4-DIMENSIONAL TQFTS FROM BRAIDED MONOIDAL CATEGORIES

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Abstract. These are notes from lectures given at the introductory workshop on higher categories and categorification at MSRI in Feburary 2020.

Introduction

These lectures will have the following two goals:

- Introduce a class of 4-dimensional topological field theories obtained from braided monoidal categories.
- Explain in practical terms how to work with the cobordism hypothesis in low dimensions.

We will concentrate on a 4-dimensional TQFT initially defined by Crane and Yetter [CY93]. One motivation for its study is its relationship to 4-dimensional quantum gravity: quantum gravity may be viewed as a Crane–Yetter (or BF) theory with constraints. A realization of such ideas gave rise to the Barrett–Crane [BC98] and EPRL [Eng+08] models. Here the input braided monoidal category is the category of representations of the quantum group $\mathrm{Rep}_q(\mathrm{SO}(1,3))$ (where q is related to the cosmological constant). Another motivation for its study is that the Crane–Yetter theory for the category $\mathrm{Rep}_q(G)$ of representations of the quantum group is a useful organization tool for the geometric Langlands program [KW07; BN18]. A final motivation we will mention is that when the input is a modular tensor category, we obtain an invertible TQFT which may be viewed as an anomaly theory for the Reshetikhin–Turaev TQFT [Wal91].

1. Top-down approach

One way to construct a TQFT is to define invariants of closed 4-manifolds, state spaces for closed 3-manifolds, relative invariants for compact 4-manifolds with boundary and so on. We begin with this approach.

1.1. **Partition function.** Let k be a field and $\mathcal{C}$ a k-linear modular tensor category. Recall that this means the following:

- $\mathcal{C}$ is a braided monoidal category.
- The Hom-spaces are finite-dimensional vector spaces.
- $\mathcal{C}$ is rigid as a monoidal category, i.e. every object has a dual.
- $\mathcal{C}$ has a balancing (also known as a twist) $\theta_x\colon x \to x$.
- $\mathcal{C}$ is a semisimple category with finitely many simple objects. Our convention is that the unit is simple and any simple object has only scalar endomorphisms (automatic if k is algebraically closed).

- Any object $x \in \mathcal{C}$ such that $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{id}_{x \otimes y}$ for every $y \in \mathcal{C}$ has to be a sum of the unit object.

Let $\{x_i\}$ be the collection of simple objects of $\mathcal{C}$.

We are going to define the partition function of the 4d TQFT on a closed oriented 4-manifold as a state sum. Namely, for a triangulated 4-manifold M it is a sum over labelings of simplices of the triangulation of certain weights computed from the labeling. We consider the following labelings:

- We label a 2-simplex (a triangle) by a simple object x_i.
- A 3-simplex (a tetrahedron) has 4 faces labeled by simple objects x_a, x_b, x_c, x_d. We then label the 3-simplex by a basis vector of $\mathrm{Hom}(1, x_a \otimes x_b \otimes x_c \otimes x_d)$.

Given a labeled triangulation, to each 4-simplex we can associate a weight $W \in k$ (known as a $15j$ symbol) computed using the ribbon structure on $\mathcal{C}$ (see [BB18, Section 5.1]). Then the partition function is given by

$$Z(M) = \sum_{\text{labelings of } M} W. \tag{1}$$

To see that $Z(M)$ is well-defined, on has to show independence of the above expression from the choice of a triangulation. This can be verified by checking the invariance of $Z(M)$ under Pachner moves [Pac91].

If $\mathcal{C}$ is a modular tensor category, we denote by

$$\dim(\mathcal{C}) = \sum_i \dim(x_i)^2$$

its ***global dimension*** and by

$$p^{\pm} = \sum_i \theta_i^{\pm 1} \dim(x_i)$$

its ***Gauss sums***, where θ_i is the value of the balancing on x_i.

The following is shown in [CKY93; CKY97].

Proposition 1.1 (Crane–Yetter–Kauffman)**.** *Let M be a closed oriented manifold and $\mathcal{C}$ a modular tensor category. Then*

$$Z(M) = \dim(\mathcal{C})^{\chi(M)/2}(p_+/p_-)^{\sigma(M)/2}.$$

Remark 1.2. More generally, it is shown in [Sch17] that a fully extended invertible 4d TFT Z is determined by two complex numbers λ_1, λ_2, so that the value on a closed oriented 4-manifold M is

$$Z(M) = \lambda_1^{\chi(M)} \lambda_2^{3\sigma(M)}.$$

So, we see that the above state sum is a bit boring: it is simply a local way to compute the signature and the Euler characteristic. We will see that weakening assumptions on $\mathcal{C}$ in lower dimensions will give rise to a more interesting TFT.

1.2. **State spaces.** For a state sum model there is a standard procedure to define state spaces for closed oriented 3-manifolds N, so that if M is a 4-manifold with boundary ∂M, the partition function $Z(M)$ is a functional $Z(M)\colon Z(\partial M) \to \mathbf{C}$.

Namely, suppose N is a triangulated 3-manifold. We define the vector space

$$H(N) = \bigoplus_{\text{labelings of } N} \mathbf{C}.$$

Given a compatible triangulation of $N \times [0,1]$ we may define a linear map

$$Z(N \times [0,1])\colon H(N) \to H(N)$$

by generalizing the state sum (1) to include the boundary. We then define the state space

$$Z(N) = \mathrm{im}(Z(N \times [0,1])\colon H(N) \to H(N)).$$

Remark 1.3. A related definition of the state space $Z(N)$ was given by Walker and Wang [WW11] who use a dual graph to the triangulation.

