Cobordism Categories, Classifying Spaces, and (invertible) TQFTs I

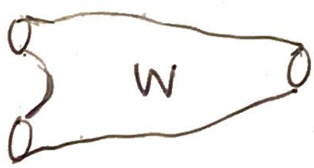
Ulrike Tillman

MOTIVATION.

- Higher categories
- TQFTs
- Classification of manifolds & families thereof.

COBORDISMS

Def. Two closed smooth oriented compact d-dim manifolds M_0 and M_1 are cobordant iff $\exists$ d+1 diml oriented manifold W with boundary $\partial W = M_0 \amalg M_1$



M_0 M_1

- equivalence reln
- equiv. classes Ω_d^{SO} form a group under $\amalg$ & a ring under $\times$

Ex. $\Omega_0^{SO} \simeq \mathbb{Z}$

$\Omega_1^{SO} \simeq 0$

$\Omega_2^{SO} = 0$

Thm (1950s). $\Omega_*^{SO} = \pi_*(\Omega^\infty MSO)$

$$\Omega^\infty MSO = \operatorname*{colim}_{\substack{n\to\infty \\ k\to\infty}} \Omega^n (U_{n,k}{}^c)$$

$\mathrm{map}_*(S^n, -)$

$$\begin{array}{c} \mathbb{R}^n \to P = U_{n,k} \\ \downarrow \\ P \in Gr(n,k) \end{array}$$

$= $ Grassmanian of n-planes in $\mathbb{R}^{n+k}$

Cor. $\Omega_*^{SO} \otimes \mathbb{Q} \simeq \mathbb{Q}[y_{4i} \mid i \geq 0]$ $\quad y_{4i} = \mathbb{C}P^{2i}$

Proof.
$$\begin{aligned} \Omega_*^{SO} &= \pi_* \Omega^\infty MSO \\ &= \pi_* \Omega^n (U_{n,k}{}^c) \qquad k, n \gg 0 \\ &= \pi_{*+n} (U_{n,k}{}^c) \\ &\underset{\mathbb{Q}}{\simeq} H_{*+n} (U_{n,k}{}^c) \quad (\text{Serre}) = H_*(Gr(n,k)) \quad \square \end{aligned}$$

TQFTs.

Def. Let Cob_d be the <u>Cobordism category</u> (discrete) w/.

<u>objects</u>: $d-1$ closed manifold M

<u>morph.</u> d-cobordisms / up to diffeomorph. relative to ∂

Def. (Atiyah)

A <u>TQFT</u> of dim. d is a symmetric monoidal functor $Cob_d \xrightarrow{Z} Vect_{\mathbb{C}}, \otimes$

$\emptyset \longmapsto \mathbb{C}$ units

Why did Atiyah care?

Z gives a topo. invariant of closed oriented d-dim'l manifolds: $W : \emptyset \to \emptyset$

$\rightsquigarrow Z(W) : \mathbb{C} \to \mathbb{C}$

Folk Theorem.

2-dim'l TQFTs $\xleftrightarrow{1:1}$ finite dim'l commutative Frobenius algebras

$Z \longmapsto Z(S^1) = A$

1-dim'l TQFTs $\leftrightsquigarrow$ f.d. vector spaces (w/ non degenerate inner product)

Classifying Spaces

$$\underset{\text{categories}}{\mathrm{CAT}} \longrightarrow \Delta\text{-}\overset{\text{simplicial}}{\text{sets}} \longrightarrow \mathrm{TOP} = \text{topological spaces}$$

$$\mathcal{C} \longmapsto N_\cdot\mathcal{C} \longmapsto \|N_\cdot\mathcal{C}\| := |\mathcal{C}| \text{ or } B\mathcal{C}$$

Nerve $N_q\mathcal{C} = \{(f_1, \dots, f_q) \mid \text{composable}\}$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

pt for each object
$\to$ for morphs

So $|\mathcal{C}| := \coprod_{q \geq 0} N_q\mathcal{C} \times \Delta^q/\sim$

- Products: $B(\mathcal{C}\times\mathcal{D}) \simeq B\mathcal{C} \times B\mathcal{D}$
- Functorial: $F: \mathcal{C} \to \mathcal{D} \rightsquigarrow$ cont. map
- ntrl trans: $\eta: F \to G \rightsquigarrow$ homotopy $|F| \sim |G|$

  $\mathcal{C} \times (0<1) \to \mathcal{D}$

  $(id_A, <) \longmapsto \eta_a: F(a) \to G(a)$ $\rightsquigarrow |(0<1)| = [0,1]$

$\rightsquigarrow$ $\mathcal{C}$ w/ initial or final obj. $\Rightarrow |\mathcal{C}| \sim *$

$\rightsquigarrow$ $\mathcal{C}, \mathcal{D}$ adjoint pair $\Rightarrow |\mathcal{C}| \simeq |\mathcal{D}|$

