Synthetic

∞-Category Theory

Introductory Workshop:
Higher Categories & Categorification

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AIM: GIVE A PRECISE AND MODEL INDEPENDENT ACCOUNT OF ∞-CATEGORY THEORY.

RESOURCES

"∞-CATEGORY THEORY FROM SCRATCH"

https://journals.mq.edu.au/index.php/higher_structures/article/view/38

"ELEMENTS OF ∞-CATEGORY THEORY"

http://www.math.jhu.edu/~eriehl/elements.pdf

$(\infty,1)$-CATEGORIES

SCHEMATIC	MODELS
"DEFINITION"	QUASI-CATEGORIES
AN $(\infty,1)$-CATEGORY IS A CATEGORY WEAKLY ENRICHED IN SPACES	COMPLETE SEGAL SPACES
∞-GROUPOIDS	SEGAL CATEGORIES
	MARKED QUASI-CATEGORIES
	ALSO FIBRED VARIANTS OF THESE NOTIONS

∞-CATEGORIES

AUXILIARY AIM: BUILD A FRAMEWORK THAT CAN BE EXTENDED TO GIVE FOUNDATION'S FOR THE CATEGORY THEORY OF (∞, n)-CATEGORIES

MODELS:

COMPLICIAL SETS

ITERATED SEGAL

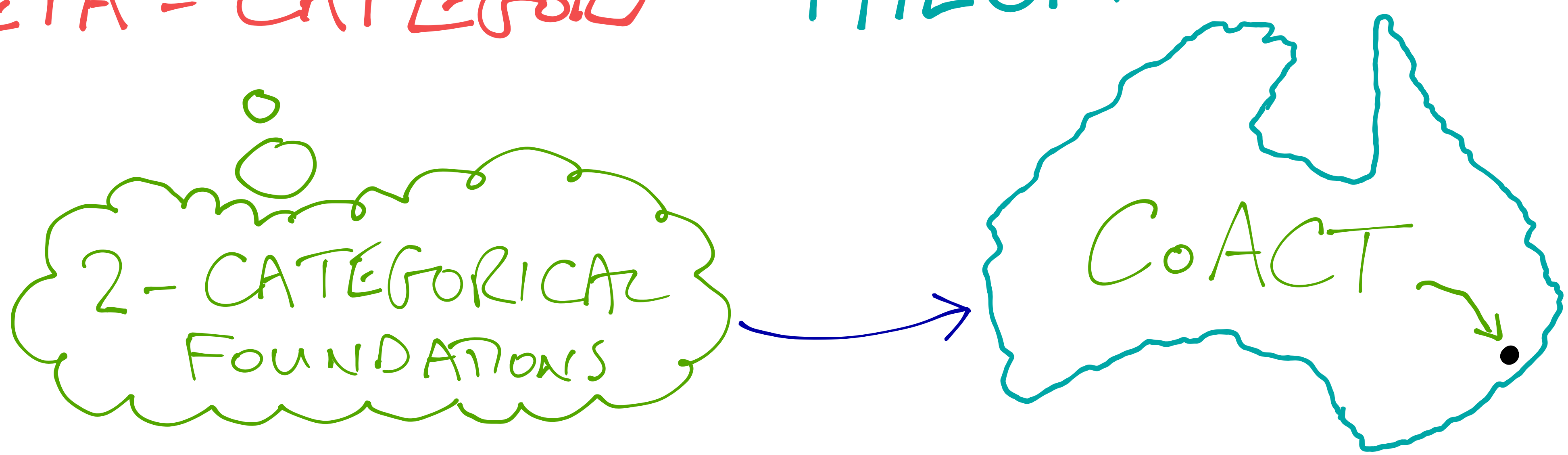
Θ_n-SETS AND SPACES

CATEGORIES WEAKLY ENRICHED IN $(\infty, n-1)$-CATEGORIES

WE WILL USE THE TERM ∞-CATEGORY TO MEAN ANY STRUCTURE THAT INHABITS AN INSTANCE OF THE FRAMEWORK WE BUILD HERE.

BENEFITS

- GIVE SYNTHETIC PROOFS OF ∞-CATEGORICAL RESULTS, WHICH THEN APPLY IN ALL MODELS.
- TRANSFER CATEGORICAL RESULTS DERIVED ANALYTICALLY IN ONE MODEL TO RELATED MODELS.
- ADAPT AND APPLY INTUITIONS FROM "META-CATEGORY" THEORY.



PLAN

PART 1: AN AXIOMATIC FRAMEWORK FOR ∞-CATEGORY THEORY.

PART 2: LIMITS AND ADJUNCTIONS.

PART 3: HOMOTOPY COHERENT MONADS AND BECK MONADICITY.

DIVING IN: 2-CATEGORIES

MODELLED ON THE TOTALITY OF CATEGORIES, FUNCTORS + NATURAL TRANSFORMATIONS

OF THE STRICT VARIETY

AKA CATEGORIES ENRICHED IN $(\underline{Cat}, \times, \mathbb{1})$

- OBJECTS A, B, C
- 1-CELLS $A \xrightarrow{f} B$
- 2-CELLS
$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

2-CELLS COMPOSE

VERTICALLY

$$\begin{tikzcd}[column sep=huge] A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\beta"] & B \end{tikzcd}$$

$\beta \bullet \alpha$

HORIZONTALLY

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left=50, "l"] \arrow[r, bend right=50, "m"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

$\gamma \circ \alpha$

BOTH STRICTLY ASSOCIATIVE

MIDDLE-4-INTERCHANGE

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\lambda"] & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "m"] \arrow[r, bend right, "n"'] \arrow[r, phantom, "\Downarrow\delta"] & C \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}[column sep=large] A \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\sigma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \end{tikzcd} \qquad \begin{tikzcd}[column sep=large] B \arrow[r, bend left=60, "l"] \arrow[r, "m"] \arrow[r, bend right=60, "n"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\lambda"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & C \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\lambda"] & C \end{tikzcd}$$

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$\rightsquigarrow$

$$\begin{tikzcd}[column sep=large] A \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\sigma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \arrow[r, bend left=60, "l"] \arrow[r, "m"] \arrow[r, bend right=60, "n"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\lambda"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & C \end{tikzcd}$$

$$(\delta \bullet \lambda) \circ (\beta \bullet \sigma)$$

$$\mathbin{/\!/}$$

$$(\delta \circ \beta) \bullet (\lambda \circ \sigma)$$

WHISKERING

$$\lambda g \bullet l\sigma = \lambda \circ \sigma = \sigma m \bullet f\lambda$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, "l"] & C \end{tikzcd}$$

$$\bullet$$

$$\begin{tikzcd} A \arrow[r, "g"'] & B \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\lambda"] & C \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\lambda"] & C \end{tikzcd}$$

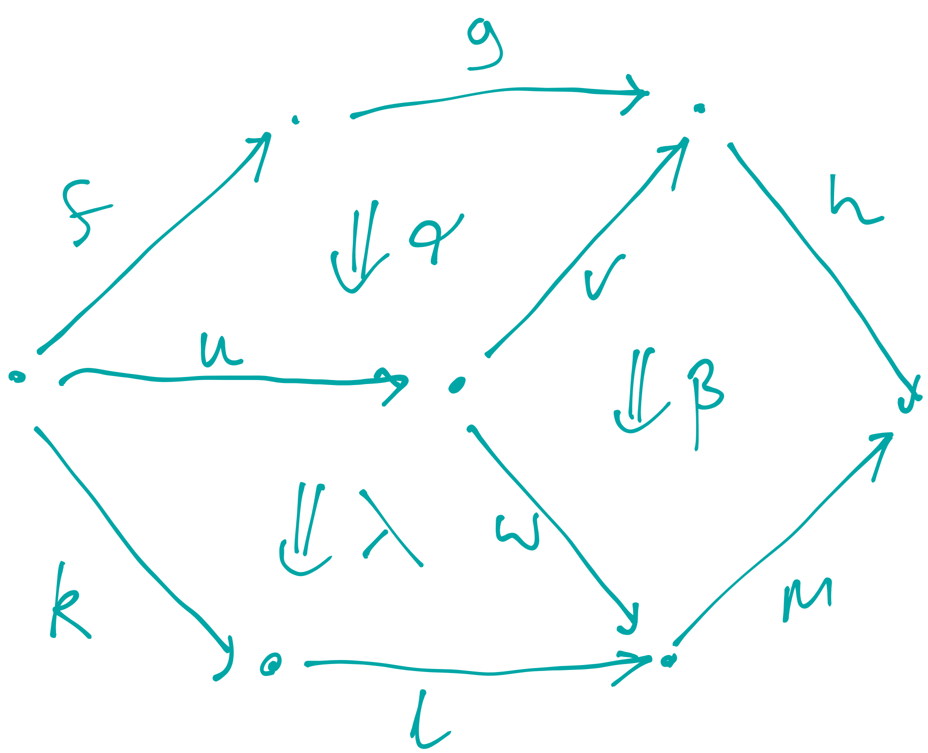
$\rightsquigarrow$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\lambda"] & C \end{tikzcd}$$

$$\bullet$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, "m"] & C \end{tikzcd}$$

