Cobordism Categories, Classifying Spaces, and (invertible) TQFTs II

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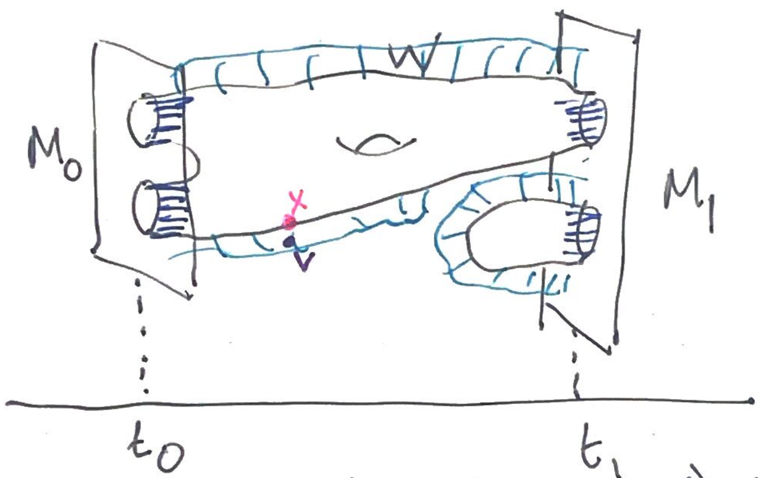
Cobordism category $\mathrm{Bord}_{\langle d-1, d\rangle}$

topologically enriched $\equiv$ $(\infty,1)$-cat

Ob: $d-1$ closed smooth oriented manifolds $M \subseteq \mathbb{R}^\infty \times \{t\}$

some point

morph: d-diml cobordisms $W \subseteq \mathbb{R}^\infty \times [t_0, t_1]$



$$\coprod_{/\partial W = \bar{M}_0 \amalg M_1} \mathrm{Emb}^{\partial}(W^d, \mathbb{R}^\infty \times [t_0, t_1]) / \mathrm{Diff}^{\partial}(W)$$

$$\simeq \coprod_{/W} B\mathrm{Diff}(W; \partial)$$

Fine Print: $d = 2$ $\quad B\mathrm{Diff}(W; \partial) \underset{\neq\emptyset}{\simeq} \mathcal{M}(W)$

GMTW. $|\mathrm{Bord}^{or}_{\langle d-1,d\rangle}| \simeq \Omega^{\infty}\Sigma\, MTSO(d)$

$$= \lim_{n\to\infty} \Omega^{d+n+1}(U^{\perp}_{d,n})^{c}$$

compactification

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & U^{\perp}_{d,n} \arrow[d] \\ & Gr(d,n) \end{tikzcd}$$

Take nice nbhd of W (shown on previous page) & collapse the rest to a pt.

$$\mathbb{R}^{d-1+n}\times[t_0,t_1] \longrightarrow (U^{\perp}_{d,n})^{c}$$

$$(x,v) \longmapsto (T_xW, v)$$

Compare them: $(x,v) \mapsto (N_xW, V)$

Tangential Structures

Def. Let $\theta: X(d) \to BO(d)$ be a fiber bundle/fibration. Then a θ-structure on a manifold M of dim $k\le d$ is a $\boxed{\text{lift}}$

$$\begin{tikzcd} & X(d) \arrow[d, "\theta"] \\ M \arrow[ur, dashed] \arrow[r, "g_E"'] & BO(d) \end{tikzcd}$$

(classifying map)

$TM \oplus \mathbb{R}^{d-k} = E$

framed: $\theta(d): EO(d) \to BO(d)$

k-connected: $BO(d)\langle k\rangle \to BO(d)$

Ex: S^k is d-framed $\forall\, k<d$: $TS^k \oplus \mathbb{R}$

We can now define new cat. $\mathrm{Bord}^{\theta}_{\langle d-1,d\rangle}$
$\theta\langle k\rangle$

pullback

$$\rightsquigarrow \Omega^{\infty-1} MT\theta_{(d)} = \lim_{n\to\infty} \Omega^{d-1+n}\, \theta^*(U^{\perp}_{d,n})^{c}$$

Ex: $\theta = fr$

$$\begin{tikzcd}
\theta^*(U^{\perp}_{d,n}) \arrow[d] & S^n \arrow[l] \arrow[d] \\
\mathrm{Stiefel}(d,n) & * \arrow[l]
\end{tikzcd}$$

getting higher connected as $n \to \infty$

$$\Omega^{\infty} MT(fr(d)) \simeq \Omega^{\infty+d} S^{\infty}$$

we will use this computation later

Relation to Characteristic Classes of W^d-bundles

W^d closed oriented manifold.

Then $BDiff(W) \longrightarrow \Omega_\emptyset Bord_{\langle d-1,d\rangle} \simeq \Omega^\infty MTSO(d)$

$\cap$
morphs. of category

Universal characteristic classes for d-dim. mfld bundles:

$$H^*(\Omega^\infty_0 MTSO(d), \mathbb{Q}) = \Lambda^*(H^{*>0}(BSO(d),\mathbb{Q})[d])$$

Ex. $d=2$

$$H^*BSO(2) = H^*\mathbb{CP}^\infty = \mathbb{Z}[e]$$

$$H^*(\Omega^\infty MTSO(2), \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \dots]$$

$\deg \kappa_i = 2i$

$$\kappa_i = \int_W e^{i+1} \in H^{2i}(B)$$

$$\begin{tikzcd} W \arrow[r] & E \arrow[r] \arrow[d] & {} \\ & B & \end{tikzcd}$$

Question: How close to an H_*-isom is α, for given $\theta(d)$?

