Higher Categorical Traces in geometric representation theory III

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Def. Let $\mathcal{A}$ be a monoidal DG category.
$\mathcal{A}$ is rigid if:

(i) $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$ is compact

(ii) $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ admits a colimit preserving right adjoint m^R

(iii) $m^R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a strict functor of $\mathcal{A} \otimes \mathcal{A}$-module categories.

Exercise. If $\mathcal{A}$ is compactly generated, then
$\mathcal{A}$ is rigid $\Leftrightarrow$
- unit cpct.
- $\otimes$ preserves cpct. objects
- every cpct. obj. has a left & right dual.

Ex. If Z is a perfect stack, then $QCoh(Z)$ is rigid.

If $\mathcal{A}$ is rigid, as a plain DG category, $\mathcal{A}$ is self-dual:

unit:

$$\mathrm{Vect} \xrightarrow{\mathbb{1}_{\mathcal{A}}} \mathcal{A} \xrightarrow{m^R} \mathcal{A} \otimes \mathcal{A}$$

counit:

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\mathrm{Hom}(\mathbb{1}_{\mathcal{A}}, -)} \mathrm{Vect}$$

Exercise. check that this defines a self-duality of $\mathcal{A}$.

Ex. $\mathcal{A} = \mathrm{QCoh}(X)$ X scheme

$\varphi : X \to X$

$F_{\mathcal{A}} = \varphi^*$ $HH(\mathcal{A}, F_{\mathcal{A}}) = ?$

$$\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(X \times X)}{\otimes} \mathrm{QCoh}(X) = \mathrm{QCoh}(\underbrace{X \times X}_{\Delta \hookrightarrow X \times X \hookleftarrow (id, \varphi)}) = \mathrm{QCoh}(X^{\varphi})$$

$\qquad \| $

$HH(\mathcal{A}, F(\mathcal{A}))$

Suppose $\mathcal{A}$ is a rigid $\otimes$ category w/

$F_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ $\otimes$-functor

$\mathcal{M}$ dualizable module category w/ compatible endofunctor

$F_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$

$\Rightarrow tr^{enh}_{(\mathcal{A}, F_{\mathcal{A}})}(\mathcal{M}, F_{\mathcal{M}}) \in \mathcal{HH}(\mathcal{A}, F_{\mathcal{A}})$

THM. $HH(\mathcal{M}, F_{\mathcal{M}}) = \mathrm{Hom}_{\mathcal{HH}(\mathcal{A}, F_{\mathcal{A}})}(\mathbb{1}, tr^{enh}(\mathcal{M}, F_{\mathcal{M}}))$

ex.
$$\begin{tikzcd} X \arrow[r] \arrow[loop below, "\varphi_X"] & Y \arrow[loop below, "\varphi_Y"] \end{tikzcd}$$
map of schemes

$QCoh(X)$ is a module for $QCoh(Y)$

We saw $\mathcal{HH}(QCoh(Y), \varphi_Y^*) = QCoh(Y^{\varphi_Y})$

compatibility $\Rightarrow$

$\psi : X^{\varphi_X} \longrightarrow Y^{\varphi_Y}$

Exercise. $tr^{enh}(QCoh(X), \varphi_X^*) = \psi_X(\mathcal{O}_{X^{\varphi_X}})$

We have a functor

$z : \mathcal{A} \longrightarrow \mathcal{HH}(\mathcal{A}, F_{\mathcal{A}}) = \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathcal{A}_F$

Also, if $\mathcal{M}$ is an $\mathcal{A}$-module category, $a \in \mathcal{A}$ gives an endofunctor

$H_a : \mathcal{M} \longrightarrow \mathcal{M} \qquad H_a(m) = a \otimes m$

Obs/exercise: $\mathrm{H}_a \circ F_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is also compatible w/ $F_{\mathcal{A}}$.

THM. $HH(\mathcal{M}, \mathrm{H}_a \circ F_{\mathcal{M}}) = \mathrm{Hom}_{HH(\mathcal{A}, F_{\mathcal{A}})}(\mathbb{1}, \iota(a) \otimes tr^{enh}(\mathcal{M}, F_{\mathcal{M}}))$

Our case of interest:

$$\begin{array}{c} \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(X)) \circlearrowright \mathrm{Shv}(\mathrm{Bun}_G(X)) \\ \cup \\ \mathcal{A}^{\otimes Y} \end{array}$$

Exercise: If $\mathcal{A}$ is a rigid $\otimes$-category then $\mathcal{A}^{\otimes Y}$ is also a rigid $\otimes$-category (Y finite CW-complex)

Suppose we have $\varphi : Y' \to Y$

$\Rightarrow \mathcal{A}^{\otimes \varphi} : \mathcal{A}^{\otimes Y'} \to \mathcal{A}^{\otimes Y}$

and suppose that $\mathcal{M}$ is a dualizeable $\mathcal{A}^{\otimes Y}$-module category with a compatible endofunctor

$$F_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}.$$

$\Rightarrow \mathrm{tr}^{\mathrm{enh}}(\mathcal{M}, F_{\mathcal{M}}) \in HH(\mathcal{A}^{\otimes \mathcal{Y}}, \mathcal{A}^{\otimes \mathcal{Y}})$

$$\begin{gathered} \| \\ \mathcal{A}^{\otimes \mathcal{Y}/\varphi} \end{gathered}$$