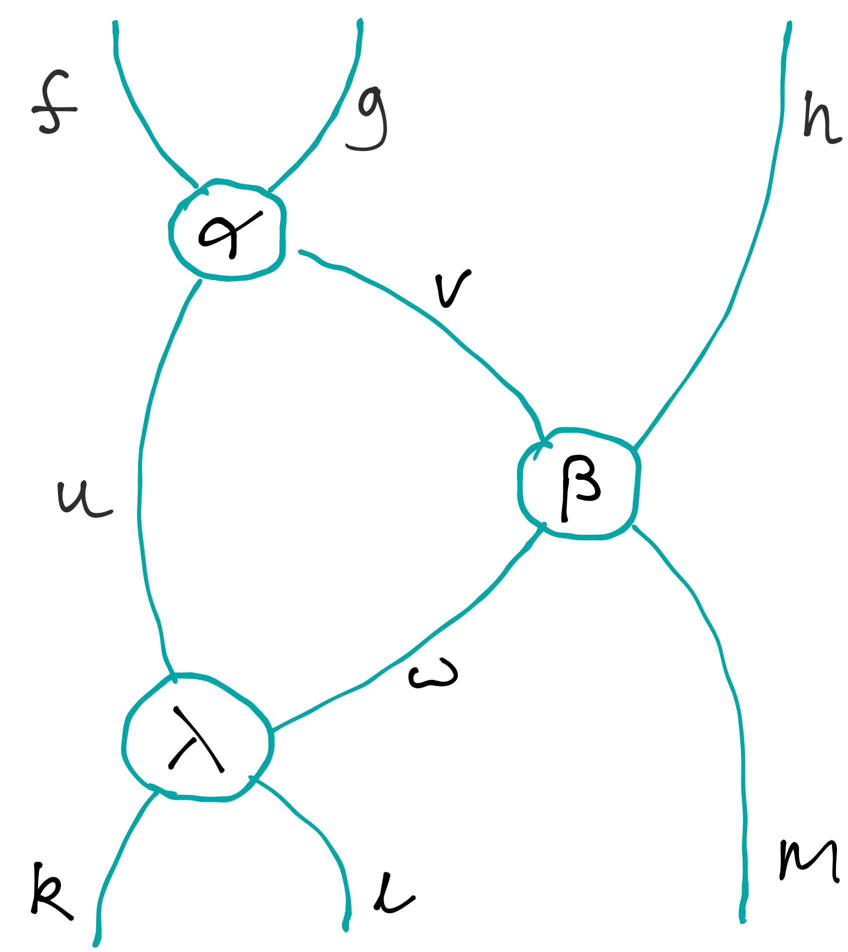
PASTING



$m\lambda \bullet \beta u \bullet h\alpha$

STRING DIAGRAMS

PLANAR
DUAL



ADJUNCTIONS IN 2-CATEGORIES

DEFN An adjunction in a 2-category comprises

- a pair of objects A and B
- a pair of 1-cells $A \xrightarrow{u} B$ and $B \xrightarrow{f} A$
- a pair of 2-cells:

$\eta : id_B \Rightarrow uf$

$$\begin{tikzcd} B \arrow[rr, "id_B"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

$\varepsilon : fu \Rightarrow id_A$

$$\begin{tikzcd} & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ A \arrow[ur, "u"] \arrow[rr, "id_A"'] & {} & A \end{tikzcd}$$

NOTATION

$$\varepsilon : f \dashv u : \eta$$

- TRIANGLE IDENTITIES

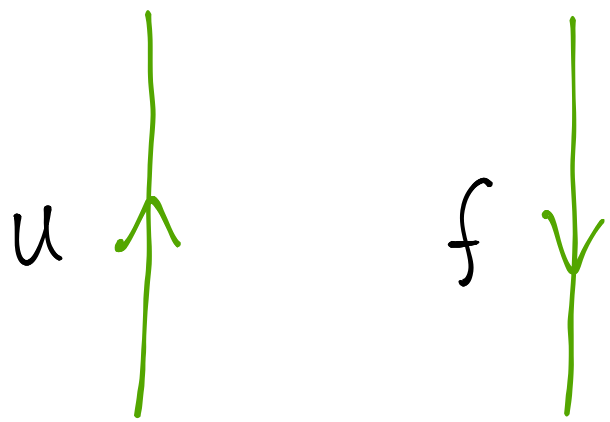
Identity 2-cells

$$\begin{tikzcd} B \arrow[rr, "id_B"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, "id_A"'] & {} & A \end{tikzcd} = id_f$$

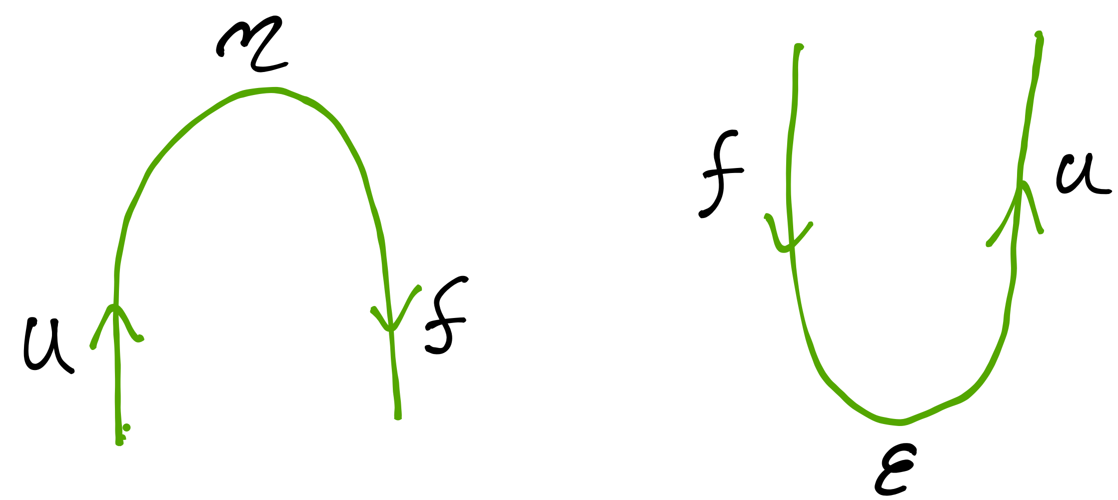
$$id_u = \begin{tikzcd} & B \arrow[rr, "id_B"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[rr, "id_A"'] & {} & A \arrow[ur, "u"'] & \end{tikzcd}$$