$d=2$. Madson-Weiss, GMTW

$$\begin{tikzcd}
B\mathrm{Diff}(S_g, D^2) \arrow[r] \arrow[dr, hook] \arrow[dd] & {|\mathrm{Bord}_{\langle d-1,d\rangle}|} \simeq \Omega^\infty MTSO(2) \arrow[d, phantom, "\simeq"] \\
 & \Omega\,|\mathrm{Bord}^\partial_{\langle d-1,d\rangle}| \\
\lim_{g\to\infty} B\mathrm{Diff}(S_g, D^2) \arrow[ur, bend left, "\simeq"] &
\end{tikzcd}$$

in H_*

$d=2K$. $K\geq 2$

$$W_g = \#_g S^K \times S^K$$

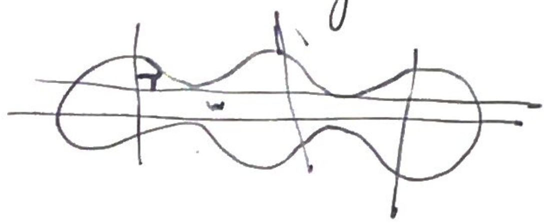
Galatius-Randal-Williams $B\mathrm{Diff}^{\theta\langle K\rangle}(W_g, D^{2K}) \simeq \Omega^\infty MT\theta\langle K\rangle$

H_*-b∂ for $* < \frac{g-4}{2}$

BUT: odd dimension tricky.

Locality & TQFT

Schematically



chop up manifold
get manifolds w/ corners

Baez-Dolan: extended TQFTs:

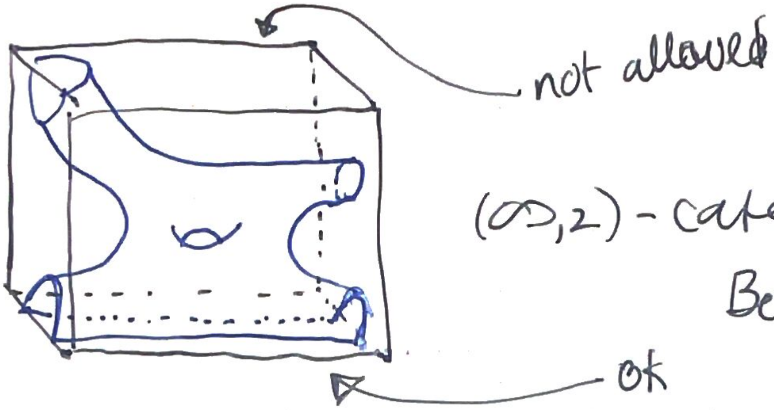
$$\mathrm{Cob}_{\langle 0,1,2,\dots,d\rangle} \xrightarrow{Z} \mathcal{V}$$ ← suitable symm. mon. d-cat

is determined by $Z(*)$ Cobordism hypothesis

Hopkins-Lurie, Lurie:

extend TQFT: $\mathrm{Bord}^{fr}_{\langle 0,\dots,d\rangle} \xrightarrow{Z} \mathcal{V}$
is det. by $Z(*)$

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{\langle 0,\dots,d\rangle}, \mathcal{V}) \simeq \mathcal{V}^{\times}$$



$(\infty,2)$-category $\mathrm{Bord}_{\langle 0,1,2\rangle}$

Madsen-Bøkstedt / Schommer-Pries

$$|\mathrm{Bord}^{or}_{\langle 0,\dots,d\rangle}| \simeq \Omega^{\infty-d}\, MTSO(d)$$

Invertible TQFTs

$$\mathrm{Bord}_{\langle 0,\dots,d\rangle} \xrightarrow{Z} \mathcal{V}_d$$

$$\cup$$

$$\pi_{\leq\infty}|\mathrm{Bord}_{\langle 0,\dots,d\rangle}| \longrightarrow \mathrm{Pic}(\mathcal{V}_d)$$

Ex:

$\vdots$	$*$
$\vdots$	$\vdots$
	$*$
$d-1$	$*$
d	A abelian gp

interpretation: $\mathcal{D}$ (∞,d)-cat $\quad |\mathcal{D}| \quad \underbrace{\pi_{\leq\infty}|\mathcal{D}|}$

∞-gpoid ~~obtained~~ by inverting all morphisms $1 \leq k \leq d$

$\Rightarrow$ invertible dim d TQFTs

$$= \mathrm{Fun}^{\otimes}(\mathrm{Bord}_{\langle 0,\dots,d\rangle}, \mathrm{Pic}(\mathcal{V}^d))$$

$$= \mathrm{Fun}^{\otimes}(\pi_{\leq\infty}|\qquad|, \mathrm{Pic}(\mathcal{V}^d))$$

$= \mathrm{Map}^{\Omega\infty}(\Sigma^d MTSO(d), \quad |\mathrm{Pic}(V^d)|)$

$\Rightarrow$ invertible framed d-diml TQFTs:

$\mathrm{Map}^{\infty}(\underbrace{\Sigma^d MT\mathrm{fr}(d)}_{\shortparallel}, |\mathrm{Pic}(V^d)|)$

$= \mathrm{Map}^{\infty}(\underbrace{\Omega^\infty S^\infty}_{\substack{\shortparallel \\ \text{free } \Omega^\infty \text{ spaces on } *}}, |\mathrm{Pic}(V^d)|) \quad \simeq |\mathrm{Pic}(V^d)|$