If M is a triangulated 4-manifold with boundary, we have a functional

$$Z(M)\colon Z(\partial M) \longrightarrow \mathbf{C}$$

defined as before, but where we sum over the labelings of the interior of M.

There is an alternative presentation of the state space $Z(N)$ due to Walker [Wal06] which we only sketch. Here we assume that $\mathcal{C}$ is merely a ribbon category which is not required to be modular or even semisimple. The informal definition of the skein module is given as follows.

Definition 1.4. Let $\mathcal{C}$ be a ribbon category and N an oriented 3-manifold. The $\mathcal{C}$-***skein module*** $\mathrm{Sk}_{\mathcal{C}}(N)$ is the k-vector space spanned by $\mathcal{C}$-colored ribbon graphs modulo isotopy and local relations coming from $\mathcal{C}$.

We refer to [Tur94, Section I.2] for the precise definition of a $\mathcal{C}$-colored ribbon graph. The rough idea is that it is an oriented graph (with edges being bands) embedded in N with edges labeled by objects of $\mathcal{C}$ and vertices labeled by morphisms in $\mathcal{C}$. Such graphs can be interpreted in terms of the graphical calculus for ribbon categories and we impose relations coming from such an interpretation.

Expectation 1.5. *Let $\mathcal{C}$ be a ribbon fusion category. Then $Z(N) \cong \mathrm{Sk}_{\mathcal{C}}(N)$.*

Remark 1.6. There is an analogous state-sum TQFT in 3 dimensions due to Turaev and Viro [TV92] whose input is a spherical fusion category. There is a "state sum" definition of the state space $Z_{\mathrm{TV}}(\Sigma)$ for an oriented surface Σ. There is also a definition of the space of string nets $H^{\mathrm{string}}(\Sigma)$ due to Levin and Wen [LW05] which is analogous to the above definition of the skein module $\mathrm{Sk}_{\mathcal{C}}$. It is then a theorem of Kirillov [Kir11] that there is a natural isomorphism $H^{\mathrm{string}}(\Sigma) \cong Z_{\mathrm{TV}}(\Sigma)$. See also [Goo18] where this statement was extended to an equivalence of 1-2-3 TQFTs.

Here are some computations of skein modules. Let us recall that if $\mathcal{C}$ is a k-linear category, its Hochschild homology is given by the coend

$$\mathrm{HH}_0(\mathcal{C}) = \int^{x \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x, x).$$

Proposition 1.7. *Let $\mathcal{C}$ be a ribbon category. Then:*

- $\mathrm{Sk}_{\mathcal{C}}(B^3) \cong \mathrm{Sk}_{\mathcal{C}}(S^3) \cong k$.
- $\mathrm{Sk}_{\mathcal{C}}(S^2 \times S^1) \cong \mathrm{HH}_0(\mathrm{Z}_{Müg}(\mathcal{C}))$, *where* $\mathrm{Z}_{Müg}(\mathcal{C})$ *is the* **Müger center***: the full subcategory of* $\mathcal{C}$ *of objects* $x \in \mathcal{C}$ *such that* $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{id}_{x\otimes y}$ *for every* $y \in \mathcal{C}$.

1.3. **Extension to lower-dimensional manifolds.** We may also extend the TQFT to lower-dimensional manifolds. Let us sketch how such an extension works in the "skein" approach:

- Let Σ be an oriented surface and $\mathcal{C}$ a balanced monoidal category. The ***skein category*** $\mathrm{SkCat}_{\mathcal{C}}(\Sigma)$ is the category whose objects are collections of points on Σ labeled by objects of $\mathcal{C}$ and morphisms are given by the relative $\mathcal{C}$-skein module of $\Sigma \times [0,1]$ where the ribbon graphs end on the given points in $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. We refer to [Coo19, Section 1] for more details.
- Let S be an oriented 1-manifold. The ***skein 2-category*** $\mathrm{Sk2Cat}_{\mathcal{C}}(C)$ is the 2-category with a single object, endomorphisms of which are given by the monoidal category $\mathrm{SkCat}_{\mathcal{C}}(S \times (0,1))$.

1.4. **Relationship to the Chern–Simons theory.** Let $\mathcal{C}$ be a modular tensor category. Reshetikhin and Turaev [RT91; Bar+15] associate to it a 3-2-1 TQFT Z_{RT}, so that

$$Z_{\mathrm{RT}}(S^1) = \mathcal{C}.$$

Remark 1.8. If we take $\mathcal{C}$ to be the semisimplification of the category of tilting modules over the quantum group $\mathrm{U}_q(\mathfrak{g})$ where q is a root of unity, the corresponding Reshetikhin–Turaev theory is expected to be a formalization of the topological Chern–Simons theory [Wit89].

However, Z_{RT} is not an oriented TQFT. Instead, it requires a choice of an additional structure on the 3-manifold to define the partition function. This problem is known as a *framing anomaly*.

Definition 1.9. Let M be an oriented 3-manifold. A p_1***-structure*** is a homotopy class of the trivialization of the composite

$$M \xrightarrow{\mathrm{T}_M} \mathrm{BSO}(3) \xrightarrow{p_1} K(\mathbf{Z}, 4).$$

Let $\mathrm{Triv}_{p_1}(M)$ be the set of p_1-structures on M.