REPRISE WITH STRINGS

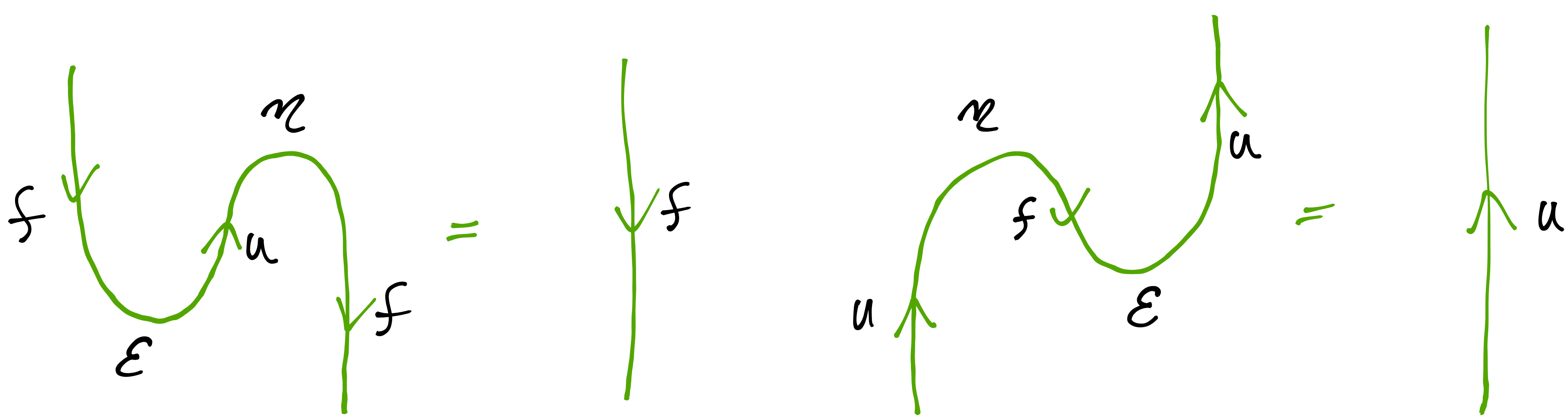
1-CELLS



2-CELLS



TRIANGLE IDENTITIES



SOME CONSEQUENCES

- COMPOSITION OF ADJUNCTIONS
- ADJOINTS ARE UNIQUE UPTO, AND STABLE UNDER, ISOMORPHISM.
- EQUIVALENCES MAY BE PROMOTED TO ADJOINT EQUIVALENCES.
- ADJUNCTIONS ARE STABLE UNDER EQUIVALENCES.

SLOGAN

CATEGORIES LIVE INSIDE 2-CATEGORIES*

* IT IS MORE ACCURATE TO SAY THAT 2-CATEGORIES PROVIDE A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP CATEGORY THEORY.

∞-SLOGAN

∞-CATEGORIES LIVE INSIDE $(\infty,2)$-CATEGORIES*

$*$ IT IS MORE ACCURATE TO SAY THAT $(\infty,2)$-CATEGORIES PROVIDE A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP ∞-CATEGORY THEORY.

$(\infty,2)$-CATEGORIES REALLY?!!

THE TRICK HERE IS TO:

- PICK A MODEL OF $(\infty,2)$-CATEGORIES THAT IS FAMILIAR.
- SIMPLIFY MANY ARGUMENTS BY WORKING, WHERE POSSIBLE, IN THE HOMOTOPY 2-CATEGORY ASSOCIATED WITH OUR AMBIENT $(\infty,2)$-CATEGORY.

COMMITTING TO AN $(\infty,2)$-CATEGORICAL MODEL

AN **∞-COSMOS** IS, IN ESSENCE, A CATEGORY OF FIBRANT OBJECTS ENRICHED IN THE JOYAL MODEL STRUCTURE

QUASI-CATEGORIES

RECALL:

QUASI-CATEGORY IS A SIMPLICIAL SET A SATISFYING:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "h"] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

$0<k<n$

INNER HORN

JOYAL MODEL STRUCTURE

ON $\underline{SSet}$ HAS

•) FIBRANT OBJECTS THE QUASI-CATEGORIES

•) FIBRATIONS, CALLED ISOFIBRATIONS

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted, "\exists"'] & B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dotted, "\exists"'] & B \end{tikzcd}$$

LIFTING OF COMPOSITES

LIFTING OF ISOMORPHISMS $\mathbb{I} := \{\bullet \cong \bullet\}$

•) COFIBRATIONS $\equiv$ MONOS

∞-COSMOS IN ONE PAGE...

A SIMPLICIAL CATEGORY $\mathcal{K}$

EQUIPPED WITH CLASSES OF

- ISOFIBRATIONS
- WEAK EQUIVALENCES

PRODUCTS AND PULLBACKS OF ISOFIBRATIONS

COFIBRANT REPLACEMENT $A_c \xrightarrow[\sim]{r_A} A$

ALL OBJECTS FIBRANT $!: A \twoheadrightarrow 1$

FIBRATION NOTIONS STABLE UNDER PULLBACK

ALL LIMITS SIMPLICIALLY ENRICHED

COTENSORS SATISFYING "SM7" WRT JOYAL MODEL STRUCTURE

FUNCTOR SPACE

AXIOM "JM7"

$$\mathrm{Fun}_{\mathcal{K}}(A,B)^X \cong \mathrm{Fun}_{\mathcal{K}}(A,B^X)$$

COTENSOR

INCLUSION $i : X \hookrightarrow Y \in \underline{sSet}$

ISOFIBRATION $p : E \twoheadrightarrow B \in \mathcal{K}$

$$\begin{tikzcd}
E^Y \arrow[dr, dotted, "\exists!\, i \hat{\pitchfork} p"] \arrow[drr, bend left, "E^i"] \arrow[ddr, bend right, "p^Y"'] & & \\
& E^X \times_{B^X} B^Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E^X \arrow[d, "p^X"] \\
& B^Y \arrow[r, "B^i"'] & B^X
\end{tikzcd}$$

"JM 7"

$i \hat{\pitchfork} p$ IS AN ISOFIBRATION AND IT IS A TRIVIAL FIBRATION IF i IS A WEAK EQUIVALENCE IN THE JOYAL MODEL STRUCTURE OR p IS A TRIVIAL FIBRATION IN $\mathcal{K}$.

∞-COSMOS EXAMPLES

$(\infty,1)$-MODELS

- QUASI-CATEGORIES
- SEGAL CATEGORIES
- COMPLETE SEGAL SPACES
- MARKED QUASI-CATS

(∞,n)-MODELS

- SEGAL (∞,n)-SPACES
- Θ_n-SETS + SPACES
- n-COMPLICIAL SETS

MOST COMMON EXAMPLES HAVE ALL OBJECTS COFIBRANT

$$\begin{tikzcd} & E \arrow[d, "p", "\sim"'] \\ X \arrow[ur, dashed, "\exists"] \arrow[r, "b"'] & B \end{tikzcd}$$

DERIVED EXAMPLES

- SLICES
- ∞-CATEGORIES FIBRED OVER A BASE.
- (CO)COMPLETE ∞-CATS

HOMOTOPY 2-CATEGORIES

$$
\begin{tikzcd}
\underline{\mathcal{C}at} \arrow[rr, bend right, "N"'] & \arrow[phantom, "\perp"] & \underline{sSet} \arrow[ll, bend right, "h"']
\end{tikzcd}
$$

HOMOTOPY CATEGORY FUNCTOR - PRESERVES FINITE PRODUCTS

NERVE
"ALL CATEGORIES ARE SIMPLICIAL SETS"

APPLY h TO HOM-SPACES

$$
\begin{tikzcd}
\underline{sSet}\text{-}\mathcal{C}at \arrow[r] & 2\text{-}\mathcal{C}at \\
\mathcal{K} \arrow[r, mapsto] & h\mathcal{K}
\end{tikzcd}
$$

THE HOMOTOPY 2-CATEGORY OF $\mathcal{K}$

HOMOTOPY 2-CATEGORIES OF ∞-COSMOI

WHEN A IS COFIBRANT IN $\mathcal{K}$ THEN ANY HOMSPACE $\mathrm{Fun}_{\mathcal{K}}(A,B)$ IS A QUASI-CAT

$\Rightarrow$

"WE SHALL RESTRICT THE HTY 2-CAT $h\mathcal{K}$ OF AN ∞-COSMOS TO COFIBRANT OBJECTS

EXPLICIT DESCRIPTION OF $h\mathcal{K}$

∞-CATEGORIES

(∞-)FUNCTORS

- 0-CELLS THE COFIBRANT OBJECTS IN $\mathcal{K}$
- 1-CELLS ARROWS BETWEEN COF OBJECTS IN $\mathcal{K}$
- 2-CELLS HOMOTOPY CLASSES OF 1-SIMPLICES, THAT IS OF (∞-) NATURAL TRANSFORMATIONS, IN $\mathrm{Fun}_{\mathcal{K}}(A,B)$.

A WILD CONJECTURE!

2-CATEGORICAL ADJUNCTIONS IN THE HOMOTOPY 2-CATEGORY $h\mathcal{K}$ OF AN ∞-COSMOS $\mathcal{K}$ ARE THE CORRECT NOTION WITH RESPECT TO ∞-CATEGORY THEORY.

