \xrightarrow{d} V^{n-2} \to \cdots\cdots$

morphs: chain maps/homotopy

$$K_0(\mathrm{Kom}(\mathrm{GrVect})) \cong \mathbb{Z}[q,q^{-1}]$$

$$V^\bullet \longmapsto [V^\bullet] := \sum_{s\in\mathbb{Z}} (-1)^s \,\mathrm{grdim}\, V^s$$

$$V^s = \bigoplus_{t\in\mathbb{Z}} (V^s)_t$$

GOAL: Take $\mathfrak{sl}_n$-polynoms $\in \mathbb{Z}[q,q^{-1}]$ $\longleftarrow$ (K_0 = graded Euler char) chain cplxs of graded vector spaces

$\mathrm{Hom}_{\dot{U}}(\lambda,\lambda')$ $\longleftarrow$ (K_0 of addt. cat.) graded additive category

$$\parallel$$

$1_{\lambda'} \dot{U}_q(\mathfrak{gl}_n) 1_\lambda$

$\mathbb{C}(q)$ - vector sp

$\mathbb{Z}[q,q^{-1}]$-module

$\dot{U}(\mathfrak{gl}_m)$ category $\xleftarrow{\ K_0\ }$ graded additive 2-cat

What makes a "good" categorification?

- An analogy:

  If X is a nice space, say finite CW cplx.

  Euler Char.

  $\chi(X)$

  top. invariant.

  We ❤ this.

But better! Homology groups $H_*(X, \mathbb{Q})$

Notice: The Euler char is K_0 of hom. gps.

Why are homology gps better invariants?

- comult, dually mult. on cohomology
- It's better at telling apart spaces
- It's functorial: $X \to Y \Rightarrow H_n(X, \mathbb{Q}) \to H_n(Y, \mathbb{Q})$

It deeply knows more about X and has useful structure to use to study X.

$$\chi(X) \xleftarrow{\ K_0\ } H_*(X, \mathbb{Q})$$

- In our setting:

$\mathfrak{sl}_n$-poly $\xleftarrow{K_0}$ link homology theory

Jones poly. — aka Khovanov homology, Khovanov-Rozansky

like χ — better, like H_*

quantum groups $\xleftarrow{K_0}$ categorified quantum gps

(Step one: write everything in curly letters... ü)

DEF: $\mathcal{U}_q(\mathfrak{gl}_m)$ (graded, additive, $\Bbbk$-linear) 2-cat

- Ob: $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$
- Morphs: gen'd by $\mathbb{1}_\lambda : \lambda \to \lambda$

  $\mathcal{E}\mathbb{1}_\lambda : \lambda \to \lambda + \alpha_i$

  $\mathcal{F}\mathbb{1}_\lambda : \lambda \to \lambda - \alpha_i$

Graded $\Rightarrow$ $\forall$ 1-morph X we have $X\langle t\rangle$, $t \in \mathbb{Z}$

i.e. $\mathcal{E}_i \mathbb{1}_\lambda \langle t\rangle : \lambda \to \lambda + \alpha_i$

Additive $\Rightarrow$ formal direct sums

i.e. $\mathcal{E}_i \mathbb{1}_\lambda\langle t\rangle \oplus \mathcal{E}_i \mathcal{F}_i \mathcal{E}_i \mathbb{1}_\lambda\langle t'\rangle : \lambda \to \lambda + \alpha_i$

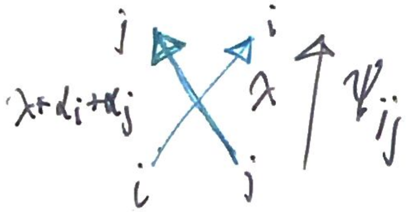
2-morphs:

convention $\mathrm{Id}_{\mathcal{E}_i 1_\lambda} = \lambda+\alpha_i \uparrow_i \lambda$

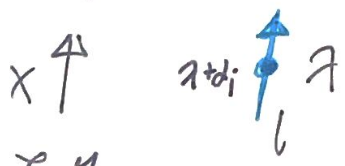
$\mathrm{Id}_{\mathcal{F}_i 1_\lambda} = \lambda-\alpha_i \downarrow^i \lambda$

(string diagrams (read bottom to top, R to L))

