What is a homotopy coherent $SO(3)$ action?

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MSRI: Tensor categories and TQFTs

Outline

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Local Topological Quantum Field Theories

TQFTs informally

An n-dimensional TQFT is an invariant of n-manifolds which can be computed via cutting and pasting.

Local TFTs informally

- Better to cut up along lower dimensional pieces
- Best of all cut down to points.

Formal Definition

An n-dimensional fully local topological field theory with values in a symmetric monoidal (not necessarily discrete) n-category $\mathcal{C}$ is a symmetric monoidal functor:

$$\mathcal{F} : \mathrm{Bord}_n \to \mathcal{C}.$$

Topological structures

Flavors of TFT

Different topological structures on bordisms give different TFTs:

- oriented
- spin
- framed (choice of trivialization of the tangent bundle)
- etc.

Lower dimensions

In order to glue structures, we need to pick a structure on small collars of boundaries.

For example, a 3-framing on a 1-manifold is a trivialization of $TM \oplus \mathbb{R}^2$.

Baez-Dolan Cobordism Hypothesis

Dualizable objects

A symmetric monoidal n-category is fully dualizable if every object has a dual and ever k-morphism for $1 \leq k < n$ has a left adjoint and a right adjoint.

Every symmetric monoidal n-category has a maximal fully dualizable subcategory $\mathcal{C}^{fd}$.

Theorem (Lurie(-Hopkins) 09)

$$\mathrm{TFT}^{fr}(\mathcal{C}) \tilde{\to} \mathcal{C}^{fd}$$

as <u>*spaces*</u> *via*

$$\mathcal{F} \mapsto \mathcal{F}(\mathrm{pt}_+).$$

$SO(n)$ action

$SO(n)$ action on framed TFTs

$SO(n)$ acts on the space of framed TFTs via changing the framing.

$SO(n)$ action on fully dualizable objects

By the cobordism hypothesis there's a "homotopy coherent action" of $SO(n)$ on the space of fully dualizable objects.

Warnings:

- By "space" we mean keep only the invertible morphisms. This is an ∞-groupoid, i.e. a "space" by Grothendieck's homotopy hypothesis.
- Since we've translated across a homotopy equivalence, this action is only defined homotopy coherently.

Main question

Main question

What does it mean explicitly to give a homotopy coherent action on a $SO(3)$ (discrete) 3-category?

Why we care

- Use TFTs to give applications from topology to algebra.
- Fully understand all consequences of the cobordism hypothesis in simple models.
- Important if you want to study oriented TFTs instead of framed.

Theorem (Lurie)

The homotopy fixed points of the $SO(n)$ action on Bord_n are exactly the oriented local TFTs.

Main example

Theorem (Haugseng, Scheimbauer–Johnson-Freyd)

There's a symmetric monoidal "Morita" 3-category $\mathcal{TC}$ of tensor categories, bimodule categories, bimodule maps, and bimodule natural transformations.

Theorem (DSPS 13)

Fusion categories (of non-zero global dimension) are fully dualizable objects in $\mathcal{TC}$.

Conjecture (DSPS 13)

Non-semisimple finite tensor categories are not fully dualizable, but there's still an $SO(3)$-action.

Other examples

Example

$SO(2)$-action on separable algebras. (Action is trivial. Davidovich over $\mathbb{C}$, Patrick Chu over all fields.)

Example

$SO(n)$-action on E_n-algebras (Gwilliam–Scheimbauer).
Comes from a natural action of $SO(n)$ on the E_n-operad

Example

$SO(3)$-action on cp-rigid braided tensor categories (Brochier-Jordan-S.)
Warning: this is a $(4,3)$-category.

Outline:

Algebraic topology of the 4-truncation of $BSO(3)$.

Action is $BG \to B\mathrm{Aut}(X)$. The RHS is a 4-type. Want to give a "minimal" CW description of the 4-truncation of $BSO(3)$.

Bordism model of the braided monoidal 2-groupoid coming from $BSO(3)$.

On the categorical side, this is a functor of braided monoidal 2-categories. Describe this "quantum topologically" using a Pontryagin-Thom bordism model.