Recall. $\mathcal{A}^{\otimes \mathcal{Y}/\varphi}$ is rigid $\Rightarrow (\mathcal{A}^{\otimes \mathcal{Y}/\varphi})^\vee = \mathcal{A}^{\otimes \mathcal{Y}/\varphi}$

$$\begin{gathered} \| \\ \mathrm{Funct}(\mathcal{A}^{\otimes \mathcal{Y}/\varphi}, \mathrm{Vect}) \end{gathered}$$

$$\mathrm{Funct}(\mathcal{A}^{\otimes \mathcal{Y}/\varphi}, \mathrm{Vect})$$
$$\|$$
$$\{\psi_I : \mathcal{A}^{\otimes I} \longrightarrow \mathrm{Shv}_{\mathrm{lc}}((\mathcal{Y}/\varphi)^I) \text{ natural in } I \in \mathrm{Fin}\}$$

Suppose $\mathcal{S} \in \mathcal{A}^{\otimes \mathcal{Y}}$. What are the corresponding functors $\mathcal{A}^{\otimes I} \longrightarrow \mathrm{Shv}_{\mathrm{lc}}(\mathcal{Y}^I)$ for $\underline{y} \in \mathcal{Y}^I$, $\underline{a} \in \mathcal{A}^{\otimes I}$?

$\underline{y}$ gives a functor $\mathcal{A}^{\otimes I} \longrightarrow \mathcal{A}^{\otimes \mathcal{Y}}$

Let $r(\underline{y}, \underline{a}) \in \mathcal{A}^{\otimes \mathcal{Y}}$ denote the image of $\underline{a}$

Exercise. $\psi_I(\underline{a}, \underline{y}) \simeq \mathrm{Hom}_{\mathcal{A}^{\otimes \mathcal{Y}}}(\mathbb{1}, r(\underline{y}, \underline{a}) \otimes \mathcal{S})$

Apply this in our case of interest:

$$\mathcal{A}^{\otimes Y} \circlearrowright \mathcal{M}$$

$$\Updownarrow$$

$$\mathcal{A}^{\otimes I} \xrightarrow[I \in Fin]{} \mathrm{End}(\mathcal{M}) \otimes \mathrm{Shv}_{\ell\ell}(Y^I)$$

Want to compute:

$$\mathrm{Sht}_{\mathcal{M},univ} \; tr^{enh}(\mathcal{M}, F_{\mathcal{M}}) \in \mathcal{A}^{\otimes Y/\varphi}$$

GOAL. describe the corresponding functor

$$\mathcal{A}^{\otimes I} \longrightarrow \mathrm{Shv}_{\ell\ell}(Y^I)$$

Upshot. for $\underline{y} \in Y^I$, $\underline{a} \in \mathcal{A}^I$, the corresponding vector space is given by

$$\mathrm{Hom}_{\mathcal{A}^{\otimes Y}}(\mathbb{1}, r(\underline{y}, \underline{a}) \otimes \mathrm{Sht}_{\mathcal{M},univ})$$

$$\|$$

$$\boxed{HH(\mathcal{M}, H_{\underline{a}} \circ F_{\mathcal{M}})}$$

very computable $\Rightarrow$:)

Let's compute it:

$\mathcal{A} = \mathrm{Rep}(G^\vee)$

$\mathcal{M} = \mathrm{Shv}_{nilp}(\mathrm{Bun}_G)$

$$\begin{tikzcd}
 & \mathrm{Hecke}_I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times X^I
\end{tikzcd}$$

$$\left(p_{2*}(p_1^*) \otimes \mathrm{Sat}_I(a)\right)_y$$

$$\begin{tikzcd}
\mathrm{Sht}_I \arrow[rr] \arrow[dd, "f"'] & & \mathrm{Bun}_G \arrow[r] \arrow[d, "\Delta"] & * \arrow[ddl, bend left, "\mathrm{Graph}(\mathrm{Frob})"] \\
 & & \mathrm{Bun}_G \times \mathrm{Bun}_G \arrow[d, "(\mathrm{Frob}, \mathrm{id})"] & \\
\mathrm{Hecke}_{I,y} \arrow[r] \arrow[d] & \mathrm{Hecke}_{I,y} \times \mathrm{Bun}_G \arrow[r] \arrow[d] & \mathrm{Bun}_G \times \mathrm{Bun}_G & \\
\mathrm{Bun}_G \arrow[r, "\Delta"] \arrow[d] & \mathrm{Bun}_G \times \mathrm{Bun}_G & & \\
\mathrm{pt} & & &
\end{tikzcd}$$

$HH(\mathcal{M}, H_{\underline{a}} \circ \mathrm{Frob}) = H^*(f^*(\mathrm{Sat}_I(\underline{a})))$

"cohomologies on the moduli of shtukas"

$$\underline{a} \in \mathrm{Rep}(G^\vee)^{\otimes I} \xrightarrow{\mathrm{Sat}_I} \mathrm{Shv}(\mathrm{Hecke}_I)$$

geometric satake.