$\mathcal{E}_j \mathcal{E}_i 1_\lambda$

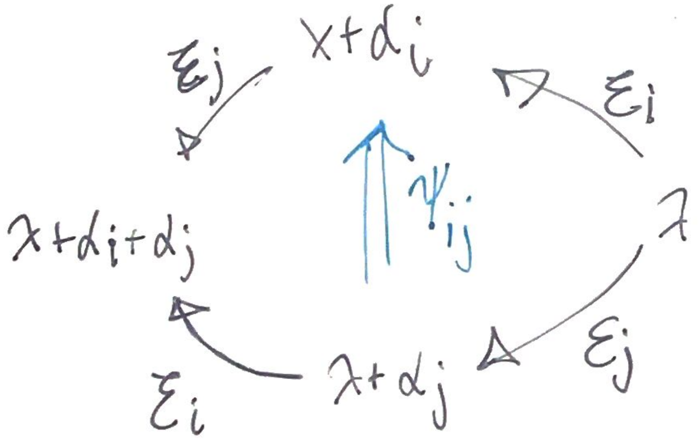


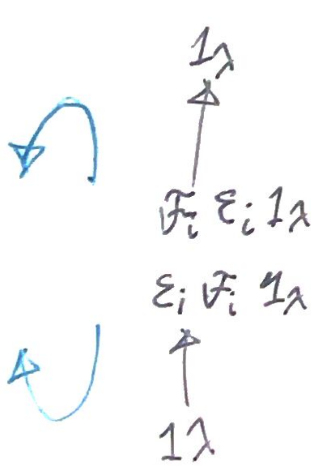
$\mathcal{E}_i 1_\lambda$

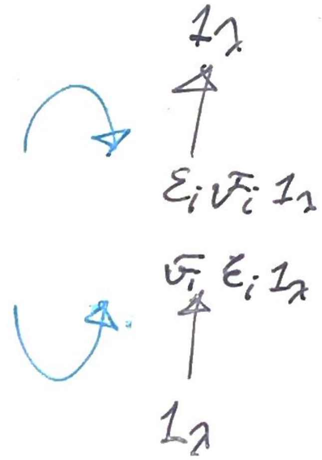


$\mathcal{E}_i 1_\lambda$

"dot" not the identity

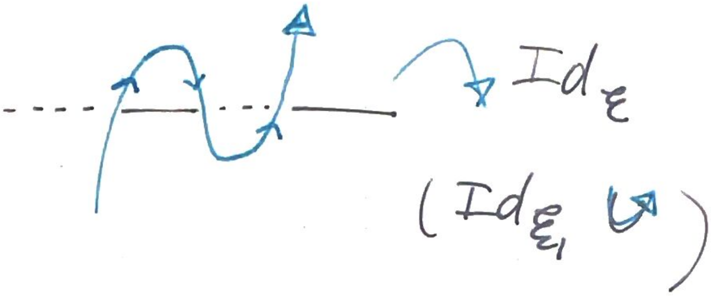


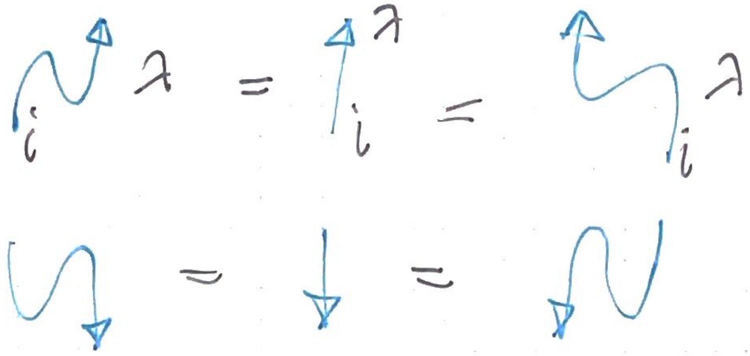




Relations:

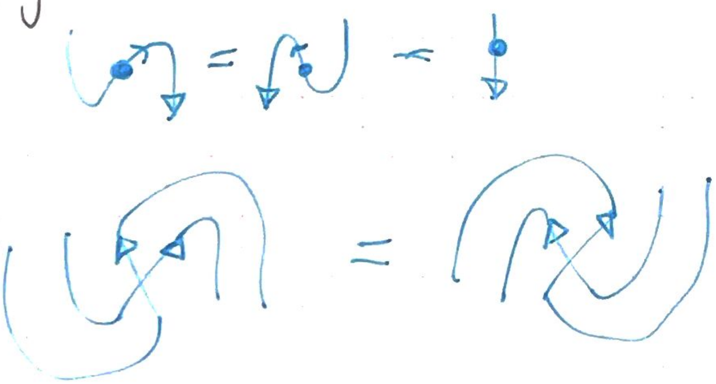
- $\mathcal{E}_i$, $\mathcal{F}_i$ are biadjoint