FIRST STEPS

... TOWARDS ∞-CATEGORY THEORY

LEFT ADJOINT
RIGHT INVERSE
(LARI)

$$\begin{tikzcd} A \arrow[r, bend right, "!"'] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend right, "i"'] \end{tikzcd}$$

$\leadsto$ INITIAL ELEMENT

TERMINAL ELEMENT $\leadsto$

$$\begin{tikzcd} A \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "t"] \end{tikzcd}$$

RIGHT ADJOINT
RIGHT INVERSE
(RARI)

MINIMAL DATA PRESENTING A TERMINAL ELEMENT

(i) AN ELEMENT $t : 1 \longrightarrow A$

(ii) A 2-CELL

$$\begin{tikzcd} A \arrow[rr, "id_A"] \arrow[dr, "!"'] & \Downarrow \xi & A \\ & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

$\leadsto$ NATURAL FAMILY OF MAPS FROM THE ELEMENTS OF A TO t

(iii) $\xi t = id_t$ $\leadsto$ WHOSE COMPONENT AT t IS THE IDENTITY

EXERCISE

SUPPOSE THAT A IS A QUASI-CATEGORY; AN OBJECT IN THE ∞-COSMOS $\underline{QCat}$. LET t BE A VERTEX OF A, THEN

$t : 1 \longrightarrow A$ IS A TERMINAL ELEMENT

$$\Updownarrow$$

THIS IS JOYAL'S TERMINAL OBJECT NOTION

A ADMITS FILLERS FOR SIMPLEX BOUNDARIES WITH LAST VERTEX t

$$\begin{tikzcd}
1 \arrow[r, "\langle n \rangle"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r, "\partial a"'] \arrow[d, hook] & A \\
& \Delta[n] \arrow[ur, dotted, "\exists a"'] &
\end{tikzcd}$$

IMMEDIATE CONSEQUENCES

- TERMINAL ELEMENTS PRESERVED BY RIGHT ADJOINTS.

- $t : 1 \to A$ IS TERMINAL IFF FOR ALL ∞-CATS B AND FUNCTORS $f : B \to A$ WE HAVE

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow \exists !"] & A \\ & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

IN $h\mathcal{K}$.

- $t : 1 \to A$ IS TERMINAL IFF FOR ALL ∞-CATS B THE GENERALISED ELEMENT $B \xrightarrow{!} 1 \xrightarrow{t} A$ IS TERMINAL IN THE QUASI-CAT $\mathrm{Fun}_{\mathcal{K}}(B, A)$.

LIMITS AND COLIMITS

$$\begin{array}{ccc} \underline{sSet}^{op} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ X \quad\quad A & \longmapsto & A^X \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{K}^{op} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ U \quad\quad A & \longmapsto & A^U \end{array}$$

COTENSOR

(CO)LIMITS OF SIMPLICIAL SET SHAPED DIAGRAMS

CLOSURE

(WHEN $\mathcal{K}$ CARTESIAN CLOSED)

LIMITS OF DIAGRAMS WHOSE SHAPES ARE ∞-CATEGORIES IN $\mathcal{K}$

$$\begin{tikzcd} A \arrow[rr, "\Delta" description] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[rr, phantom, bend right=20, "\perp"] & & A^X \arrow[ll, bend right=40, "\mathrm{colim}"'] \arrow[ll, bend left=40, "\lim"] \end{tikzcd}$$

COLIMITS OF DIAGRAMS OF SHAPE X

CONSTANT DIAGRAM FUNCTOR

LIMITS OF DIAGRAMS OF SHAPE X

FAMILIES OF DIAGRAMS

WE MAY WISH TO ASK THAT AN ∞-CATEGORY HAS SOME, BUT NOT ALL, LIMITS OF A GIVEN SHAPE.

EXAMPLE

- POINTED ∞-CATEGORY HAS AN ELEMENT $* : 1 \longrightarrow A$ WHICH IS BOTH INITIAL + TERMINAL

POINTS

$$\begin{tikzcd} & 1 \arrow[dr, "*"] & \\ A \arrow[ur, "!"] \arrow[rr, equal, "id_A"'] & \arrow[u, phantom, "\Downarrow \rho"] & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow \xi"] & A \\ & 1 \arrow[ur, "*"'] & \end{tikzcd}$$

COPOINTS

- IT IS COMMON ONLY TO ASK FOR CERTAIN PULLBACKS AND PUSHOUTS

LOOP SPACE

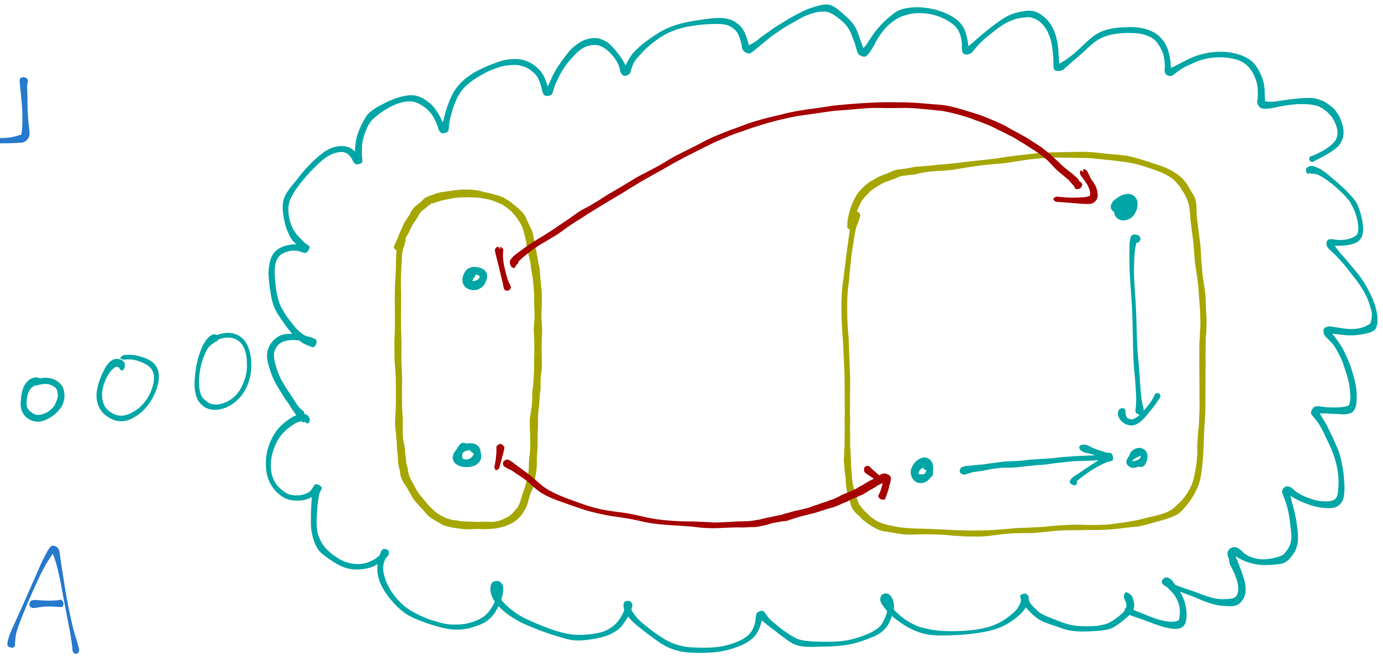
$$\begin{tikzcd} \Omega a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "p_a"] \\ * \arrow[r, "p_a"'] & a \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "\xi_a"] \arrow[d, "\xi_a"'] & * \arrow[d] \\ * \arrow[r] & \Sigma a \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

SUSPENSION

LOOP SPACES

$$\begin{tikzcd} A_*^{\lrcorner} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\lrcorner} \arrow[d] \\ 1 \arrow[r, "(*,*)"'] & A \times A \end{tikzcd}$$



- LOOP SPACES ARE PULLBACKS, SO THEIR UNIVERSAL PROPERTY LIVES IN THE DIAGRAM ∞-CATEGORY $A^{\lrcorner}$....
- ... BUT WE ONLY WANT TO ASK FOR LIMITS OF DIAGRAMS IN $A_*^{\lrcorner}$
- $A_*^{\lrcorner} \xrightarrow{f} A^{\lrcorner}$ SPECIFIES A FAMILY OF DIAGRAMS IN $A^{\lrcorner}$ WHOSE LIMITS ARE LOOP SPACES.