- monoidal $\rightsquigarrow$ htpy. ass. product $|\mathcal{C}| \times |\mathcal{C}| \xrightarrow{\otimes} |\mathcal{C}|$
  - symm. monoidal $\rightsquigarrow$ $E\Sigma_j \underset{\Sigma_j}{\times} |\mathcal{C}|^j \to |\mathcal{C}|$

What is a symmetric monoidal category?

comes w/ natural transformation

$c : \otimes \to \otimes \circ T : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$\tilde{\Sigma}_j \times \mathcal{C}^j \to \mathcal{C}$

obs: $\sigma, a_1, \ldots, a_j \mapsto a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$

morphs: $\sigma \to \mu; (f_1, \ldots, f_j) \mapsto c_{\mu\sigma^{-1}} \circ (f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(j)})$

FACT. Every space X is a $B\mathcal{C}$ for some $\mathcal{C}$.

Vice Versa. $\mathrm{Top} \to \mathrm{CAT}$

$X \mapsto \pi_{\leq 1} X : \begin{cases} \text{ob} = \text{pts in } X \\ \text{morph: homotopy classes of paths} / \text{rel } \partial \end{cases}$

Partial inverse:

$|\pi_{\leq 1} X| \simeq 1 \text{ type of } X$

$\pi_0 X, \pi_{\leq 1} X \sim \pi_1(X, *)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, mapsto] \arrow[d] & {|\mathcal{C}|} \arrow[r, mapsto] & \pi_{\le 1}|\mathcal{C}| \text{ is a localization.} \\ \underbrace{\mathcal{C}[\mathcal{C}^{-1}]}_{\substack{\text{forcibly}\\ \text{invert all}\\ \text{arrows}}} \arrow[urr, "\text{equivalence of categories}"'] & & \end{tikzcd}$$

Remarks. All works for Top-enriched categories, and for categories in Top.

Invertible TQFTs

Invertible functors: $\mathcal{V}$ category

$\mathcal{V}$ cat., $\mathcal{V}^{\times}$ = maximal subgroupoid of $\mathcal{V}$
(ignore all non-invertible.)

$\mathcal{V}$ symm. mon., $\mathrm{Pic}(\mathcal{V})$ = max. subgpoid of $\mathcal{V}$ w/obs. a s.t. $\exists\, \bar{a} : a \otimes \bar{a} \simeq e$

$$\mathrm{vect}^{\times} \simeq \begin{cases} \text{ob: } \mathbb{C}^n \quad n \geq 0 \\ \text{morph: } GL_n\,\mathbb{C} \end{cases}$$

$$\mathrm{Pic}(\mathrm{Vect}) \simeq \begin{cases} \text{ob: } \mathbb{C} \\ \text{morph: } GL\,\mathbb{C} = \mathbb{C}^{\times} \end{cases}$$

Category of d-TQFTs

$= \mathrm{Fun}^{\otimes}(\mathcal{C}ob_d, \mathrm{Vect}_{\mathbb{C}})$

Invertible cat. of d-TQFTs

$= \mathrm{Fun}^{\otimes}(\mathcal{C}ob_d, \mathrm{Pic}(\mathrm{Vect}_{\mathbb{C}}))$

$= \mathrm{Fun}^{\otimes}(\mathcal{C}ob_d[\mathcal{C}ob_d^{-1}], \mathrm{Pic}(\mathrm{Vect}_{\mathbb{C}}))$

$\boxed{d=2}$: calculate these

$\mathcal{C}ob_2[\mathcal{C}ob_2^{-1}] \longrightarrow \mathbb{Z}$

${}_nS_m \longmapsto n-m-\chi(S)$

equivalence to $\mathbb{Z}$

invert. 2-TQFTs

discrete category

$\mathrm{Fun}^{\otimes}[\mathbb{Z}, \mathbb{C}^\times] = \begin{cases} \cdot\, \mathbb{C}^* \\ \cdot \text{ identities} \end{cases}$

Invertible 1-TQFTs:

$\mathcal{C}ob_1[\mathcal{C}ob_1^{-1}] \simeq \begin{cases} * \\ \mathbb{Z}/2 \end{cases}$

$\mathrm{Fun}^{\otimes}[\mathbb{Z}/2, \mathbb{C}^\times] = \begin{cases} \cdot\, \mathbb{Z}/2 \\ \cdot \text{ ids} \end{cases}$

$\pi_1 |\mathcal{C}ob_2| = \mathbb{Z}$

$|\mathcal{C}ob_2| = ? = S^1 \times [\text{simply connected}]$

$|\mathcal{C}ob_2^\partial| \simeq S^1$

$\pi_1 |\mathcal{C}ob_1| = \mathbb{Z}/2$

$|\mathcal{C}ob_1| =$ complicated ☹

$= \mathrm{htpy}(\Sigma^{\infty-2} MTSO(2) \to \mathbb{C}P^\infty)$

Steinebrunner