SYNTHETIC ∞-CATEGORY THEORY

PROBLEM SET

1) PROVE THAT THE COMPOSITE OF TWO ADJUNCTIONS IN A 2-CATEGORY $\mathcal{K}$ IS AGAIN AN ADJUNCTION. IN OTHERWORDS, GIVEN

$$\begin{tikzcd} A \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "g"'] \\ {} \end{tikzcd}$$

$$A \xrightarrow{u} B \xrightarrow{v} C \quad \text{IN } \mathcal{K}$$

$\eta : id_B \Rightarrow uf$ $\quad\quad$ $\mu : id_C \Rightarrow vg$ $\quad\quad$ + TRIANGLE

$\varepsilon : fu \Rightarrow id_A$ $\quad\quad$ $\nu : gv \Rightarrow id_B$ $\quad\quad$ IDENTITIES

WRITE DOWN CANDIDATE UNIT + COUNIT FOR AN ADJUNCTION $fg \dashv vu$ AND SHOW THEY SATISFY THE TRIANGLE IDENTITIES.

HINT TRY USING STRING DIAGRAMS TO EXPRESS / PLAN YOUR ARGUMENT

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2) GIVEN AN ADJUNCTION $\varepsilon : A \underset{u}{\overset{f}{\rightleftarrows}} B : \eta$ ($\perp$) IN A 2-CATEGORY $\mathcal{K}$, SHOW THAT IT GIVES RISE TO ADJUNCTIONS OF HOM-CATEGORIES.

$$\mathcal{K}(X, f) \dashv \mathcal{K}(X, u) \quad + \quad \mathcal{K}(u, Y) \dashv \mathcal{K}(f, Y)$$

$$\frac{fa \Rightarrow b}{a \Rightarrow ub} \qquad\qquad \frac{cu \Rightarrow d}{c \Rightarrow df}$$

HMMM A LITTLE WEIRD!

3) GIVEN AN EQUIVALENCE IN A 2-CATEGORY $\mathcal{K}$

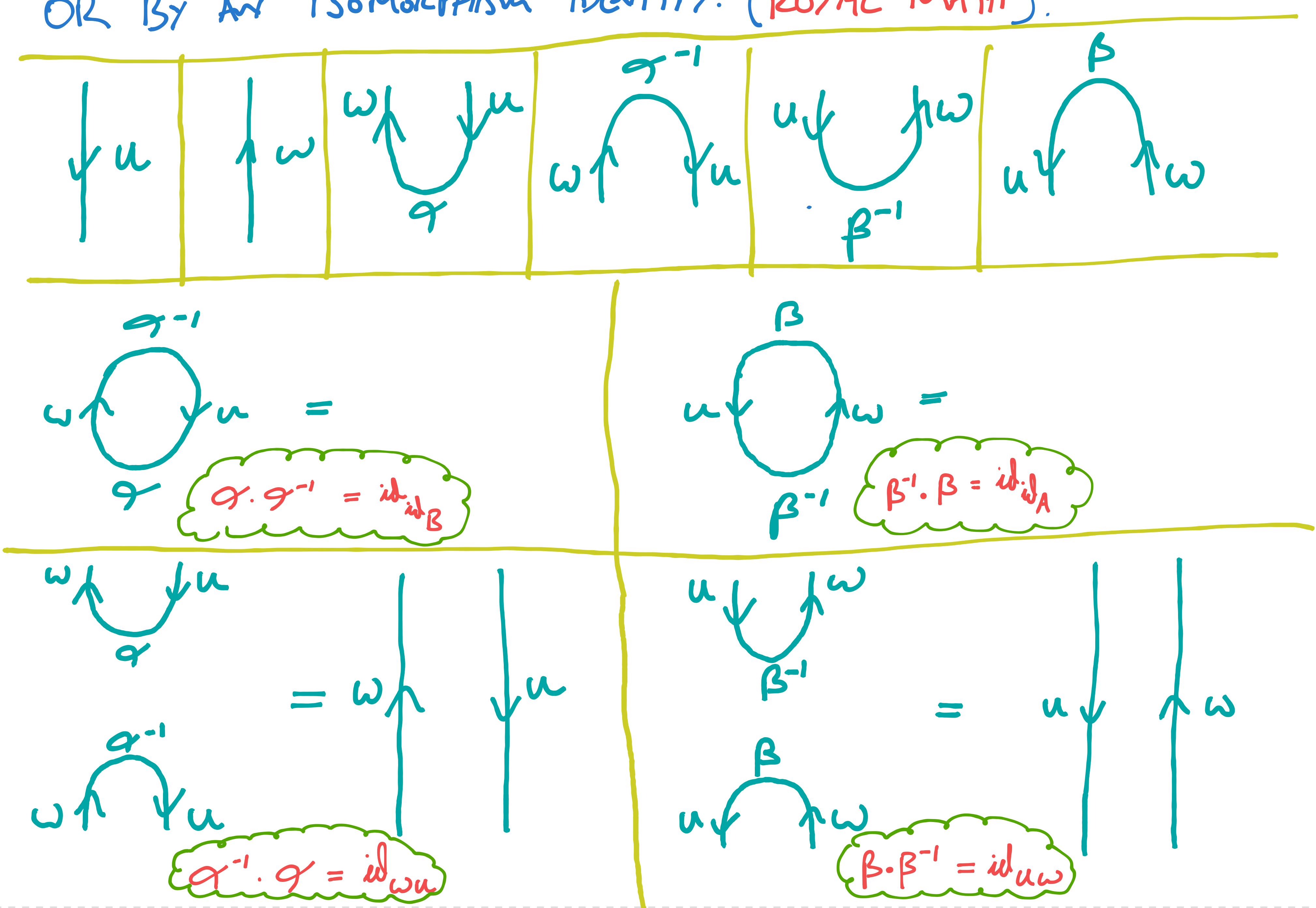
$$\begin{tikzcd} A \arrow[r, bend right, "w"'] & B \arrow[l, bend right, "u"'] \end{tikzcd}$$

