Higher Categorical Traces in geometric representation theory I

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LANGLANDS.

G reductive group $\rightsquigarrow G^\vee$ Langlands dual gp.

$$\left\{ \begin{matrix} G\text{-automorphic} \\ \text{forms} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} G^\vee\text{-Galois reps} \\ \Gamma \to G^\vee \end{matrix} \right\}$$

X is a smooth projective curve $/\mathbb{F}_q$

$$\Gamma = \pi_1^{\text{ét}}(X)$$

$$\begin{matrix} G\text{-automorphic} \\ \text{forms on } X \end{matrix} \simeq \mathrm{Fun}_c(\underbrace{\mathrm{Bun}_G(X)}_{\substack{\text{moduli stack of} \\ \text{principal } G\text{-bundles} \\ \text{on } X}}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

Main Constraints:

Bun_G has lots of symmetries

$x \in X$ $\exists$ an action of the "Hecke algebra"

$$\mathcal{H}_x \circlearrowright \text{Aut. forms}$$

Satake isom. $\Pi_X \simeq \{\text{class fcns on } G^\vee\}$

As a result, we have a map

$$\Pi_X \longrightarrow \text{Functions}\left(\left\{\begin{matrix} G^\vee\text{-Galois} \\ \text{reps} \end{matrix}\right\}\right)$$

evaluate at Frob_x.

One Strategy?

Extend the action of Π_X on Aut. forms to an action of "functions on Galois reps"

Grothendieck's sheaf-function correspondence:

X variety $/\mathbb{F}_q$

Let $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$

$\text{Frob}: \overline{X} \longrightarrow \overline{X}$

$\overline{X}^{\text{Frob}} = X(\mathbb{F}_q)$

Suppose $\mathcal{F} \in \text{Shv}(\overline{X})$ w/ Weil structure.

$\alpha: \mathcal{F} \longrightarrow \text{Frob}_* \mathcal{F}$

$\Rightarrow$ can build a function

$x \in X(\mathbb{F}_q)$

$\alpha : \mathcal{F}_x \to \mathcal{F}_x$

$x \longmapsto tr(\alpha : \mathcal{F}_x \to \mathcal{F}_x)$

Instead of automorphic forms, we can look for automorphic sheaves

Now we can ask the same question over $\mathbb{C}$.

X smooth projective curve$/\mathbb{C}$

$Bun_G(X)$

Over $\mathbb{C}$, we have moduli spaces of Galois reps $(\pi_1^{top}(X) \to G^\vee)$

$LocSys^{dR}_{G^\vee}(\cancel{X})$: de Rham local systems

$LocSys^{Betti}_{G^\vee}(X)$: Betti local systems.

Categorical Langlands conjecture (Beilinson-Drinfeld)

$\exists$ equivalence $Dmod(Bun_G(X)) \simeq QCoh(LocSys^{dR}_{G}(X))$.

known to be false, but not in a serious way.

Conjecture. (Ben-Zvi–Nadler)

$$\text{“}Shv_{Nilp}(Bun_G(X))\text{”} \simeq \text{“}QCoh(LocSys_{\check{G}}^{Betti}(X))$$

(False in the same way.)

Analogy of having an action of $Fun(\{\check{G}\text{-Galois reps}\}) \circlearrowright$ Autom. Forms is a categorical ation of $QCoh(LocSys_{\check{G}}(x)) \circlearrowright Shv(Bun_G)$

TAM

Heuristic. Spectral action is the categorical trace of Frobenius of the categorical action.

Recall. Suppose $\mathcal{C}$ is a symm. monoidal cat.

$c \in \mathcal{C}$ dualizable object

$$\mathbb{1}_{\mathcal{C}} \xrightarrow{\text{unit}} c \otimes c^\vee \xrightarrow{\text{counit}} \mathbb{1}_{\mathcal{C}}$$

Suppose $F : c \to c$ is an endomorphism

$$Tr(F, c) \in End_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$$

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\text{unit}"] \arrow[rrr, bend right, "Tr(F,c)"'] & c \otimes c^\vee \arrow[r, "F \otimes id"] & c \otimes c^\vee \arrow[r, "\text{counit}"] & \mathbb{1}_{\mathcal{C}} \end{tikzcd}$$

Take $\mathcal{O} = \text{DGCat}, \otimes$

Ex A alg.

