An Introduction to Categorification of Quantum Groups & Link Invariants II

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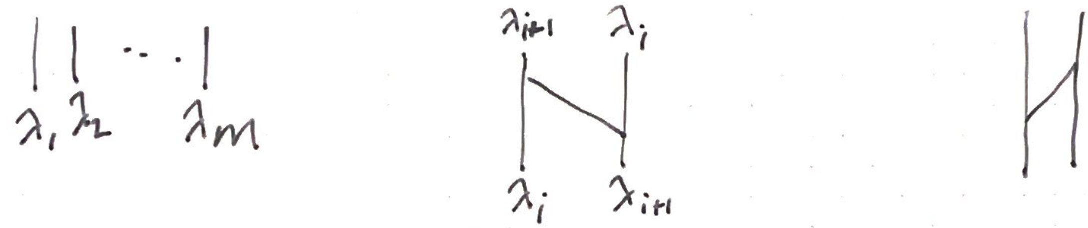
RECALL

("Idempotent form of $U_q(sl_2)$)

DEF. $\dot{\mathcal{U}} = \dot{U}_q(gl_m)$ is the $\mathbb{C}(q)$-linear category with

- objects: $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m$
- 1-morphisms generated by $1 \leq i \leq m-1$

$1_\lambda : \lambda \to \lambda$ $\qquad E_i 1_\lambda : \lambda \to \lambda + \alpha_i$ $\qquad F_i 1_\lambda : \lambda \to \lambda - \alpha_i$

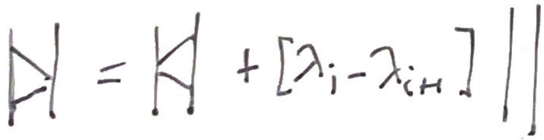


$\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$

- sl_2 relations $\boxed{\text{very important}}$

$$E_i F_i 1_\lambda = F_i E_i 1_\lambda + [\lambda_i - \lambda_{i+1}] 1_\lambda$$

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n}$$



- Serre relations

$E_i^{(2)} E_j 1_\lambda - E_i E_j E_i 1_\lambda + E_j E_i 1_\lambda$ if $j = i \pm 1$

$E_i E_j 1_\lambda = E_j E_i 1_\lambda$ $\qquad E_i F_j 1_\lambda = F_j E_i 1_\lambda$ $j \neq i$

Def. A <u>representation of $\dot{\mathcal{U}}$</u> is a functor

$$\dot{\mathcal{U}}_q(\mathfrak{gl}_m) \longrightarrow \mathbb{C}(q)\text{-Vect}$$

$$\lambda \longmapsto V_\lambda \quad \text{vect. sp.}$$

$$\left.\begin{array}{l} E_i 1_\lambda \\ F_i 1_\lambda \end{array}\right. \longmapsto \text{linear maps that preserve the relations}$$

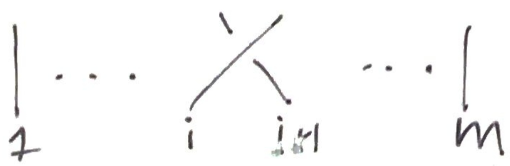
<u>Aside</u>: $V = \bigoplus_\lambda V_\lambda$

$$V_\lambda \xrightarrow{E_i 1_\lambda} V_{\lambda+\alpha_i}$$

<u>Quantum Weyl Group action</u>

Key Fact: Any $\dot{\mathcal{U}}$ rep. admits a braid group action.

B_m m-strand braid gp.

$\sigma_i =$ 

$\sigma_i^{-1} =$ 

<u>Relations</u>: $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j|>1$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