Homotopy fixed points.

Perhaps a follow-up talk.

Outline

What is a homotopy coherent action?

Definition

A homotopy coherent action of G on X is a map $BG \to BAut(X)$.

Example

A homotopy coherent action of $SO(2)$ is a map $\mathbb{C}P^\infty \to BAut(X)$. $\mathbb{C}P^\infty$ has a CW description with a cell in every even degree. Giving a map out of $BSO(2)$ is giving the values of each of these cells in $BAut(X)$.

What about $SO(3)$?

$BSO(3)$ is complicated/difficult (but see Ayala-Francis). We can simplify the question using that our category is discrete.

Truncation and actions

Truncation

If X is a homotopy n-type (i.e. $\pi_k(X)$ vanishes for $k > n$). Maps to an n-type only depend on the n-truncation of the source. In a CW complex you only need to go to $n+1$ cells to describe the n-type.

Truncations and actions

Under the homotopy hypothesis, n-groupoids correspond to homtopy n-types. If X is an n-type, $Aut(X)$ is an n-type, and $BAut(X)$ is an $n+1$-type.

Example

For the $SO(2)$-action on algebras we only need to understand the 3-type of $BSO(2)$, so we only need the first two cells.

Main topological result

Theorem (DSPS)

$BSO(3)$ has the same 4-type as the CW complex with:

- *One 2-cell,*
- *One 3-cell attached to S^2 via the degree 2 map (yielding $\Sigma\mathbb{R}P^2$),*
- *One 4-cell attached to S^2 via the Hopf map (yielding $\mathbb{C}P^2$),*
- *One 5-cell attached to $\Sigma\mathbb{R}P^2$ via the generator of $\pi_4(\Sigma\mathbb{R}P^2) = \mathbb{Z}/4\mathbb{Z}$.*

Furthermore, any other such CW complex must have at least as many cells.

Definition

Let H be the CW complex with only the above 2, 3, and 4 cells.

Brief summary of argument, Part I

Homotopy groups of $BSO(3)$

$\pi_2 = \mathbb{Z}/2$, $\pi_3 = 0$, $\pi_4 = \mathbb{Z}$.

Lemma

There's a map from H to $BSO(3)$ which is an isomorphism on $\pi_{\leq 3}$

Proof.

Generator of π_2 gives a map from S^2. Relation in π_2 means this extends to $\Sigma\mathbb{R}P^2$ where it's an iso on $\pi_{\leq 2}$. Since $\mathbb{Z} = \pi_3(S^2) \to \pi_3(\Sigma\mathbb{R}P^2) = \mathbb{Z}/4$ is surjective, this maps extends to H where it is an iso on $\pi_{\leq 3}$. □

Homotopy groups of $\Sigma\mathbb{R}P^2$

Homotopy groups of $\Sigma^k\mathbb{R}P^2$ are described in Jie Wu's monograph "Homotopy theory of the suspensions of the projective plane."

Brief summary of argument, Part II

Hard part

Computing $\pi_4(H)$ is difficult. Instead calculate π_4 without the 4-cell.

Definition

$BOrp(3)$ is the homotopy fiber of $BSO(3) \to BSO \xrightarrow{p_1} K(\mathbb{Z}, 4)$.

Lemma

$BOrp(3)$ has the same 4-type as the CW complex given by attaching a 5-cell to $\Sigma\mathbb{R}P^2$ via the generator of $\pi_4(\Sigma\mathbb{R}P^2) = \mathbb{Z}/4$.

Proof.

Serre spectral sequences shows $\Sigma\mathbb{R}P^2 \to BSO(3) \to BOrp(3)$, is an iso on $H_{\leq 4}$ and thus by Hurewicz an iso on $\pi_{\leq 3}$. The 5-cell then assures that the map is an iso on π_4. □

Brief summary of argument, Part III

We have a map $BOrp(3) \to BSO(3)$ using the 2, 3, and 5 cells, and a map $BSO(2) \to BSO(3)$ using the 2 and 4 cells. So we need to prove.