LIMITS OF FAMILIES

ABSOLUTE RIGHT LIFTING
(IN A 2-CATEGORY)

$$\begin{tikzcd} & A \arrow[d, "a"] \\ C \arrow[ur, bend left, "\ell"] \arrow[r, "c"'] \arrow[ur, phantom, shift left=3, "\Downarrow\lambda"] & B \end{tikzcd} \quad \text{S.T. } \forall \quad \begin{tikzcd} I \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\sigma"] & A \arrow[d, "a"] \\ C \arrow[r, "c"'] & B \end{tikzcd} \rightsquigarrow = \begin{tikzcd} I \arrow[r, "u"] \arrow[d, "v"'] \arrow[d, phantom, shift left=4, "\Downarrow\exists!\hat{\sigma}"] & A \arrow[d, "a"] \\ C \arrow[ur, "\ell"] \arrow[r, "c"'] \arrow[ur, phantom, shift right=5, "\Downarrow\lambda"] & B \end{tikzcd}$$

EXAMPLE

$$\epsilon : A \underset{u}{\overset{f}{\rightleftarrows}} B : \eta \qquad (\perp)$$

$\rightsquigarrow$

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, bend left, "u"] \arrow[r, equal, "\mathrm{id}_A"'] \arrow[ur, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

IS AN ABSOLUTE RIGHT LIFTING

DEFN A FAMILY OF DIAGRAMS $f : I \longrightarrow A^{\times}$ ADMITS A LIMIT IF THERE IS AN ABSOLUTE RIGHT LIFTING

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ I \arrow[ur, bend left, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift left=3, "\Downarrow\lambda"] & A^{\times} \end{tikzcd} \quad \text{IN } hK$$

PRESERVATION

RIGHT ADJOINTS PRESERVE LIMITS OF FAMILIES

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

GIVEN AN ADJUNCTION

IF

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ I \arrow[ur, bend left, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\lambda"] & A^X \end{tikzcd}$$

IS AN ABSOLUTE RIGHT LIFTING ....

$\Longrightarrow$

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\ I \arrow[ur, bend left, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\lambda"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

IS ALSO AN ABSOLUTE RIGHT LIFTING.

Hom-Spaces

THE HOM SPACE FROM f TO g

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(f,g) \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] \arrow[ur, phantom, "\swArrow"] & \mathcal{A} \end{tikzcd}$$

A PARAMETERISED OBJECT IN THE ∞-CATEGORY $\mathcal{A}$

$\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{2} \arrow[d] \\ C \times B \arrow[r, "g \times f"'] & \mathcal{A} \times \mathcal{A} \end{tikzcd}$$

WEAK COMMA OBJECTS

HOM-SPACES POSSESS A WEAK 2-UNIVERSAL PROPERTY IN THE HOMOTOPY 2-CATEGORY.

SPECIFICALLY FOR EACH X THE INDUCED FUNCTOR

$$h\mathcal{K}(X, \mathrm{Hom}_A(f,g)) \longrightarrow h\mathcal{K}(X,f) \downarrow h\mathcal{K}(X,g)$$

IS

- SURJECTIVE ON OBJECTS,
- FULL, AND
- CONSERVATIVE.

AKA SMOTHERING

WE SAY THAT $\mathrm{Hom}_A(f,g)$ IS THE WEAK COMMA OF THE FUNCTORS f AND g IN $h\mathcal{K}$.

HOM-WISE CHARACTER OF ADJUNCTIONS

DEFN A MODULE $\mathrm{Hom}_A(f,g) \to C \times B$ IS

- LEFT REPRESENTABLE IF THERE EXISTS A FUNCTOR $\ell : B \to C$ AND AN EQUIVALENCE $\mathrm{Hom}_C(\ell, C) \simeq \mathrm{Hom}_A(f,g)$ IN $\mathcal{K}/_{C\times B}$.

AN EQUIVALENCE FIBRED OVER $C \times B$

- RIGHT REPRESENTABLE IF THERE EXISTS A FUNCTOR $r : C \to B$ AND AN EQUIVALENCE $\mathrm{Hom}_B(B, r) \simeq \mathrm{Hom}_A(f,g)$ IN $\mathcal{K}/_{C\times B}$

LIFTINGS HOM-WISE

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[ur, bend left, "l"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & C \end{tikzcd}$$

LEMMA THE TRIANGLE ON THE LEFT IS AN ABSOLUTE RIGHT LIFTING IFF THE INDUCED FUNCTOR $\mathrm{Hom}_A(A,l) \xrightarrow{\omega} \mathrm{Hom}_C(f,g)$ IS AN EQUIVALENCE

$$\begin{tikzcd} & \mathrm{Hom}_A(A,l) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ B \arrow[rr, "l"] \arrow[dr, "g"'] & \overset{\lambda}{\Leftarrow} & A \arrow[dl, "f"] \\ & C & \end{tikzcd} \rightsquigarrow = \begin{tikzcd} & \mathrm{Hom}_A(A,l) \arrow[d, dashed, "\omega"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\ & \mathrm{Hom}_C(f,g) \arrow[dl, "q_1"'] \arrow[dr, "q_0"] & \\ B \arrow[dr, "g"'] & \overset{\psi}{\Leftarrow} & A \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

LIFTINGS HOM-WISE

KEY OBSERVATION IT IS A CONSEQUENCE OF THE WEAK 2-UNIVERSAL PROPERTY OF $\mathrm{Hom}_C(f,g)$ THAT THERE EXISTS A BIJECTION

$$
\begin{tikzcd}
 & X \arrow[dl, "b"'] \arrow[dr, "a"] & \\
B \arrow[dr, "g"'] & \Leftarrow \sigma & A \arrow[dl, "f"] \\
 & C &
\end{tikzcd}
\quad \overset{\sim}{\longleftrightarrow}_{\text{BIJECTION}} \quad
\begin{tikzcd}
X \arrow[r, "\bar{\sigma}"] \arrow[dr, "{(b,a)}"'] & \mathrm{Hom}_C(f,g) \arrow[d, "{(q_1,q_0)}"] \\
 & B \times A
\end{tikzcd}
$$

2-CELLS

ISO-CLASSES OF 1-CELLS OVER $B \times A$

THEOREM

A FUNCTOR $u : A \to B$ HAS A LEFT ADJOINT IFF $\mathrm{Hom}_B(B, u)$ IS LEFT REPRESENTABLE.

$$\begin{tikzcd}
\mathrm{Hom}_B(B,u) \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Hom}_A(f,A) \arrow[dl] \\
& A \times B &
\end{tikzcd}$$

THIS IS THE USUAL "HOM-WISE" CHARACTER OF AN ADJUNCTION

THREE VIEWS OF ADJUNCTIONS

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

$\eta : id_B \Rightarrow uf$

$\varepsilon : fu \Rightarrow id_A$

2-CATEGORICAL IN $h\mathcal{K}$

$$\begin{tikzcd} Hom_A(f,A) \arrow[r] \arrow[dr] & Hom_B(B,u) \arrow[d] \\ & A \times B \end{tikzcd}$$

"MODULES" + REPRESENTABILITY

HOMOTOPY COHERENCE?

(PART 3)

$A \xrightarrow{u} B$ IS RIGHT ADJOINT IN $\mathcal{K}$ IFF

$$Fun_{\mathcal{K}}(X,A) \xrightarrow{u\circ -} Fun_{\mathcal{K}}(X,B)$$

IS A RIGHT ADJOINT FUNCTOR OF QUASI-CATEGORIES FOR ALL $X \in \mathcal{K}$

RETURN TO LOOPS

WE SAY THAT A POINTED ∞-CAT A ADMITS LOOPS + SUSPENSIONS IF THERE EXISTS:

$$
\begin{tikzcd}
 & A \arrow[d] \\
A_*^{\lrcorner} \arrow[ur, "\Omega"] \arrow[r, hook] \arrow[ur, phantom, "\Downarrow"'] & A^{\lrcorner}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & A \arrow[d] \\
A_*^{\ulcorner} \arrow[ur, "\Sigma"] \arrow[r, hook] \arrow[ur, phantom, "\Uparrow"'] & A^{\ulcorner}
\end{tikzcd}
$$

ABSOLUTE RIGHT LIFTING

ABSOLUTE LEFT LIFTING

(CO)LIMITS AS KAN EXTENSIONS

X A DIAGRAM SHAPE (IN <u>sSet</u>)

$X^{\triangleleft}$ OBTAINED BY ADDING INITIAL OBJECT

$$
\begin{tikzcd}
 & A \arrow[d] \\
I \arrow[r, "d"'] \arrow[ur, dotted, "\ell"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A^X
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & A^{X^{\triangleleft}} \arrow[d] \\
I \arrow[r, "d"'] \arrow[ur, dotted, "\hat{i}"] \arrow[ur, phantom, shift right=3ex, "\Vert"] & A^X
\end{tikzcd}
$$

ADMITS AN ABSOLUTE RIGHT LIFTING $\Longleftrightarrow$ ADMITS AN ABSOLUTE RIGHT LIFT WHOSE 2-CELL IS THE IDENTITY.

