Categorifications & Lie Algebra Actions on categories arising from representation theory III

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Why Representation Theorists should care about 2-morphisms

Deligne Category: "Universal tensor category interpolation representations of algebraic (super) groups"

tensor category: $\mathbb{C}$-linear, symmetric monoidal cat, $\mathrm{End}(\mathbb{1}) = \mathbb{C}$

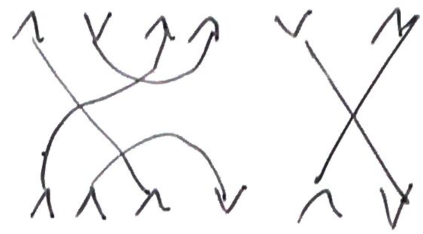
ribbon. $V^* \otimes V \xrightarrow{ev} \mathbb{1}$ $\quad \mathbb{1} \longrightarrow V \otimes V^*$

dim object $V =$  $\in \mathrm{End}(\mathbb{1}) = \mathbb{C}$

$\boxed{\text{Fix } \delta \in \mathbb{C}!}$

I $\mathrm{Rep}_0(GL_\delta) :=$ universal $\otimes$-cat gen'd by an object V of dim. δ and its dual V^*
$(V \not\cong V^*)$

objects: finite words in $\{\vee, \wedge\}$

morphisms: eg. 

relations: isotopies & loops evaluate to δ.

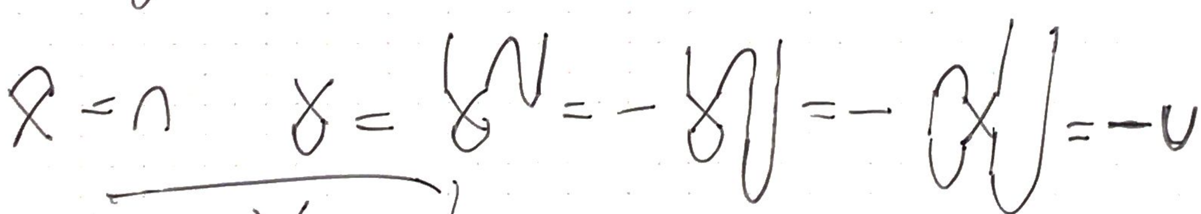
$\boxed{\text{II}}$ $\mathrm{Rep}_0(O_\delta)$ same as $\boxed{\text{I}}$ except that $V \cong V^*$

objects: $\mathbb{N}$

morphs: eg 

relations: isotopies & loops evaluate to δ

$\boxed{\text{III}}$ $\mathrm{Rep}_0(P_\delta)$ same as $\boxed{\text{II}}$, but enriched in super vector spaces (Coulembier-Ehrig)
w/ deg $\cup, \cap$ is 1 (Brundan-Ellis-?)



$\Rightarrow$ $\boxed{\dim V = 0}$

$\underline{Rep}(\cdot) :=$ Karoubian envelope of $\underline{Rep}_0(\cdot)$

Representation Categories

$\boxed{\text{I}}$ $\underbrace{\mathfrak{gl}(m|n)}_{\text{Lie superalgebra}} \curvearrowright \mathbb{C}^{m|n}$ (super vector sp. of dim $(m|n)$)

$\rightsquigarrow G := GL(m|n)$ supergroup

$Rep(G) :=$ fin. diml. repns of G

$\boxed{\text{II}}$ $V = \mathbb{C}^{r|m}$ $r = 2m$ or $2m+1$ w/ nondegenerate bilinear form $(-,-)$ which is supersymmetric (i.e., symmetric on even part, skew-symmetric on odd)

$\dim V := r - n = \delta$ superdimension = categorical dimension

$\mathfrak{osp}(r|2n) = \{ A \in \mathfrak{gl}(r|2n) \mid (Av, w) + (-1)(v, Aw) = 0 \}$

$\rightsquigarrow G = OSp(r|2n)$ Lie supergroup $\rightsquigarrow Rep(G)$

$\boxed{\text{III}}$ $V = \mathbb{C}^{n|n}$

$p(n) := \{A \in gl(n|n) \mid `` \}$ but now $(-,-)$ is an <u>odd</u> symmetric bilinear form

Up to one more family ($Q(n)$) and a few exceptional cases, these are all basic simple Lie super-algebras (Kac)

* In our cases, $\exists$

a) tensor functor $F : \underline{Rep}(`` _\delta) \longrightarrow Rep(G)$

V of $\dim_\delta \longmapsto V$ is <u>full</u>

(Comes-Wilson, Brundan-S, Lehrer-Zhang, Colembler, Moon)

b) V generates $Rep(G)$ as a tensor category. In particular, any indecomp. projective object appears as a summand in some $V^{\otimes r}$ or $\oplus V$'s

Comes-Wilson, Brundan-S, Ehrig-S,
[BDEHHILNSS]

$\Rightarrow$ Understanding $V^{\otimes r} \otimes V^{* \otimes s}$ w/ morphs between them gives a good understanding of Rep(G) (as abelian cat)

Rep($\mathfrak{g}_\delta$) is not semisimple $\Leftrightarrow$ $\delta \in \mathbb{Z}$ and r, n chosen in a good way

$\boxed{\text{Fix } \delta \in \mathbb{Z}}$ Which simple objects appear in a Jordan-Hölder series of a projective?
$[P(\lambda) : L(\mu)] = ??$
in Rep(G) multiplicities