If M is an oriented 3-manifold, we may consider the $\mathrm{SO}(6)$-bundle $\mathrm{T}_M \oplus \mathrm{T}_M$. It has a canonical spin structure, so it gives rise to a $\mathrm{Spin}(6)$-bundle which we denote by $2\mathrm{T}_M$. The following notion is introduced in [Ati90].

Definition 1.10. Let M be an oriented 3-manifold. A ***2-framing*** is a homotopy class of the trivialization of

$$M \xrightarrow{2\mathrm{T}_M} \mathrm{BSpin}(6).$$

Let $\mathrm{Triv}_{At}(M)$ be the set of 2-framings on M.

Theorem 1.11 (Atiyah)**.** $\mathrm{Triv}_{p_1}(M)$ *and* $\mathrm{Triv}_{At}(M)$ *are naturally isomorphic* $\mathbf{Z}$*-torsors.*

Here is another possible extra structure one may consider.

Definition 1.12. Let M be a closed oriented 3-manifold. A ***componentwise signature structure*** is a choice of a bounding oriented 4-manifold for each connected component of M considered up to an oriented 5-cobordism. Let $\mathrm{Triv}_{csgn}(M)$ be the set of such.

Proposition 1.13. *There is an isomorphism* $3\mathrm{Triv}_{csgn}(M) \cong \mathrm{Triv}_{p_1}(M)$ *of* $\mathbf{Z}$*-torsors.*

Given a $\mathbf{Z}$-torsor P and a complex number $\lambda \in \mathbf{C}^\times$, there is a complex line λ^P constructed as

$$\lambda^P = \mathbf{C}[P]/\sim,$$

where we identify $p \sim \lambda^{p-q}q$ for every $p, q \in P$.

Proposition 1.14. *Let* $\mathcal{C}$ *be a modular tensor category and* M *a closed oriented 3-manifold. Then we have an isomorphism of lines*

$$Z(M) \cong \left(\sqrt{p_+/p_-}\right)^{\mathrm{Triv}_{csgn}(M)}.$$

Let M be a connected closed oriented 3-manifold. We have a canonical vector $1 \in \mathrm{Sk}_{\mathcal{C}}(M)$ given by embedding the unknot labeled by the unit into M. If W is an oriented 4-manifold with boundary M, then it gives rise to a functional

$$Z(W)\colon Z(M) \to \mathbf{C}.$$

We then obtain the Reshetikhin–Turaev invariant of M as

$$Z_{\mathrm{RT}}(M) = Z(W)(1).$$

We refer to [Lic93; Bla+95; Wal91] for further details.

2. Bottom-up approach

2.1. **Crane–Yetter as a fully extended TQFT.** Our goal will be to explain a fully local (i.e. fully extended) approach to constructing the Crane–Yetter TQFT. From section 1.3 we expect the following kind of assignment:

- To the point we assign a braided monoidal category.
- To the circle we assign a monoidal category.
- To a surface we assign a category.
- ...

We may organize braided monoidal categories into a Morita 4-category BrTens (see [Hau17; Sch14a; JS17; BJS18]) as follows:

- Its objects are braided monoidal categories $\mathcal{A}$.
- 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by monoidal categories $\mathcal{C}$ together with a braided monoidal functor $\mathcal{A} \otimes \mathcal{B}^{\sigma\mathrm{op}} \to \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ into the Drinfeld center. In this case we will say that $\mathcal{C}$ is an $\mathcal{A} \otimes \mathcal{B}^{\sigma\mathrm{op}}$***-monoidal category***.
- 2-morphisms from ${}_{\mathcal{A}}\mathcal{C}_{\mathcal{B}}$ to ${}_{\mathcal{A}}\mathcal{D}_{\mathcal{B}}$ are given by $(\mathcal{C}, \mathcal{D})$-bimodule categories compatible with the action of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{C}$ and $\mathcal{D}$.
- 3-morphisms are given by functors of bimodule categories.
- 4-morphisms are given by natural transformations.

Remark 2.1. The above definition makes sense for braided monoidal objects of an arbitrary symmetric monoidal 2-category. We may then further specify that 2-category to be the 2-category of locally presentable k-linear categories with k-linear colimit-preserving functors as morphisms in which case we recover the definition of [BJS18]. In these notes the background 2-category will not be made explicit.

We will also need the (∞, n)-category $\mathrm{Bord}_n^{\mathrm{or}}$ of n-dimensional bordisms whose objects are compact oriented 0-manifolds, 1-morphisms are oriented 1-cobordisms and so on up to dimension n, $(n+1)$-morphisms are diffeomorphisms of n-dimensional cobordisms, $(n+2)$-morphisms are isotopies and so on. We refer to [Lur09; CS19] for more details.

We expect the Crane–Yetter theory to define a functor

$$Z\colon \mathrm{Bord}_4^{\mathrm{or}} \to \mathrm{BrTens}.$$

Constructing such a functor directly would amount to providing an infinite collection of data. We will, however, use the cobordism hypothesis [BD95; Lur09] to give a hands-on construction of such a functor.

2.2. **Cobordism hypothesis.**

Definition 2.2. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. We say ***it has adjoints*** if every object is dualizable and every k-morphism has left and right adjoints for every $0 < k < n$.