LOOP – SUSPENSION ADJUNCTION

Σ oO (SUSPENSION)

$$\begin{tikzcd}[column sep=huge]
A \simeq A_*^{\lrcorner} \arrow[r, bend right, "\mathrm{ran}"'] \arrow[r, phantom, "\perp"] \arrow[rr, dotted, bend right=70, "\Omega"'] & A_*^{\boxdot} \arrow[l, bend right, "\mathrm{res}"'] \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\perp"] & A_*^{\ulcorner} \simeq A \arrow[l, bend right, "\mathrm{lan}"'] \arrow[ll, dashed, bend right=70, "\Sigma"']
\end{tikzcd}$$

Ω oO (LOOP)

$$\bullet\!\lrcorner^{\triangleleft} = \boxdot = \ulcorner\!\bullet^{\triangleright}$$

NEW ∞-COSMOI FROM OLD.

IMPORTANT: MANY (2-)CATEGORICALLY INSPIRED CONSTRUCTIONS APPLY TO ∞-COSMOI GIVING NEW STRUCTURES WHICH ARE AGAIN ∞-COSMOI.

SLICES

$\mathcal{K}_{/A}$

OBJECTS ARE ISOFIBRATIONS $p : E \twoheadrightarrow A$

COSMOS OF ISOFIBRATIONS $\mathcal{K}^2$

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, two heads, "p"'] & F \arrow[d, two heads, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

OBJECTS ARE ISOFIBRATIONS

ARROWS ARE COMMUTATIVE SQUARES

(CO)COMPLETE ∞-CATS

$\mathcal{L}ex_{\omega}(\mathcal{K})$

OBJECTS: THOSE OF $\mathcal{K}$ ADMITTING SOME CLASS OF LIMITS

ARROWS: FUNCTORS IN $\mathcal{K}$ THAT PRESERVE THOSE LIMITS

POINTED ∞-CATS

$\mathcal{K}_*$

POINTED OBJECTS OF $\mathcal{K}$

+ POINT PRESERVING FUNCTORS

Aside: Comparing $h(\mathcal{K}/B)$ and $h\mathcal{K}/B$

The slice constructions are called the internal and external slice respectively.

But we do have a canonical comparison

Ugh! These are not, in general, equivalent 2-cats

$$h(\mathcal{K}/B) \longrightarrow h\mathcal{K}/B$$

which is 2-smothering

In particular adjunctions lift along this comparison.

Obscurity corner: In the 2-derivator world this is a key axiom.

THREE VIEWS OF GROTHENDIECK FIBRATIONS

$p : E \twoheadrightarrow B$ AN ISOFIBRATION IN $\mathcal{K}$.

$$\begin{tikzcd} E \arrow[rr, "i"] \arrow[dr, "p"'] & & \mathrm{Hom}_B(B,p) \arrow[dl, "p_0"] \\ & B & \end{tikzcd}$$

HAS A RIGHT ADJOINT IN $h\mathcal{K}_{/B}$

$$E^{\mathbb{2}} \xrightarrow{k} \mathrm{Hom}_B(B,p)$$

HAS A RIGHT ADJOINT RIGHT INVERSE (RARI) IN $h\mathcal{K}$.

"CHEVALLEY CRITERION"

OFTEN CALLED CARTESIAN FIBRATIONS IN THE ∞-CAT LITERATURE

$$\mathrm{Fun}_{\mathcal{K}}(X,E) \xrightarrow{p\circ -} \mathrm{Fun}_{\mathcal{K}}(X,B)$$

ADMITS LIFTS OF ALL ARROWS $\beta : a \to pb$ TO A P-CARTESIAN ARROW $\chi_\beta : \beta^* b \to b$

More ∞-Cosmoi

Cartesian fibrations

over a fixed base

$$\mathrm{Cart}(\mathcal{K})_{/A} \subseteq \mathcal{K}_{/A}$$

Cartesian fibrations with base A + cartesian functors between them

2-sided cartesian fibrations

$${}_{A\backslash}\mathrm{Fib}(\mathcal{K})_{/B} \subseteq \mathcal{K}_{/A\times B}$$

$$\mathrm{coCart}\left(\mathrm{Cart}(\mathcal{K})_{/B}\right)_{/A\times B \xrightarrow{\pi} B}$$

All cartesian fibrations

$$\mathrm{Cart}(\mathcal{K}) \subseteq \mathcal{K}^{2}$$

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Cartesian fibration

Cartesian functor: preserves cartesian 1-arrows

Groupoidal ∞-cats

$$\mathrm{Gpd}(\mathcal{K}) \subseteq \mathcal{K}$$

$A \in \mathrm{Gpd}(\mathcal{K})$ iff $\forall X \in \mathcal{K}$

$$\mathrm{Fun}_{\mathcal{K}}(X, A)$$

is a Kan complex.

MODULES

AKA PROFUNCTORS

$$ {}_A\mathrm{Mod}(\mathcal{K})_B := \mathrm{Gpd}\left({}_A\mathrm{Fib}(\mathcal{K})_B\right) $$

GIVEN $A \xrightarrow{f} C \xleftarrow{g} B$

$$\begin{tikzcd} & \mathrm{Hom}_C(g,f) \arrow[dl, two heads] \arrow[dr, two heads] & \\ A & & B \end{tikzcd}$$

IS A MODULE

ALL MODULES

$\mathrm{Mod}(\mathcal{K})$ + MAPS

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, two heads] & F \arrow[d, two heads] \\ A \times B \arrow[r, "f \times g"'] & C \times D \end{tikzcd}$$

THIS IS AGAIN AN ∞-COSMOS.

STABLE ∞-CATEGORIES

AN ∞-CATEGORY A IS SAID TO BE STABLE IF

- A HAS A ZERO OBJECT $* : 1 \to A$
- EVERY X-INDEXED FAMILY $f : a \to b$ OF ARROWS IN A ADMITS KERNELS + COKERNELS

$\{ f : X \to A^2 \}$

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\rho"] \\ a \arrow[r, "f"'] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\xi"] \arrow[d, "f"'] & * \arrow[d] \\ b \arrow[r] & \operatorname{cok} f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & * \arrow[d] \\ b \arrow[r, "g"'] & c \end{tikzcd}$$

DIAGRAMS ARE CALLED TRIANGLES

- A TRIANGLE IS A PUSHOUT (COKERNEL) IFF IT IS A PULLBACK (KERNEL).

A HOM-WISE CHARACTERISATION OF PULLBACKS

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{Hom}_A(A, aj) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathrm{Hom}_A(A, bj) \arrow[d] \\ \mathrm{Hom}_A(A, cj) \arrow[r] & \mathrm{Hom}_A(A, dj) \end{tikzcd}$$

I-INDEXED COMMUTATIVE SQUARES IN A

ISO-SQUARES IN 2-CAT $\mathrm{Mod}_{h\mathcal{K}}(J, A)$

THIS SQUARE IS AN I-INDEXED FAMILY OF PULLBACKS $\iff$ THIS ISO-SQUARE IS A WEAK ISO-COMMA FOR ALL $j : J \to I$.

APPLICATION: THE COMPOSITION + CANCELLATION LAWS HOLD FOR PULLBACKS IN ANY ∞-CAT.

ELEMENTARY RESULTS ...