$(\cap\ \mathrm{Id}_{\mathcal{E}_i 1_\lambda}) \circ (\mathrm{Id}_{\mathcal{E}_i 1_\lambda}\ \cup) = \mathrm{Id}_{\mathcal{E}_i 1_\lambda}$

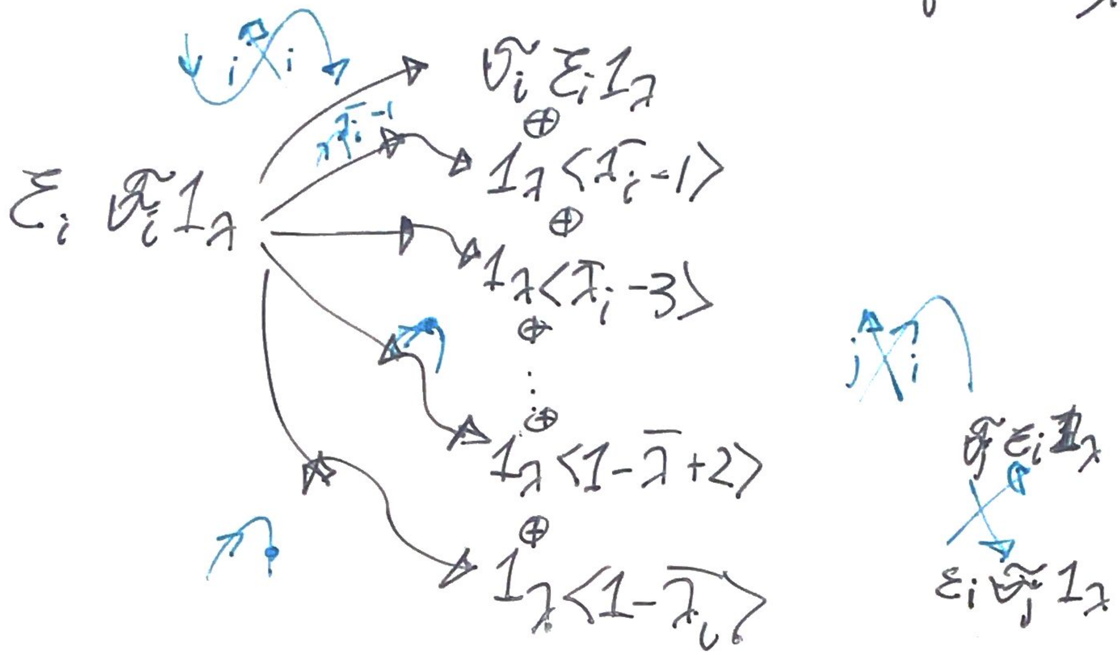
- cyclic



top. def. gives same 2-morphisms

• $\mathfrak{sl}_2$-iso

$$E_iF_i1_\lambda = F_iE_i1_\lambda + [\underbrace{\lambda_n-\lambda_{n+1}}_{\bar\lambda}]1_\lambda = F_iE_i1_\lambda + q^{\bar\lambda_i-1}1_\lambda + q^{\bar\lambda_i-3}1_\lambda + \dots + q^{1-\bar\lambda_i}1_\lambda$$



Relations on upward strands governed by KLR-algebra ⇒ Serre relation isos

THM. (Khovanov, L)

$$K_0(\dot{\mathcal{U}}) \cong \dot{U}_q(\mathfrak{gl}_m)$$

↑

slight enlargement where we add images of idempotents

$$\sigma_i1_\lambda = \sum(-q)^sE_i^{(s)}F_i^{(s+\bar\lambda_i+1)}1_\lambda = F_i^{(\bar\lambda_i+1)}1_\lambda - qE_iF_i^{(\bar\lambda_i+2)}1_\lambda \dots$$

$$= \tilde{\vartheta}_i^{(\bar{\lambda}_i)} \xrightarrow{d} \textstyle\sum_i \tilde{\vartheta}_i^{(\bar{\lambda}_i+1)} 1_{\lambda}\langle 1\rangle \xrightarrow{d} \cdots$$

$$d^2 = 0$$