Theorem

The following square is a 4-pushout.

$$\begin{array}{ccc} S^2 & \longrightarrow & BOrp(3) \\ \downarrow & & \downarrow \\ BSO(2) & \longrightarrow & BSO(3) \end{array}$$

This is again proved by a cohomology argument, but it requires some tricks beyond the Serre spectral sequence. Let P be the pushout, we check that $H_{\leq 5}(BSO(3), P) = 0$, thus $\pi_{\leq 5}(BSO(3), P) = 0$, and the map $P \to BSO(3)$ is an iso on $\pi_{\leq 4}$.

How useful is this?

Corollary

Description of the space of maps from $BSO(3)$ to any 4-type.

Is this useful?

We want to understand actions on a 3-groupoid X. To apply the above literally, we'd need to study $BAut(|X|)$ and then translate back into 3-groupoids.

Why there's hope

If X is a 3-groupoid, there's a 4-groupoid whose only point is X, whose 1-morphisms are autofunctors, etc. This 4-groupoid is equivalent to the fundamental 4-groupoid of $BAut(X)$.

HoTT aside (Buchholtz–van Doorn–Rijke)

In Homotopy Type Theory $BAut(X)$ is described directly as $\Sigma_{T:\mathcal{U}}|T = X|$.

Outline

1. Introduction
2. Algebraic topology
3. Bordism models

Goal of this section

Goal

Describe a 4-groupoid by generators and relations whose fundamental 4-groupoid is equivalent to $BSO(3)$.

But I'm afraid of 4-categories

Since $BSO(3)$ is 2-connected, we can rephrase everything in terms of the braided monoidal 2-category $\Omega^2 BSO(3)$. Braided monoidal 2-categories are less scary.

Is this rigorous?

Most of this can be made rigorous in existing models of higher categories, with some gaps to be filled later. In this paper, instead of working in a model we will work model independently but with clear statements about what our requirements are on the model.

Bordism model for $\Omega^2|S^2|_4$

Free braided monoidal 2-category

$|\Omega^2 S^2|_4$ is equivalent to the free braided monoidal 2-category with a single generating object.

What would a quantum topologist do?

String diagrams for $\Omega^2|S^2|_4$:

- Objects are normally framed 0-manifolds in $\mathbb{R}^2$
- 1-morphisms are normally framed 1-dimensional bordisms in $\mathbb{R}^3$
- 2-morphisms are normally framed 2-dimensional bordisms in $\mathbb{R}^4$ modulo normally framed 3-dimensional bordism in $\mathbb{R}^5$.
- Composition is gluing, and tensor product is disjoint union.

Pontryagin-Thom theory

What would a classical algebraic topologist do?

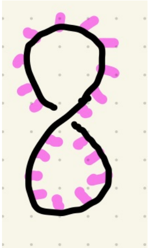
- The previous slide is a higher categorical version of the Pontryagin-Thom construction!
- Other than technical higher categorical issues, you don't need any new arguments when you're looking at appropriate loop spaces of Thom spaces.
- For CW complexes that are not Thom spaces use Buoncristiano-Rourke-Sanderson "Geometric Homology Theory."

Example (Pontryagin)

- $\pi_0(\Omega^2 S^2) = \mathbb{Z}$ generated by the positively framed point.
- $\pi_1(\Omega^2 S^2) = \mathbb{Z}$ generated by the unknot with framing 1.
- $\pi_2(\Omega^2 S^2) = \mathbb{Z}/2$ generated by the torus with non-bounding framing. (Beware of famous mistake!)

Using the bordism model

Generator of $\pi_3(S^2) = \pi_1(\Omega^2 S^2)$



$$1 \to S \otimes S^{-1} \to S^{-1} \otimes S \to 1.$$

We call this the “figure-8” construction on an object in a braided monoidal 2-category and denote it $q(S)$.

What is an $SO(2)$-action on a 3-groupoid?