Definition 2.3. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. An object $x \in \mathcal{C}$ is ***k-dualizable*** if there is a symmetric monoidal (∞, k)-subcategory $\mathcal{C}' \subset \mathcal{C}$ such that $x \in \mathcal{C}'$ and $\mathcal{C}'$ has adjoints. For $k = n$ we say $x \in \mathcal{C}$ is ***fully dualizable***.

Recall that a k-manifold M has an n-framing (here $k \leq n$) if the n-dimensional vector bundle $\mathrm{T}_M \oplus \underline{\mathbf{R}}^{n-k}$ is equipped with a trivialization. Let Bord_n^{fr} be the symmetric monoidal (∞, n)-category of bordisms equipped with an n-framing. If $\mathcal{C}, \mathcal{D}$ are symmetric monoidal (∞, n)-categories, we denote by $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ the (∞, n)-category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

We have the following important result [BD95; Lur09].

Theorem 2.4 (Cobordism hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. The functor*

$$\begin{aligned} \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) &\longrightarrow \mathcal{C} \\ Z &\mapsto Z(\mathrm{pt}) \end{aligned}$$

establishes an equivalence of (∞, n)-categories

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \cong (\mathcal{C}^{\mathrm{fd}})^{\sim},$$

where $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ is the ∞-groupoid of fully dualizable objects in $\mathcal{C}$.

Let us try to unpack what a 2-dualizable object $x \in \mathcal{C}$ is:

- $x \in \mathcal{C}$ is a dualizable object. That is, it admits a dual $x^\vee \in \mathcal{C}$, a coevaluation morphism $\mathrm{coev}\colon \mathbf{1} \to x \otimes x^\vee$ and an evaluation morphism $x^\vee \otimes x \to \mathbf{1}$ such that the composites

$$x \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} x$$

and

$$x^\vee \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} x^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} x^\vee$$

are equivalent to the identities.

- The coevaluation morphism coev has an infinite chain of right adjoints $\mathrm{coev}^{\mathrm{R}}, \mathrm{coev}^{\mathrm{RR}}, \dots$ and left adjoints $\mathrm{coev}^{\mathrm{L}}, \mathrm{coev}^{\mathrm{LL}}, \dots$ and similarly for the evaluation morphism.

So, the cobordism hypothesis (theorem 2.4) is not entirely satisfying from a computational perspective since already in dimension 2 we need to check an infinite amount of conditions. However, the evaluation map $\mathrm{ev}\colon x^\vee \otimes x \to 1$ is dual to the coevaluation map $\mathrm{coev}\colon 1 \to x \otimes x^\vee$. So, coev is right-adjointable iff ev is left-adjointable. This observation can be strengthened to the following statement (see [Lur09, Proposition 4.2.3] and [Pst14, Theorem 3.9]).

Theorem 2.5. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. An object $x \in \mathcal{C}$ is 2-dualizable iff the following conditions are satisfied:*

1. *It is dualizable.*
2. *The evaluation and coevaluation morphisms* $\mathrm{ev}, \mathrm{coev}$ *admit right adjoints* $\mathrm{ev}^{\mathrm{R}}, \mathrm{coev}^{\mathrm{R}}$.

We will also need a similar statement for 3-dualizable objects (see [Ara17]).

Theorem 2.6. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 3)$-category. An object $x \in \mathcal{C}$ is 3-dualizable iff the following conditions are satisfied:*

1. *It is dualizable.*
2. *The evaluation and coevaluation morphisms* $\mathrm{ev}, \mathrm{coev}$ *admit right adjoints* $\mathrm{ev}^{\mathrm{R}}, \mathrm{coev}^{\mathrm{R}}$.
3. *The unit and counit 2-morphisms* $\eta_{\mathrm{ev}}, \epsilon_{\mathrm{ev}}, \eta_{\mathrm{coev}}, \epsilon_{\mathrm{coev}}$ *admit right adjoints.*

2.3. **Oriented cobordism hypothesis.** There is an $\mathrm{SO}(n)$-action on Bord_n^{fr} given by acting on the n-framing $\mathrm{T}_M \oplus \underline{\mathbf{R}}^{n-k} \cong \underline{\mathbf{R}}^n$. This induces an action on $\mathrm{Fun}^\otimes(\mathrm{Bord}_n^{fr}, \mathcal{C})$ and hence, by the cobordism hypothesis (theorem 2.4), on the ∞-groupoid of fully dualizable objects.

Theorem 2.7 (Oriented cobordism hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. There is an equivalence of (∞, n)-categories*

$$\mathrm{Fun}^\otimes(\mathrm{Bord}_n^{\mathrm{or}}, \mathcal{C}) \cong ((\mathcal{C}^{\mathrm{fd}})^\sim)^{\mathrm{SO}(n)},$$

where $((\mathcal{C}^{\mathrm{fd}})^\sim)^{\mathrm{SO}(n)}$ is the ∞-groupoid of homotopy $\mathrm{SO}(n)$-fixed points on $(\mathcal{C}^{\mathrm{fd}})^\sim$.