- COMPOSITION + CANCELLATION OF PULLBACK / PULLBACK SQUARES $\Rightarrow$ STABLE ∞-CATS ADMIT ALL PULLBACKS / PUSHOUTS:

$$\begin{tikzcd}
\ker kg \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & a \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d] \\
c \arrow[r, "g"'] & b \arrow[r, "k"'] & \operatorname{cok} f \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

- THE LOOP-SUSPENSION ADJUNCTION OF A STABLE ∞-CAT IS AN EQUIVALENCE.
- A POINTED ∞-CATEGORY IS STABLE IFF IT ADMITS KERNELS + ITS LOOP FUNCTOR IS AN EQUIVALENCE.

STABILISATION

THE SUBCATEGORY OF

- POINTED ∞-CATEGORIES ADMITTING KERNELS
- ∞-FUNCTORS THAT PRESERVE POINTS AND KERNELS.

IS A SUB-∞-COSMOS SO IT ADMITS A HOMOTOPY LIMIT OF EACH CHAIN:

$$\cdots\cdots \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A$$

<u>PROP</u> LOOPS FUNCTOR ON THIS LIMITING ∞-CATEGORY IS AN EQUIVALENCE.

$\Rightarrow$ THIS ∞-CATEGORY $\mathrm{Stab}(A)$ IS STABLE.

SPECTRA LIVE HERE!

The 2-Category Adj

$$\begin{tikzcd} - \arrow[loop left, "\Delta_+^{op}"] \arrow[rr, bend right, "\Delta_\top \cong \Delta_\perp^{op}"'] & & + \arrow[ll, bend right, "\Delta_\perp \cong \Delta_\top^{op}"'] \arrow[loop right, "\Delta_+"] \end{tikzcd}$$

$[0] : - \longrightarrow +$ $\quad \underline{u}$ $\qquad [0] : + \longrightarrow -$ $\quad \underline{f}$

$\underline{f}\,\underline{u} = [0] : - \longrightarrow - \qquad id_- = [-1] : - \longrightarrow - \qquad [-1] \to [0] \in \Delta_+$

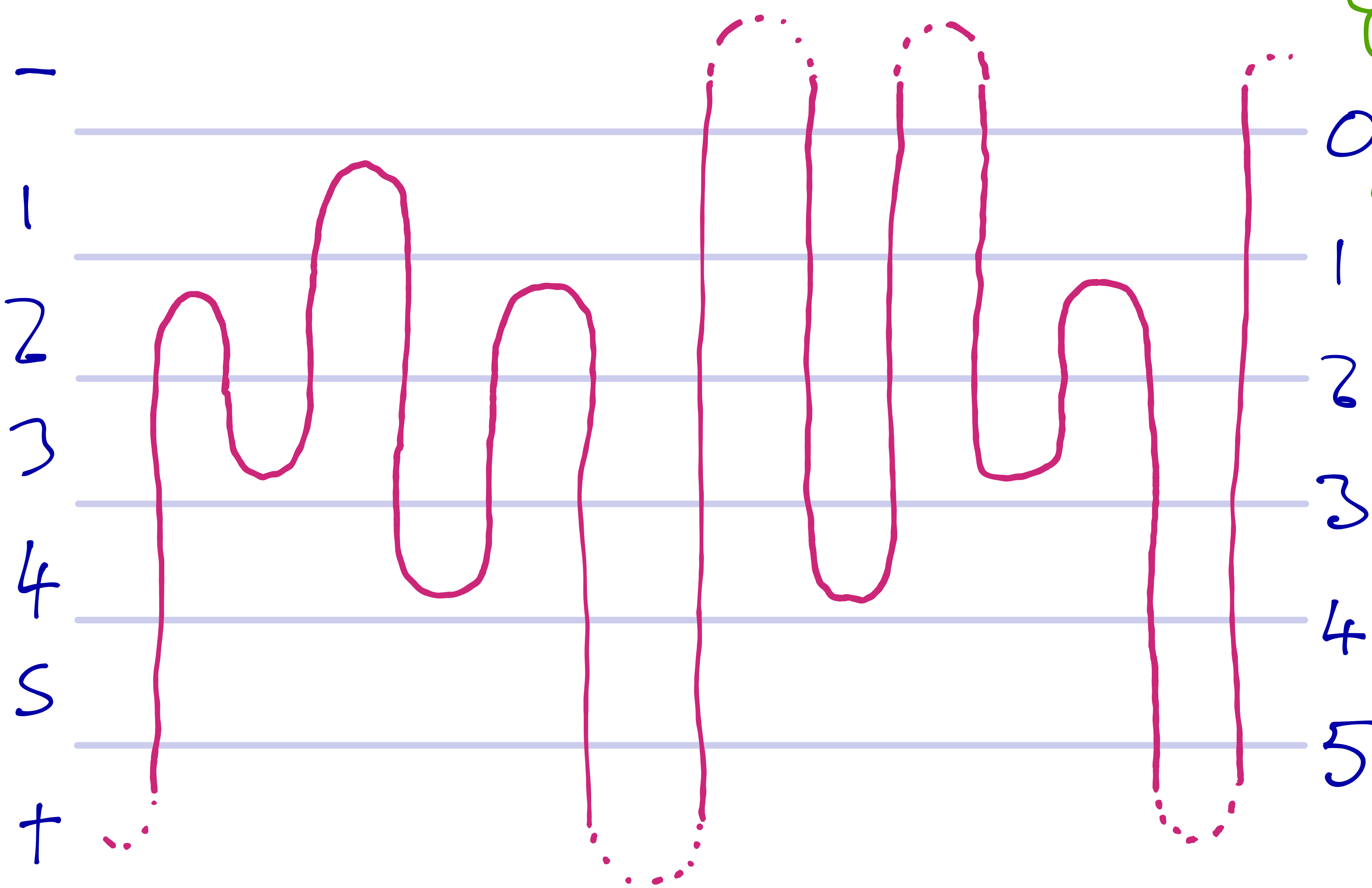
$\underline{\varepsilon} : \underline{f}\,\underline{u} \Rightarrow id_- \in Adj(-,-) = \Delta_+^{op}$

$\underline{u}\,\underline{f} = [0] : + \longrightarrow + \qquad id_+ = [-1] : + \longrightarrow + \qquad [-1] \to [0] \in \Delta_+$

$\underline{\eta} : id_+ \Rightarrow \underline{u}\,\underline{f} \in Adj(+,+) = \Delta_+$