Lemma

An $SO(2)$ action on a 3-groupoid consists of:

- *A natural assignment to every object 1-morphism $S_x : x \to x$..*
- *A natural assignment to every object x a 3-morphism $q(S)_x \to \mathbf{1}_{\mathbf{1}_x}$*

Note that the naturality is used in defining $q(S)$, which is:

$$\mathbf{1}_{\mathbf{1}_x} \to S_x \circ (S_x)^{-1} \to (S_x)^{-1} \circ S_x \to \mathbf{1}_{\mathbf{1}_x}.$$

Action on algebras

Example

The $SO(2)$-action on semisimple algebras consists of:

- The bimodule ${}_A\mathrm{Hom}(A,k)_A$ which is natural in the sense that for any Morita equivalence ${}_AM_B$ there's a canonical bimodule iso

$$\mathrm{Hom}(A,k) \otimes_A M \to M \otimes_B \mathrm{Hom}(B,k).$$

- The condition that the figure-8 construction on this vanishes, i.e. the following bimodule map is the identity map:

$$A \to \mathrm{Hom}(A,k) \otimes_A \mathrm{Hom}(A,k)^{-1} \to \mathrm{Hom}(A,k)^{-1} \otimes_A \mathrm{Hom}(A,k) \to A.$$

Bordism model for $\Omega^2|\Sigma\mathbb{R}P^2|_4$, Part I

Quantum topology description

Presented by one generating object X, together with a generating 1-morphism $\phi : X \to X^{-1}$ (ϕ is not its own 180-degree rotation!).

- Objects are normally framed 0-manifolds in $\mathbb{R}^2$
- 1-morphisms are normally framed 1-bordisms in $\mathbb{R}^3$ together with oriented marked points where the framing reverses.
- 2-morphisms are normally framed 2-bordisms in $\mathbb{R}^4$ together with marked oriented 1-submanifolds modulo higher bordism.

Thom space

$\Sigma\mathbb{R}P^2$ is a Thom space for $\gamma \oplus \mathbb{R}$ over S^1.

Thom space

Bordism model for $\Omega^2|\Sigma\mathbb{R}P^2|_4$, Part II

First version

d-dimensional bordisms in $\mathbb{R}^{d+2}$ endowed with a map to S^1 and an identification of the normal bundle with the pullback of $\gamma \oplus \mathbb{R}$.

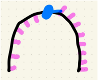
Away from the inverse image of the north pole, WLOG constant with value the south pole. Near the inverse image of the north pole we record which way around it goes.

Second version

- d-dimensional bordism M in $\mathbb{R}^{d+2}$
- A codimension 1-distinguished submanifold N.
- A normal framing of M in $\mathbb{R}^{d+2}$ away from N, which flips when it crosses N.
- A normal framing of N inside M.

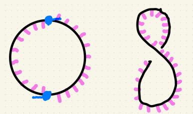
Calculations in the bordism model for $\Omega^2|\Sigma\mathbb{R}P^2|_4$, Part I

Generator of $\pi_2(\Sigma\mathbb{R}P^2)$ has order 2



Bordism M is in black. Its framing is the purple vector plus a vector out of the page. The submanifold N with its framing is in blue.

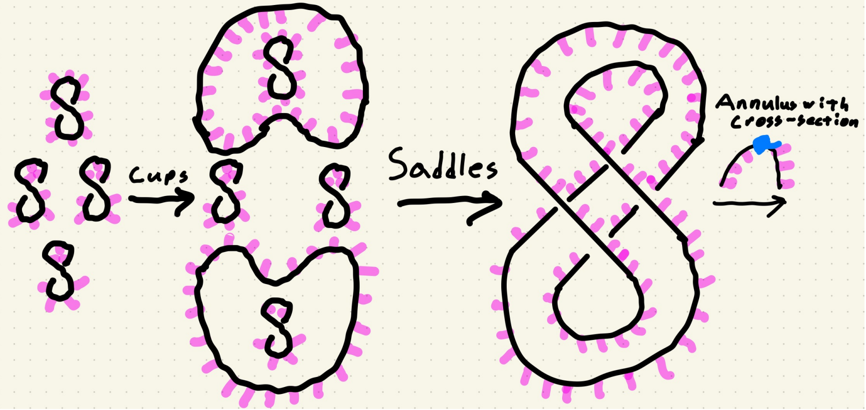
Two elements in $\pi_3(\Sigma\mathbb{R}P^2)$



First comes from $\pi_2(\mathbb{R}P^2)$, second comes from $\pi_3(S^2)$.