The data of an $\mathrm{SO}(2)$-action on the ∞-groupoid $(\mathcal{C}^{\mathrm{fd}})^\sim$ consists of a map

$$\mathbf{CP}^\infty \cong \mathrm{BSO}(2) \to (\mathcal{C}^{\mathrm{fd}})^\sim.$$

In particular, the induced map on π_2 sends the generator of $\pi_2(\mathrm{BSO}(2)) \cong \mathbf{Z}$ to a natural automorphism $S_x\colon x \to x$ for every object $x \in \mathcal{C}^{\mathrm{fd}}$ which is known as the ***Serre automorphism***.

Proposition 2.8. *Suppose $x \in \mathcal{C}$ is a 2-dualizable object. The composite*

$$x \xrightarrow{\mathrm{ev}^{\mathrm{R}}\otimes\mathrm{id}} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} x$$

coincides with the Serre automorphism $S_x\colon x\to x$*. The composite*

$$x \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} x\otimes x^\vee\otimes x \xrightarrow{\mathrm{id}\otimes\mathrm{coev}^{\mathrm{R}}} x$$

is the inverse of the Serre automorphism.

In a similar way, the data of an SO(3)-action gives rise to the Serre automorphism as before. However, since $\pi_2(\mathrm{BSO}(3))\cong\mathbf{Z}/2$, the square of the Serre automorphism has a trivialization known as the Radford isomorphism. We refer to [Sch14b] for further details.

2.4. **2-dualizability.** Our next goal is to establish when a braided monoidal category $\mathcal{C}\in\mathrm{BrTens}$ is 2-dualizable, 3-dualizable and so on, so that it defines a functor $\mathrm{Bord}^{\mathrm{fr}}_n\to\mathrm{BrTens}$.

We begin with the following statement (see [Lur09; Sch14a]).

Theorem 2.9. *Any object* $\mathcal{C}\in\mathrm{BrTens}$ *is 2-dualizable.*

In fact, the corresponding functor $\mathrm{Bord}^{\mathrm{fr}}_2\to\mathrm{BrTens}$ may be constructed using the theory of factorization homology [AF15], i.e. the value of the TFT on a surface S is $\int_S\mathcal{C}$. So, we have constructed a 2-1-0 part of a 4d TFT.

Remark 2.10. Factorization homology is defined for all surfaces S, not necessarily compact. Moreover, it satisfies a useful gluing axiom (excision) which allows one to compute it using a cover of S. We refer to [BBJ18a; BBJ18b] for many computations of the value of the Crane–Yetter TFT on surfaces.

We will now write out concretely the dualizability data following [GS18; BJS18]. Given a braided monoidal category $\mathcal{C}$ we denote by $\mathcal{C}^{\sigma\mathrm{op}}$ the same monoidal category equipped with the opposite braiding, i.e. with the braiding

$$\sigma^{-1}_{y,x}\colon x\otimes y\to y\otimes x.$$

Lemma 2.11. *Suppose* $\mathcal{C}$ *is a monoidal category. Then there is a braided monoidal equivalence*

$$\mathrm{Z}(\mathcal{C}^{\otimes\mathrm{op}})\cong\mathrm{Z}(\mathcal{C})^{\sigma\mathrm{op}}. \tag{2}$$

Moreover, there is a natural braided monoidal functor

$$\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}}\longrightarrow\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}). \tag{3}$$

If we denote objects of $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ by pairs $(x,\alpha_x\colon x\otimes(-)\xrightarrow{\sim}(-)\otimes x)$, then it sends

$$x\boxtimes\mathbf{1}\mapsto(x,\sigma_{x,-}),\qquad \mathbf{1}\boxtimes x\mapsto(x,\sigma^{-1}_{-,x}).$$

Theorem 2.12. *Every object* $\mathcal{C}\in\mathrm{BrTens}$ *is dualizable. The dual is given by* $\mathcal{C}^{\sigma\mathrm{op}}$*. The evaluation and coevaluation are given by* $\mathcal{C}$ *viewed as a* $\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}}$*-monoidal category via* $\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}}\to\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$.

Theorem 2.13. *Every 1-morphism* $\mathcal{A}\xrightarrow{\mathcal{C}}\mathcal{B}$ *in* BrTens*, i.e. an* $\mathcal{A}\otimes\mathcal{B}^{\sigma\mathrm{op}}$*-monoidal category* $\mathcal{C}$*, has a right adjoint given by* $\mathcal{C}^{\otimes\mathrm{op}}$*, the same category with the opposite monoidal structure and equipped with a braided monoidal functor* $\mathcal{B}\otimes\mathcal{A}^{\sigma\mathrm{op}}\to\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})^{\sigma\mathrm{op}}\cong\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}^{\otimes\mathrm{op}})$*. The unit* $\eta\colon\mathcal{A}\to\mathcal{C}\otimes_{\mathcal{B}}\mathcal{C}^{\otimes\mathrm{op}}$ *is given by* $\mathcal{C}$ *as an* $(\mathcal{A},\mathcal{C}\otimes_{\mathcal{B}}\mathcal{C}^{\otimes\mathrm{op}})$*-bimodule category. The counit* $\epsilon\colon\mathcal{C}^{\otimes\mathrm{op}}\otimes_{\mathcal{A}}\mathcal{C}\to\mathcal{B}$ *is given by* $\mathcal{C}$ *as a* $(\mathcal{C}^{\otimes\mathrm{op}}\otimes_{\mathcal{A}}\mathcal{C},\mathcal{B})$*-bimodule category.*