The Simplicial Category Adj

n-Arrows



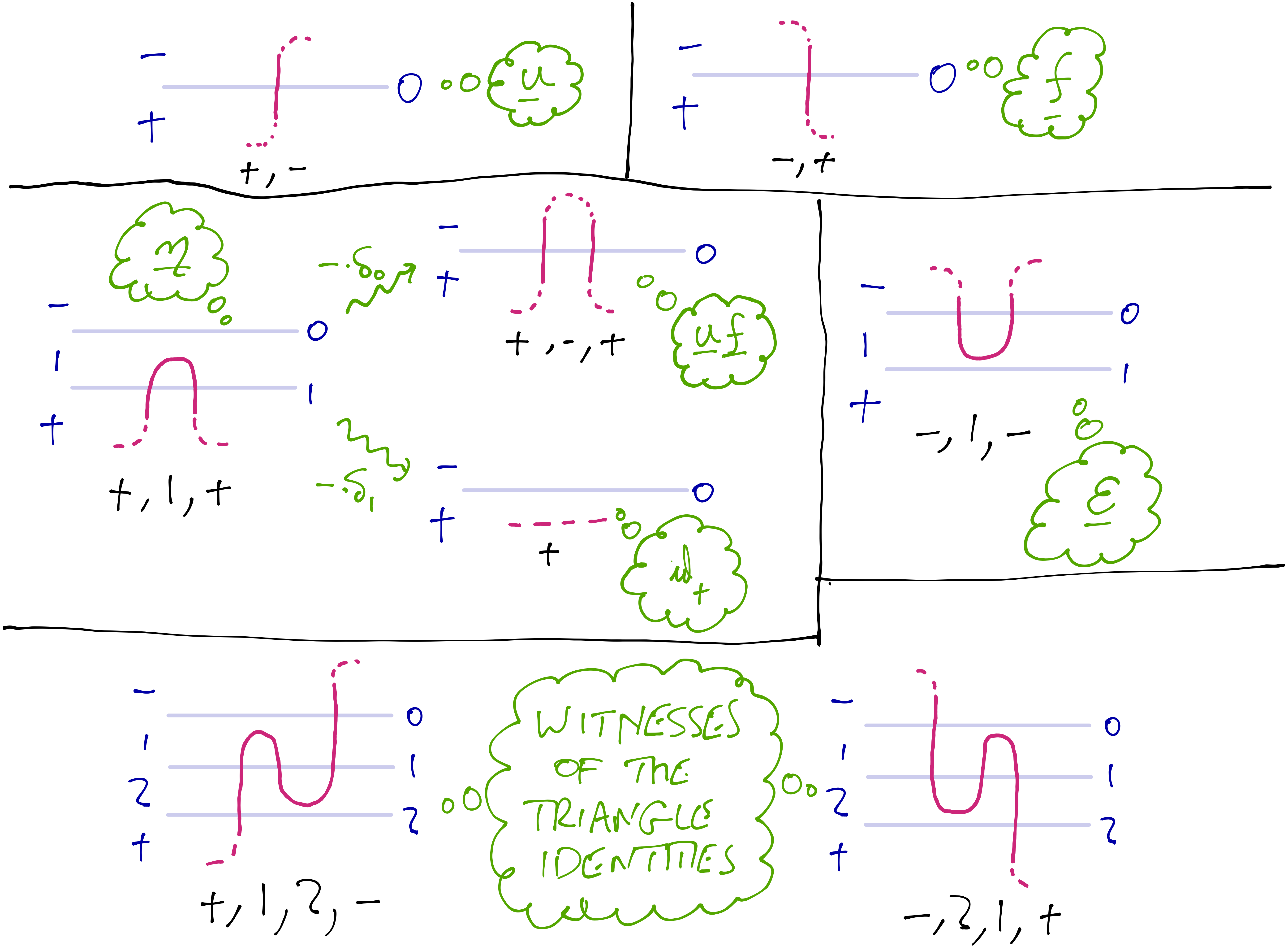
STRICTLY UNDULATING SQUIGGLE = TREMELO

THIS DIAGRAM DEPICTS A 5-ARROW

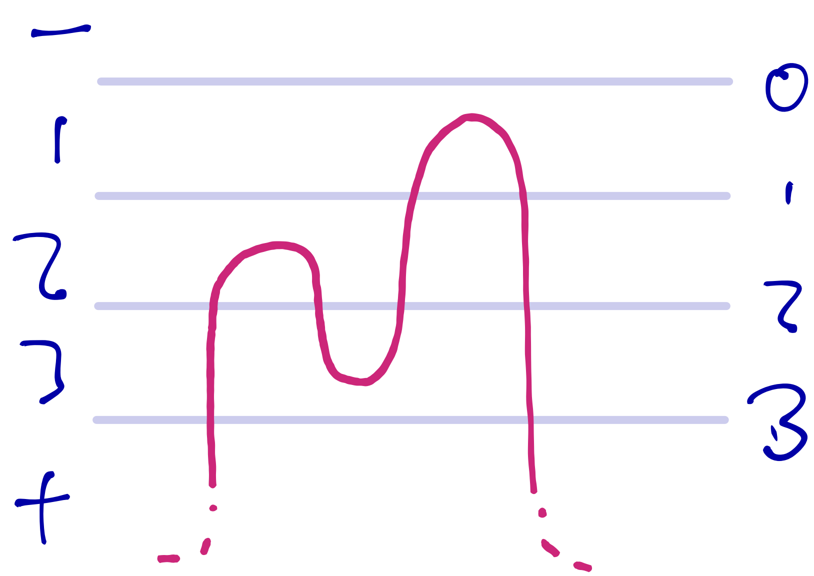
$$+,2,3,1,4,2,+,-,4,-,3,2,+,-$$

CODOMAIN

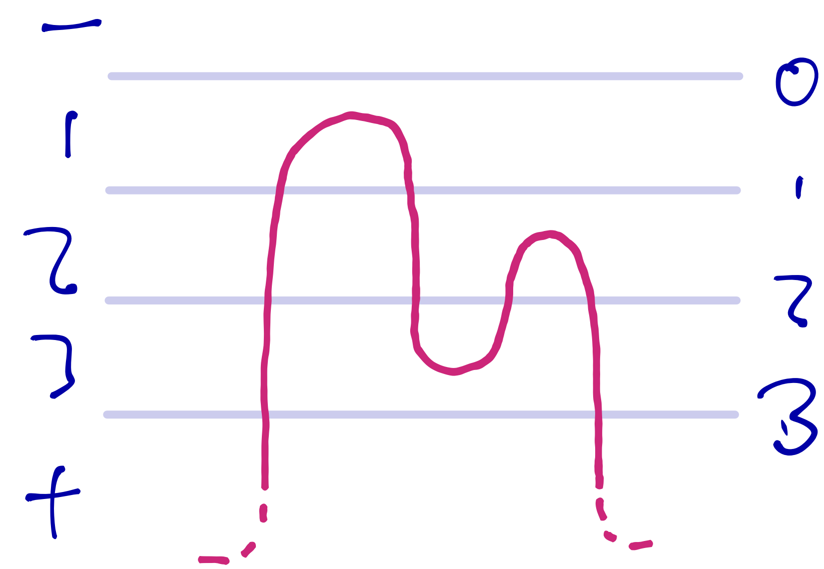
DOMAIN



+, −
u
−, +
f
η
−.δ₀
+, −, +
uf
+, 1, +
−.δ₁
+
id₊
−, 1, −
ε
+, 1, 2, −
WITNESSES OF THE TRIANGLE IDENTITIES
−, 2, 1, +



t, 2, 3, 1, t



t, 1, 3, 2, t

SOME HIGHER COHERENCES BETWEEN TRIANGLE IDENTITIES

ADJUNCTIONS ARE COHERENT

THEOREM: AN ADJUNCTION $f \dashv u$ IN $h\mathcal{K}$ GIVES RISE TO A SIMPLICIAL FUNCTOR $\mathbb{A} : \mathrm{Adj} \longrightarrow \mathcal{K}$ WHICH CARRIES THE CANONICAL ADJUNCTION $\underline{f} \dashv \underline{u}$ IN Adj TO THE ADJUNCTION $f \dashv u$ IN $h\mathcal{K}$. THE SPACE OF FUNCTORS THAT EXTEND $f \dashv u$ IN THIS WAY IS CONTRACTIBLE.

OBSCURITY CORNER: THIS IS ANOTHER IMPORTANT INTERNALISATION AXIOM IN THE WORLD OF 2-DERIVATORS.

Homotopy Coherent Monads

$\mathcal{M}\mathrm{nd}$ $\quad \boxed{+ \circlearrowleft \Delta_+}$

Full subcategory of $\mathcal{A}\mathrm{dj}$ spanning the object $+$

The "walking" homotopy coherent monad.

A homotopy coherent monad in an ∞-cosmos $\mathcal{K}$ is simply a simplicial functor

$$T : \mathcal{M}\mathrm{nd} \longrightarrow \mathcal{K}$$

$A \in \mathcal{K}$ $\quad +$

$$A \xrightarrow{t} A^{\Delta_+}$$

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[d, "t"'] & A^{\Delta_+} \arrow[d, "A^{\oplus}"] \\ A^{\Delta_+} \arrow[r, "t^{\Delta_+}"'] & A^{\Delta_+ \times \Delta_+} \end{tikzcd}$$

The homotopy coherence result for an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

in $h\mathcal{K}$ implies that such induces an HC monad on B and an HC comonad on A.

EILENBERG-MOORE OBJECTS

∞-COSMOI ADMIT ALL FLEXIBLY WEIGHTED LIMITS

$$\begin{tikzcd}
\mathrm{Mnd} \arrow[rr, hook, "\mathrm{I}"] \arrow[dr, "\mathbb{T}"'] & \arrow[d, phantom, "="] & \mathrm{Adj} \arrow[dl, dashed, "A^t := \mathrm{Ran}_{\mathrm{I}} \mathbb{T}"] \\
 & \mathcal{K} &
\end{tikzcd}$$

THIS KAN EXTENSION EXISTS BECAUSE Adj IS A COMPUTAD $\Rightarrow$ THE WEIGHT USED TO COMPUTE IT IS FLEXIBLE

FLEXIBLE $\equiv$ COFIBRANT

THE OBJECT $A^{\mathbb{T}} := A^t(-)$ IS THE ∞-CAT OF EILENBERG-MOORE ALGEBRAS.