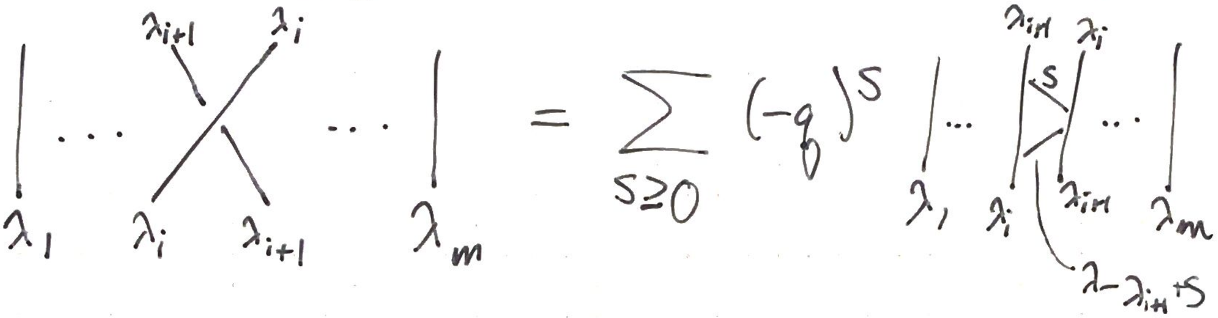
$V \circlearrowleft B_m$ $\qquad \sigma_i : V \xrightarrow{\cong} V = \bigoplus_\lambda V_\lambda$ $\quad \left(V = \bigoplus V_\lambda \right)$

$\sigma_i 1_\lambda \cdot V_\lambda \longrightarrow V_{s_i(\lambda)}$

permute labels

$\sigma_i 1_\lambda$ has an explicit formula:

$$\sigma_i 1_\lambda = \sum_{s \geq 0} (-q)^s E_i^{(s)} F_i^{(s+\lambda_i-\lambda_{i+1})} 1_\lambda$$



$$\Rightarrow = F_i^{\overbrace{\lambda_i-\lambda_{i+1}}^{\overline{\lambda_i} \text{ (assumed} \geq 0)}} 1_\lambda - q E_i F_i^{(1+\lambda_i-\lambda_{i+1})} 1_\lambda + \ldots$$

<u>Notice</u>. On any finite dimensional representation, this is a finite sum.

Link invariants live in quotients of $\mathcal{U}$

- HOMFLYPT – restrict to objects $\lambda = (\lambda_1, \dots, \lambda_m)$ w/ $\lambda_i \geq 0$

  i.e. $1_\lambda = \underset{\lambda_1}{|} \cdots \underset{\lambda_m}{|} = 0$ if some $\lambda_i < 0$

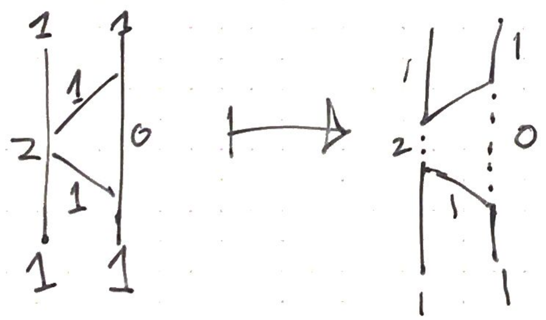
- sl_n-polynomial: restrict to λ w/ all $0 \leq \lambda_i \leq n$ Kill 1_λ outside this range.
- Alexander polynomial – further quotient of sl_n

Example. $sl(2)$ knot invariant aka Jones polynomial

Step 1. Kill lines labelled $\lambda_i < 0$ or $\lambda_i > 2$

Step 2. Erase lines labelled 0 or 2

Ex:



$\vdots$ means erase

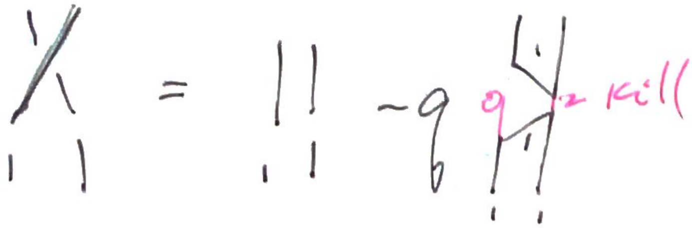
(For experts: $\lambda_i \to \Lambda^{\lambda_i}(\mathbb{C}^n) \Rightarrow \Lambda^n(\mathbb{C}^n), \Lambda^0(\mathbb{C}^n)$ are both trivial)