$\varphi : wu \cong id_B$

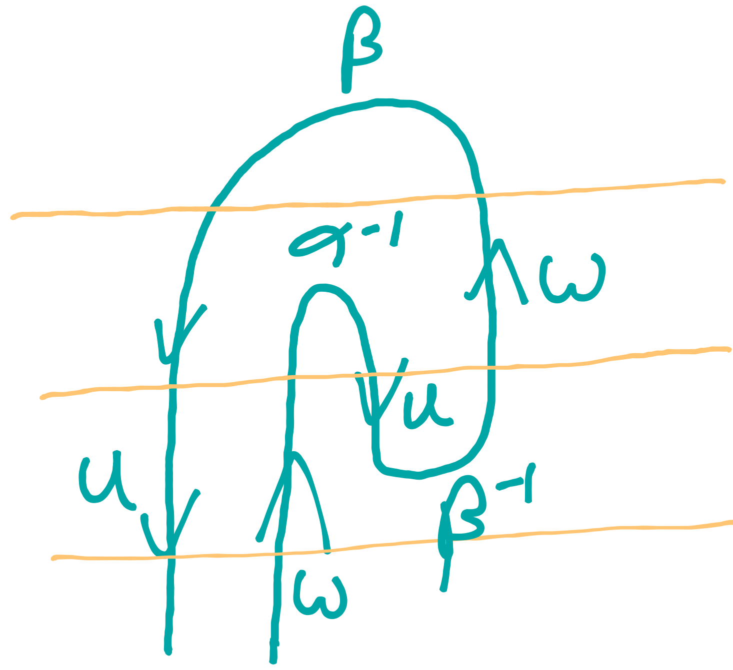
$\beta : id_A \cong uw$

SHOW THAT WE MAY CONSTRUCT A 2-CELL $\beta' : id_A \cong uw$ SUCH THAT φ AND β' SATISFY THE TRIANGLE IDENTITIES.

HINT USE SEQUENCES OF MANIPULATIONS OF STRING DIAGRAMS TO PLAN YOUR ARGUMENT (NOMADIC MATH) THEN CONVERT THOSE INTO ALGEBRAIC EQUIVALENTS IN WHICH EACH STEP IS JUSTIFIED BY EITHER BY MIDDLE 4 INTERCHANGE OR BY AN ISOMORPHISM IDENTITY. (ROYAL MATH).



$\beta' :=$



$$= \mathrm{id}_A \overset{\beta}{\cong} uw \overset{u\eta^{-1}w}{\cong} uwuw \overset{uw\beta^{-1}}{\cong} uw$$

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4) ADAPT THE PROOF GIVEN IN (3) TO PROVE THE FOLLOWING SLIGHTLY MORE GENERAL RESULT:

SUPPOSE WE ARE GIVEN 1-CELLS $A \underset{u}{\overset{f}{\rightleftarrows}} B$. IN A 2-CATEGORY $\mathcal{K}$ ALONG WITH 2-CELLS $\eta : \mathrm{id}_B \Rightarrow uf$ + $\varepsilon : fu \cong \mathrm{id}_A$ SUCH THAT $f\eta$ AND ηu ARE BOTH ISOMORPHISMS. THEN WE MAY CONSTRUCT A 2-CELL $\eta' : \mathrm{id}_B \Rightarrow uf$ SUCH THAT THE PAIR ε, η' SATISFY THE TRIANGLE IDENTITIES AND THUS DISPLAY AN ADJUNCTION $f \dashv u$

LEFT ADJOINT RIGHT INVERSE (LARI)