Let $\mathcal{C} \in \mathrm{BrTens}$ be a braided monoidal category. From the above description of adjoints we may compute the Serre automorphism as follows. By theorem 2.12 the evaluation 1-morphism is $\mathcal{C}$ viewed as a $\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}}$-monoidal category via (3). By theorem 2.13 its right adjoint is $\mathcal{C}^{\otimes\mathrm{op}}$ viewed as a $\mathcal{C}^{\sigma\mathrm{op}} \otimes \mathcal{C}$-monoidal category via

$$\mathcal{C}^{\sigma\mathrm{op}} \otimes \mathcal{C} \xrightarrow{(3)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})^{\sigma\mathrm{op}} \xrightarrow{(2)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}^{\otimes\mathrm{op}}).$$

There are two monoidal equivalences

$$F_1, F_2 \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\otimes\mathrm{op}}:$$

both have the identity underlying functor and the monoidal structure given by either the braiding or its inverse. Therefore, the right adjoint to evaluation is given by the composite

$$\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}} \xrightarrow{\mathrm{flip}} \mathcal{C}^{\sigma\mathrm{op}} \otimes \mathcal{C} \xrightarrow{(3)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})^{\sigma\mathrm{op}} \xrightarrow{(2)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}^{\otimes\mathrm{op}}) \xrightarrow{F_i} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}).$$

The underlying functor coincides with the underlying functor for coev, but the monoidal structure turns out to be twisted. We denote by

$$F_2^{-1}F_1 \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

the identity functor equipped with the monoidal structure given by $\sigma_{y,x} \circ \sigma_{x,y}$. It is not difficult to see that it is braided monoidal.

Proposition 2.14. *For a braided monoidal category $\mathcal{C}$ the Serre automorphism $S_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the $\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}}$-monoidal category $\mathcal{C}$ via*

$$\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}} \xrightarrow{F_2^{-1}F_1 \otimes \mathrm{id}} \mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}} \xrightarrow{(3)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$$

or, equivalently, via

$$\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}} \xrightarrow{\mathrm{id} \otimes F_1^{-1}F_2} \mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}} \xrightarrow{(3)} \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}).$$

2.5. **3-dualizability.** In the previous section we have seen that every braided monoidal category $\mathcal{C} \in \mathrm{BrTens}$ is 2-dualizable. We will now analyze the conditions for $\mathcal{C}$ to be 3-dualizable.

It will be convenient to introduce the following notation.

Definition 2.15. Let $\mathcal{C}$ be a braided monoidal category. The monoidal category $\mathrm{HC}(\mathcal{C})$ is

$$\mathrm{HC}(\mathcal{C}) = \mathcal{C}^{\otimes\mathrm{op}} \otimes_{\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}}} \mathcal{C}.$$

Remark 2.16. Since $\mathrm{HC}(\mathcal{C})$ is defined as a relative tensor product, it is not easy to write objects of $\mathrm{HC}(\mathcal{C})$ (or functors *into* $\mathrm{HC}(\mathcal{C})$), but it is easy to write functors out of $\mathrm{HC}(\mathcal{C})$. For instance, $\mathrm{HC}(\mathcal{C})$-module categories are the same as $\mathcal{C}$-braided module categories, see [BBJ18b, Theorem 3.11]. If $\mathcal{C}$ is cp-rigid (see [BJS18, Definition 4.1] for what this means), $\mathrm{HC}(\mathcal{C})$ is equivalent to the Drinfeld center $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ as a plain category. However, the two monoidal structures are different (for instance, the monoidal structure on $\mathrm{HC}(\mathcal{C})$ is not braided unless $\mathcal{C}$ is symmetric).

From theorem 2.12 we have the following coevaluation and evaluation morphisms:

- $\mathrm{coev} \colon \mathbf{1} \to \mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}}$ is $\mathcal{C}$ as a $\mathcal{C} \otimes \mathcal{C}^{\sigma\mathrm{op}}$-monoidal category.

- $\mathrm{ev}\colon \mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}}\to\mathbf{1}$ is $\mathcal{C}$ as a $\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}}$-monoidal category.

From theorem 2.13 the above maps are right-adjointable with the following unit and counit morphisms:

- η_{coev} is $\mathcal{C}$ as a $(\mathbf{1},\mathrm{HC}(\mathcal{C}))$-bimodule category.
- ϵ_{coev} is $\mathcal{C}$ as a $(\mathcal{C}^{\otimes\mathrm{op}}\otimes\mathcal{C},\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}})$-bimodule category.
- η_{ev} is $\mathcal{C}$ as a $(\mathcal{C}\otimes\mathcal{C}^{\sigma\mathrm{op}},\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}})$-bimodule category.
- ϵ_{ev} is $\mathcal{C}$ as an $(\mathrm{HC}(\mathcal{C}),\mathbf{1})$-bimodule category.

To analyze their right-adjointability, it will be convenient to use the following statement (see [BJS18, Proposition 5.17] and [Lur17, Proposition 4.6.2.13]).

Proposition 2.17. *Let $\mathcal{A},\mathcal{B}\in\mathrm{BrTens}$ be braided monoidal category, $\mathcal{C},\mathcal{D}\colon\mathcal{A}\to\mathcal{B}$ be $\mathcal{A}\otimes\mathcal{B}^{\sigma\mathrm{op}}$-monoidal categories and $\mathcal{M}\colon\mathcal{C}\Rightarrow\mathcal{D}$ a $(\mathcal{C},\mathcal{D})$-bimodule category. Then $\mathcal{M}$ is right-adjointable iff $\mathcal{M}$ is dualizable as a $\mathcal{D}$-module category.*

With the help of the above statement and theorem 2.6 we finally arrive at the following characterization of 3-dualizable objects.