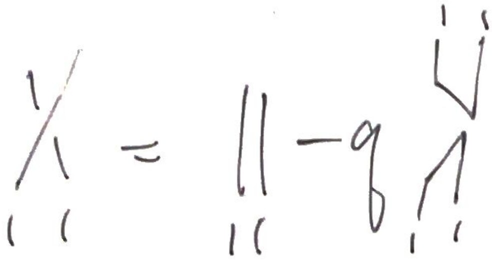
E.g. $\sigma_1 1_{(1,1)} = \sum_{s \geq 0} (-q)^s E_1^{(s)} F_1^{(s+0)} 1_\lambda$

$$= 1_\lambda - q E_1 F_1 1_{(1,1)} + q^2 E_1^{(s)} F_1^{(s)} 1_{(1,1)}$$

$$E^{(s)} F^{(s)} 1_\lambda = E^{(2)} 1_{(1,1)} F^{(2)} 1_{(1,1)}$$

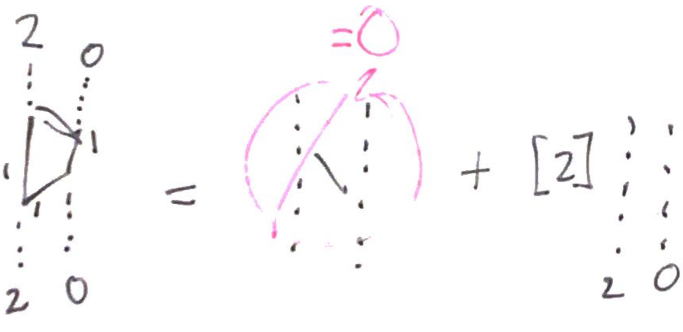
kill kill

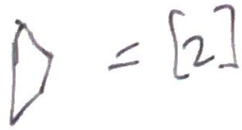




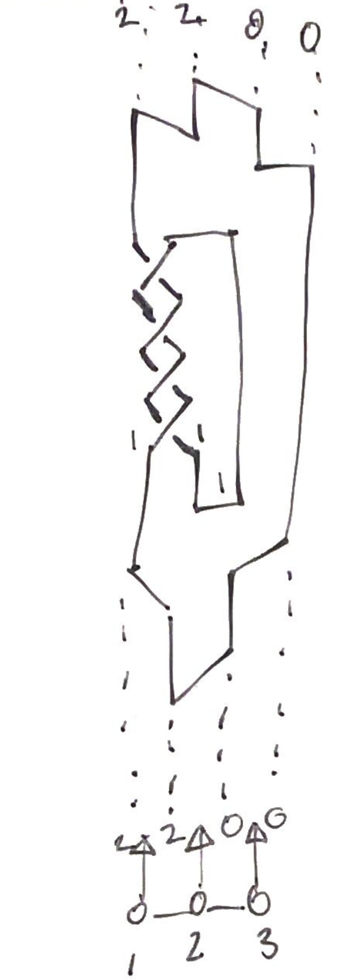
What do $\mathfrak{sl}_2$-relations look like?

$$EF 1_{(2,0)} = FE 1_{(2,0)} + [2-0] 1_{(2,0)}$$





Step 3 "ladderize" your knot



$$\longleftrightarrow \quad E_2E_1E_3E_2F_1F_2\, 1_{(2,2,0,0)}$$

$$\cap$$

$$\dot{U}(\mathfrak{gl}_4) \Big/ \begin{matrix}\lambda_i < 0 \\ \lambda_i > 2\end{matrix}$$

In the quotient where we kill λ outside $[0,2)$, this is a highest weight vector

$$E_i 1_{(2,2,0,0)} = 0 \quad \forall i$$

e.g. $E_2 1_{(2,2,0,0)} = 1_{(2,3,-1,0)} E_2$

killed $=0$

STEP 4: Compute knot invariant

$$E_2 E_1 E_3 E_2 (1_\lambda - q E_1 F_1 1_\lambda)^3 F_2 F_1 F_3 F_2 1_{(2,2,0,0)}$$

Key: PBW thm says that $\dot{U}(\mathfrak{gl}_m)$ has a basis of the form:

$$(F's)(E's)1_{(2,2,0,0)}$$

enough to use $\mathfrak{sl}_2$-relation to scoot E's right.

Process produces some q-binomials/factorials times highest weight vector $1_{(2,2,0,0)}$

i.e. Jones Polynomial:

$$f \, 1_{(2,2,0,0)}$$
$$\Uparrow$$
$$\mathbb{Z}[q, q^{-1}]$$