THIS CLASS OF ADJUNCTIONS IS SURPRISINGLY HANDY

5) PROVE THAT IN ORDER TO SHOW THAT A PAIR OF 1-CELLS
$$\begin{tikzcd} A \arrow[r, bend right, "u"'] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$
ARE ADJOINT IT SUFFICES TO GIVE 2-CELLS $\eta : id_B \Rightarrow uf$ & $\varepsilon : fu \Rightarrow id_A$ SUCH THAT THE TRIANGLE COMPOSITES

$$f \xRightarrow{f\eta} fuf \xRightarrow{\varepsilon f} f \quad \text{AND}$$
$$u \xRightarrow{\eta u} ufu \xRightarrow{u\varepsilon} u$$

ARE BOTH ISOMORPHISMS.

6) PROVE THE FOLLOWING RESULT STATED IN PART II:

RIGHT ADJOINTS PRESERVE LIMITS OF FAMILIES

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

GIVEN AN ADJUNCTION

IF
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ I \arrow[ur, bend left, "l"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A^X \end{tikzcd}$$
IS AN ABSOLUTE RIGHT LIFTING

$$\Longrightarrow$$

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\ I \arrow[ur, bend left, "l"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$
IS ALSO AN ABSOLUTE RIGHT LIFTING.

7) USE THE HOM-WISE CHARACTERISATION OF ADJUNCTIONS AND THE WELL KNOWN EQUIVALENCE BETWEEN THE THIN AND FAT SLICES OF A QUASI-CATEGORY TO COMPLETE THE EXERCISE ON SLIDE 25, VIZ:

SUPPOSE THAT A IS A QUASI-CATEGORY; AN OBJECT IN THE ∞-COSMOS $\underline{\mathcal{QC}at}$. LET t BE A VERTEX OF A, THEN

$t : 1 \longrightarrow A$ IS A TERMINAL ELEMENT

$$\Updownarrow$$

THIS IS JOYAL'S TERMINAL OBJECT NOTION

$$\begin{tikzcd}
1 \arrow[dr, "\langle n \rangle"'] \arrow[rr, bend left, "t"] & & A \\
& \partial\Delta[n] \arrow[r, "\partial a"] \arrow[d, hook] & A \\
& \Delta[n] \arrow[ur, dotted, "\exists a"'] &
\end{tikzcd}$$

A ADMITS FILLERS FOR SIMPLEX BOUNDARIES WITH LAST VERTEX t

7) USE THE RESULT CITED ON SLIDE 36, 1/17:

KEY OBSERVATION IT IS A CONSEQUENCE OF THE WEAK 2-UNIVERSAL PROPERTY OF $\mathrm{Hom}_C(f,g)$ THAT THERE EXISTS A BIJECTION

$$
\begin{tikzcd}
 & X \arrow[dl, "b"'] \arrow[dr, "a"] \arrow[dd, phantom, "\Leftarrow\tau"] & \\
B \arrow[dr, "g"'] & & A \arrow[dl, "f"] \\
 & C &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \arrow[rr, "\bar{\tau}"] \arrow[dr, "{(b,a)}"'] & & \mathrm{Hom}_C(f,g) \arrow[dl, "{(q_1,q_0)}"] \\
 & B \times A &
\end{tikzcd}
$$

BIJECTION

2-CELLS

ISO-CLASSES OF 1-CELLS OVER $B\times A$

TO PROVE THE HOM-WISE CHARACTER OF ABSOLUTE RIGHT LIFTINGS:

$$
\begin{tikzcd}
 & A \arrow[d, "f"] \\
B \arrow[ur, bend left, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\lambda"] & C
\end{tikzcd}
$$

LEMMA THE TRIANGLE ON THE LEFT IS AN ABSOLUTE RIGHT LIFTING IFF THE INDUCED FUNCTOR $\mathrm{Hom}_A(A,\ell) \xrightarrow{\omega} \mathrm{Hom}_C(f,g)$ IS AN EQUIVALENCE

$$
\begin{tikzcd}
 & \mathrm{Hom}_A(A,\ell) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\Leftarrow\phi"] & \\
B \arrow[rr, "\ell"] \arrow[dr, "g"'] & \arrow[d, phantom, "\Leftarrow\lambda"] & A \arrow[dl, "f"] \\
 & C &
\end{tikzcd}
\quad \rightsquigarrow \; = \quad
\begin{tikzcd}
 & \mathrm{Hom}_A(A,\ell) \arrow[d, dashed, "\omega"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
 & \mathrm{Hom}_C(f,g) \arrow[dl, "q_1"'] \arrow[dr, "q_0"] \arrow[dd, phantom, "\Leftarrow\psi"] & \\
B \arrow[dr, "g"'] & & A \arrow[dl, "f"] \\
 & C &
\end{tikzcd}
$$