Theorem 2.18. *A braided monoidal category $\mathcal{C}\in\mathrm{BrTens}$ is 3-dualizable iff the following conditions are satisfied:*

1. *$\mathcal{C}$ is dualizable as a plain category.*
2. *$\mathcal{C}$ is dualizable as a $\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$-module category.*
3. *$\mathcal{C}$ is dualizable as an $\mathrm{HC}(\mathcal{C})$-module category.*

The above characterization can be made a bit more explicit by introducing the notion of semi-rigidity (see [Gai15, Appendix D], [BN09, Section 3]).

Definition 2.19. A monoidal category $\mathcal{C}$ is ***semi-rigid*** if it is dualizable as a plain category and the tensor functor $\mathcal{C}\otimes\mathcal{C}\to\mathcal{C}$ viewed as a morphism of $\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$-module categories admits a right adjoint.

Remark 2.20. Suppose $\mathcal{C}$ is a locally-presentable monoidal category which has enough compact projectives. Then $\mathcal{C}$ is semi-rigid iff all compact projective objects are dualizable [BJS18, Proposition 4.1].

The following is shown in [BJS18].

Theorem 2.21. *Let $\mathcal{C}\in\mathrm{BrTens}$ be a semi-rigid braided monoidal category. Then it is 3-dualizable.*

The previous characterization allows one to construct many functors $\mathrm{Bord}_3^{\mathrm{fr}}\to\mathrm{BrTens}$, i.e. 3-dimensional TQFTs.

Example 2.22. Suppose $\mathcal{C}=\mathrm{Rep}_q(G)$ is the $\mathbf{C}[q,q^{-1}]$-linear category of representations of the Lusztig integral form of the quantum group. Choosing a root of q, we may endow $\mathcal{C}$ with a balanced monoidal structure. By [APW91] it has enough compact projectives. Moreover, every finite-dimensional representation is dualizable, so $\mathcal{C}$ is cp-rigid. Therefore, it defines a 3-2-1-0 part of a 4d TFT. It is expected to be a mathematical formalization of the Kapustin–Witten 4d TFT [KW07; BN18].

Example 2.23. Suppose $\mathcal{C}$ is a modular tensor category. Since it is semisimple, it clearly has enough compact projectives. Moreover, it is obviously cp-rigid. Therefore, it defines a 3-2-1-0 part of a 4d TFT. It is invertible and expected to coincide with the Crane–Yetter 4d TFT.

References

[AF15] D. Ayala and J. Francis. "Factorization homology of topological manifolds". *J. Topol.* 8.4 (2015), pp. 1045–1084. arXiv: 1206.5522 [math.AT].

[APW91] H. H. Andersen, P. Polo, and K. X. Wen. "Representations of quantum algebras". *Invent. Math.* 104.1 (1991), pp. 1–59.

[Ara17] M. Araújo. "Coherence for 3-dualizable objects". PhD thesis. University of Oxford, 2017.

[Ati90] M. Atiyah. "On framings of 3-manifolds". *Topology* 29.1 (1990), pp. 1–7.

[Bar+15] B. Bartlett, C. Douglas, C. Schommer-Pries, and J. Vicary. "Modular categories as representations of the 3-dimensional bordism 2-category" (Sept. 2015). arXiv: 1509.06811 [math.AT].

[BB18] M. Bärenz and J. Barrett. "Dichromatic state sum models for four-manifolds from pivotal functors". *Comm. Math. Phys.* 360.2 (2018), pp. 663–714. arXiv: 1601.03580 [math-ph].

[BBJ18a] D. Ben-Zvi, A. Brochier, and D. Jordan. "Integrating quantum groups over surfaces". *J. Topol.* 11.4 (2018), pp. 874–917. arXiv: 1501.04652 [math.QA].

[BBJ18b] D. Ben-Zvi, A. Brochier, and D. Jordan. "Quantum character varieties and braided module categories". *Selecta Math. (N.S.)* 24.5 (2018), pp. 4711–4748. arXiv: 1606.04769 [math.QA].

[BC98] J. W. Barrett and L. Crane. "Relativistic spin networks and quantum gravity". *J. Math. Phys.* 39.6 (1998), pp. 3296–3302. arXiv: gr-qc/9709028.

[BD95] J. C. Baez and J. Dolan. "Higher-dimensional algebra and topological quantum field theory". *J. Math. Phys.* 36.11 (1995), pp. 6073–6105. arXiv: q-alg/9503002.

[BJS18] A. Brochier, D. Jordan, and N. Snyder. "On dualizability of braided tensor categories" (2018). arXiv: 1804.07538 [math.QA].

[Bla+95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. "Topological quantum field theories derived from the Kauffman bracket". *Topology* 34.4 (1995), pp. 883–927.

[BN09] D. Ben-Zvi and D. Nadler. "The character theory of a complex group" (2009). arXiv: 0904.1247 [math.RT].

[BN18] D. Ben-Zvi and D. Nadler. "Betti geometric Langlands". *Algebraic geometry: Salt Lake City 2015.* Vol. 97. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, pp. 3–41. arXiv: 1606.08523 [math.RT].