$A\text{-mod} \in \mathcal{O}$ is a dualizable object

If $F: A\text{-mod} \to A\text{-mod}$ is a (colim. pres.) functor

$$\mathrm{Tr}(F, A\text{-mod}) \in \mathrm{End}(\underbrace{\mathbb{1}_{\mathcal{O}}}_{\text{Vect}})$$
$$\| $$
$$HH(A, F)$$

Ex. Suppose X is a variety$/\mathbb{C}$

$F: X \to X$

Let $\mathcal{O} = \text{Dmod}(X)$

$F = \Phi_*$

$\mathrm{tr}(\Phi_X, \text{Dmod}(X)) = ?$

counit: $\text{Dmod}(X) \otimes \text{Dmod}(X) \xrightarrow{\otimes} \text{Dmod}(X \times X)$

$\xrightarrow{\Delta^!} \text{Dmod}(X) \xrightarrow{\Gamma_{dR}} \text{Vect}$

unit: $\text{Vect} \xrightarrow{p^!} \text{Dmod}(X) \xrightarrow{\Delta_X} \text{Dmod}(X \times X) \simeq$

$\text{Dmod}(X) \otimes \text{Dmod}(X)$

$$
\begin{tikzcd}
X^{\Phi} \arrow[r] \arrow[dd] & X \arrow[r] \arrow[d, "\Delta"] \arrow[dd, bend left=60, "\mathrm{Graph}(\phi)"] & * \\
 & X \times X \arrow[d, "\mathrm{id} \times \Phi"] & \\
X \arrow[r, "\Delta"] \arrow[d] & X \times X & \\
* & &
\end{tikzcd}
$$

$$\Rightarrow \mathrm{Tr}(\Phi_*, \mathrm{Dmod}(X)) = \underbrace{\Gamma_{dR}(X^{\Phi}, \omega_{X^{\Phi}})}_{\text{Borel-Moore homology}}$$

Heuristic.

X variety $/ \mathbb{F}_q$

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\overline{X})) \cong \mathrm{Funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

In the Betti case,

$$\underbrace{\mathrm{LocSys}^{\mathrm{Betti}}_{G^\vee}(X)}_{\text{derived stack}} := \underline{\mathrm{Maps}}(\underbrace{X(\mathbb{C})}_{\text{homotopy type}}, BG^\vee)$$

i.e

functor: derived rings $\longrightarrow$ homotopy types

More generally, we can consider

$\mathrm{Maps}(Y, Z)$ $\quad$ Y homotopy type

$\quad Z$ (derived) stack

THM. (Ben Zvi - Francis - Nadler)

If Z is perfect and Y is a finite CW cplx,

$$\mathrm{QCoh}(\mathrm{Maps}(Y,Z)) = \underbrace{\int_Y \mathrm{QCoh}(Z)}_{\text{factorization homology}} =: \mathrm{QCoh}(Z)^{\otimes Y}$$

$\mathrm{QCoh}(BG^\vee) = \mathrm{Rep}(G^\vee)$

If $\mathcal{A}, \mathcal{B}$ is a symm. mon. (DG) category

$$\mathrm{Fun}^{\otimes}\left(\int_Y \mathcal{A}, \mathcal{B}\right) = \mathrm{Maps}_{\mathrm{Spc}}(Y, \mathrm{Fun}^{\otimes}(\mathcal{A}, \mathcal{B}))$$

$$\mathrm{Fun}^{\otimes}(\mathcal{A}, \underbrace{\mathcal{B} \otimes \mathrm{Shv}_{lc}(Y)}_{\text{locally constant sheaves on } Y})$$

Suppose $\Phi : \mathcal{Y} \to \mathcal{Y}$ is a map.

$\Rightarrow$ this induces a functor

$$F : \mathcal{A}^{\otimes \mathcal{Y}} \longrightarrow \mathcal{A}^{\otimes \mathcal{Y}}$$

$$Tr(F, \mathcal{A}^{\otimes \mathcal{Y}}) = End_{\mathcal{A}^{\otimes \mathcal{Y}_\Phi}}(\mathbb{1})$$

i.e. for $QCoh(Maps(\mathcal{Y}, Z))$
we get $Tr(F, QCoh(Maps(\mathcal{Y}, Z))$
$= \Gamma(\mathcal{O}, Maps(\mathcal{Y}/_F, Z))$

Expectation:

$Tr(Frob_{\mathcal{P}}, QCoh(LocSys_{G^\vee}(\overline{X}))$
$=$ functions on $LocSys_{G^\vee}(\overline{X})^{arith}$