$\boxed{\text{I}}$ Serganova, Brundan-S
given by parabolic Kazhdan-Lusztig polynomials $(S_n, S_i \times S_{n-i})$ for $n \gg 0$

$\boxed{\text{II}}$ Explicit formulas (combinatorial) [BD.....S]

$\boxed{\text{III}}$ Ehrig-S, Gruson-Serganova (B_n, A_{n-1}) KL polys $n \gg 0$

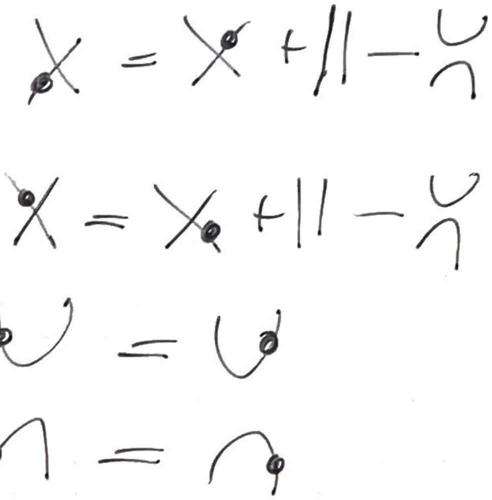
* from now on: Case II *

Categorification & super vs. classical

Def. Affine Deligne Category $\underline{Rep}^{aff}(O_\delta)$

exactly as $\underline{Rep}(O_\delta)$ but we write one extra generator ϕ for morphisms

e.g. 

relations: as before plus 

$\boxed{\text{AVW-relations}}$

$\uparrow$

$\boxed{\text{ABMW-relations}}$

Universal property of $\underline{Rep}$ / Higher Schur-Weyl / Duality

(super) (classical)

V = nat. $G = OSp(r|2n)$-module $\quad \underline{Rep}_\delta := \underline{Rep}(O_\delta)$

$\mathrm{End}_{\underline{Rep}}(d) \twoheadrightarrow \mathrm{End}(V^{\otimes d})$

$\mathrm{End}_{\underline{Rep}^{aff}}(d) \Big/ \left(\phi = \frac{\delta - 1}{2}\right) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{End}(V^{\otimes d}) \quad \underline{Rep}^{aff} = \underline{Rep}^{aff}(O_\delta)$

$\|$

$S_{d,\delta}$

On the other hand, let M be a repn of (classical) $\mathfrak{so}(N)$ $(N>>0)$

$$\underset{\mathrm{Rep}^{\mathrm{aff}}(O_N)}{\mathrm{End}(d)} \longrightarrow \mathrm{End}(M\otimes(\mathbb{C}^N)^{\otimes d})$$

ntrl repn of $\mathfrak{so}(N)$

$\mathbb{C}^N \cong (\mathbb{C}^N)^*$ via fixed bil. form.

$\underset{1}{|}\,|\,|\underset{i\ \ i+1}{X}|\,|\underset{d}{|} \longmapsto$ swap of the factor $\mathbb{C}^N$ $(i \leftrightarrow i+1)$

$|||\cap|||| \longmapsto$ evaluation $\cap : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$

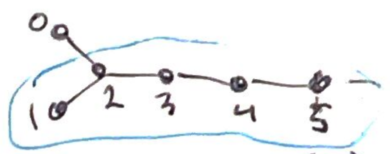
$|||\cup||| \longmapsto$ coeval. $\cup : \mathbb{C} \to \mathbb{C}^n \otimes \mathbb{C}^n$

$\underset{1}{\bullet}|||||\underset{d}{|} \longmapsto$ action of $C = \sum_i X \otimes X^*$ on $M\otimes\mathbb{C}^N$

X basis of $\mathfrak{so}(N)$

Connect N and δ

Consider for M a special module depending on δ



N even

$\to \mathfrak{p} \subseteq \mathfrak{so}(N)$ of type A_{n-1}. Consider weight

Rnd weight $\to \delta w_0 = (\frac{\delta}{2}, \dots, \frac{\delta}{2})$

$\rightsquigarrow$ "universal module"

$M^{\mathfrak{p}}(\delta w_0)$ = maximal $\mathfrak{p}$-locally finite dim. quotients of $U(\mathfrak{g}) \otimes_{\mathfrak{p}} \mathbb{C}_{\delta w_0}$

$\mathfrak{g} = \mathfrak{so}(N)$

Let $A_{d,\delta} := \mathrm{End}(d)$

THM. (Ehrig - S):

a) $A_{d,\delta} / (\phi|||_{\ell} - \alpha)(\phi|\ldots| - \beta) \xrightarrow{\sim} \mathrm{End}(M^{\mathfrak{p}}(\delta_{w_0}) \otimes (\mathbb{C}^N)^{\otimes d})$

$\alpha = \frac{\delta - 1}{2} \qquad \beta = N - \alpha$

b) Let e be the projection onto eigenspaces for ϕ with small eigenvalues.

Then $\mathrm{End}_{\mathfrak{g}}((M^{\mathfrak{p}}(\delta w_0) \otimes (\mathbb{C}^{\otimes N})^{\otimes d}) e)$

$\wr\!\uparrow$

$S_{d,\delta}$

R understood

c) The kernel of $S_{d,\delta} \twoheadrightarrow \mathrm{End}(V^{\otimes d})$ can be described explicitly dep. on r, n