[CKY93] L. Crane, L. H. Kauffman, and D. Yetter. "Evaluating the Crane-Yetter invariant". *Quantum topology.* Vol. 3. Ser. Knots Everything. World Sci. Publ., River Edge, NJ, 1993, pp. 131–138. arXiv: hep-th/9309063.

[CKY97] L. Crane, L. H. Kauffman, and D. N. Yetter. "State-sum invariants of 4-manifolds". *J. Knot Theory Ramifications* 6.2 (1997), pp. 177–234. arXiv: hep-th/9409167.

[Coo19] J. Cooke. "Excision of skein categories and factorisation homology" (2019). arXiv: 1910.02630 [math.QA].

[CS19] D. Calaque and C. Scheimbauer. "A note on the (∞, n)-category of cobordisms". *Algebr. Geom. Topol.* 19.2 (2019), pp. 533–655. arXiv: 1509.08906 [math.AT].

[CY93] L. Crane and D. Yetter. "A categorical construction of 4D topological quantum field theories". *Quantum topology.* Vol. 3. Ser. Knots Everything. World Sci. Publ., River Edge, NJ, 1993, pp. 120–130. arXiv: hep-th/9301062.

[Eng+08] J. Engle, E. Livine, R. Pereira, and C. Rovelli. "LQG vertex with finite Immirzi parameter". *Nuclear Phys. B* 799.1-2 (2008), pp. 136–149. arXiv: 0711.0146 [gr-qc].

[Gai15] D. Gaitsgory. "Sheaves of categories and the notion of 1-affineness". *Stacks and categories in geometry, topology, and algebra.* Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 127–225. arXiv: 1306.4304 [math.AG].

[Goo18] G. Goosen. "Oriented 123-TQFTs via string-nets and state sums". PhD thesis. Stellenbosch University, 2018.

[GS18] O. Gwilliam and C. Scheimbauer. "Duals and adjoints in higher Morita categories" (2018). arXiv: 1804.10924 [math.CT].

[Hau17] R. Haugseng. "The higher Morita category of $\mathbb{E}_n$-algebras". *Geom. Topol.* 21.3 (2017), pp. 1631–1730. arXiv: 1412.8459 [math.AT].

[JS17] T. Johnson-Freyd and C. Scheimbauer. "(Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories". *Adv. Math.* 307 (2017), pp. 147–223. arXiv: 1502.06526 [math.CT].

[Kir11] A. Kirillov. "String-net model of Turaev-Viro invariants" (2011). arXiv: 1106.6033 [math.AT].

[KW07] A. Kapustin and E. Witten. "Electric-magnetic duality and the geometric Langlands program". *Commun. Number Theory Phys.* 1.1 (2007), pp. 1–236. arXiv: hep-th/0604151.

[Lic93] W. B. R. Lickorish. "The skein method for three-manifold invariants". *J. Knot Theory Ramifications* 2.2 (1993), pp. 171–194.

[Lur09] J. Lurie. "On the classification of topological field theories". *Current developments in mathematics, 2008.* Int. Press, Somerville, MA, 2009, pp. 129–280. arXiv: 0905.0465 [math.CT].

[Lur17] J. Lurie. "Higher Algebra". 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[LW05] M. A. Levin and X.-G. Wen. "String net condensation: A Physical mechanism for topological phases". *Phys. Rev.* B71 (2005), p. 045110. arXiv: cond-mat/0404617.

[Pac91] U. Pachner. "P.L. homeomorphic manifolds are equivalent by elementary shellings". *European J. Combin.* 12.2 (1991), pp. 129–145.

[Pst14] P. Pstrągowski. "On dualizable objects in monoidal bicategories, framed surfaces and the Cobordism Hypothesis" (2014). arXiv: 1411.6691 [math.AT].

[RT91] N. Reshetikhin and V. G. Turaev. "Invariants of 3-manifolds via link polynomials and quantum groups". *Invent. Math.* 103.3 (1991), pp. 547–597.

[Sch14a] C. Scheimbauer. "Factorization homology as a fully extended topological field theory". PhD thesis. ETH Zurich, 2014.

[Sch14b] C. J. Schommer-Pries. "Dualizability in low-dimensional higher category theory". *Topology and field theories*. Vol. 613. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 111–176. arXiv: 1308.3574 [math.AT].

[Sch17] C. Schommer-Pries. "Invertible Topological Field Theories" (2017). arXiv: 1712.08029 [math.AT].

[Tur94] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*. Vol. 18. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1994, pp. x+588.

[TV92] V. G. Turaev and O. Y. Viro. "State sum invariants of 3-manifolds and quantum $6j$-symbols". *Topology* 31.4 (1992), pp. 865–902.

[Wal06] K. Walker. "TQFTs". 2006. URL: http://canyon23.net/math/tc.pdf.

[Wal91] K. Walker. "On Witten's 3-manifold invariants". 1991. URL: https://canyon23.net/math/1991TQFTNotes.pdf.

[Wit89] E. Witten. "Quantum field theory and the Jones polynomial". *Comm. Math. Phys.* 121.3 (1989), pp. 351–399.

[WW11] K. Walker and Z. Wang. "(3+1)-TQFTs and topological insulators" (2011). arXiv: 1104.2632 [cond-mat.str-el].

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