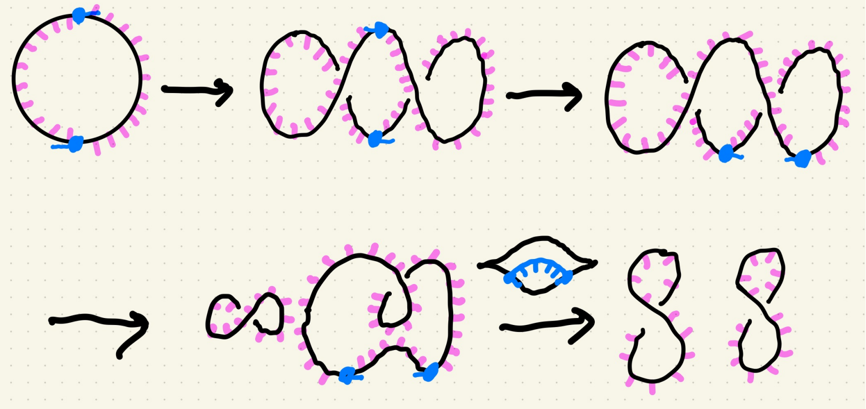
Calculations in the bordism model for $\Omega^2|\Sigma\mathbb{R}P^2|_4$, Part II

Proof that figure-8 has order 4



Calculations in the bordism model for $\Omega^2|\Sigma\mathbb{R}P^2|_4$, Part III

Proof that the image of $\pi_2(\mathbb{R}P^2)$ is 2

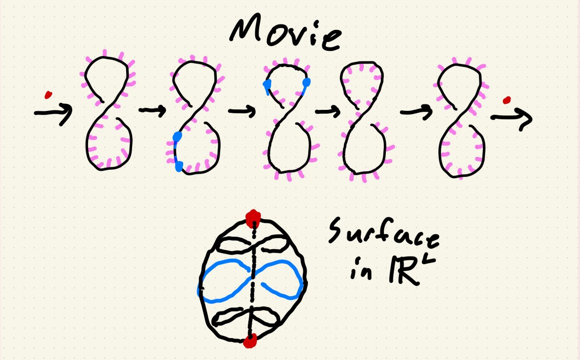


Bordism model for $\Omega^2|\mathbb{C}P^2|_4$ and $\Omega^2|H|_4$

Add codimension 2 submanifolds with a normal coframing, such that the framing of the manifold rotates appropriately around that point.

Example

$\eta q/2$ the generator of $\pi_4(H)$



Bordism model for $\Omega^2|BSO(3)|_4$

d-morphisms are d-dimensional bordisms in $\mathbb{R}^{d+2}$ which are normally framed in the bulk and decorated with:

- Normally framed codimension-1 submanifolds where the bulk framing flips.
- Normally framed codimension-2 submanifolds where the bulk framing twists around once.

modulo higher bordisms and an additional relation that connect summing with the surface from the previous slide doesn't change 2-morphisms.

What is an $SO(3)$-action?

Theorem (DSPS)

An $SO(3)$ action on a 3-groupoid consists of:

- *A natural assignment to every object x of a 1-morphism $S_x : x \to x$.*
- *A natural assignment to every object x of a 2-morphism $R_x : S_x^{-1} \to S_x$.*
- *A natural assignment to every object x of a 3-morphism $W_x : q(S)_x \to \mathbf{1}_{\mathbf{1}_x}$.*
- *An identity saying that $\frac{\eta q}{2}(S, R, W)$ vanishes.*

Finite tensor categories

Example

The $SO(3)$ action on finite tensor categories says:

- The naturally defined Serre bimodule which is $\mathcal{C}$ with the action $x \triangleright y \triangleleft z := x \otimes y \otimes z^{**}$.
- Radford's Theorem (see Etingof-Nikshych-Ostrik) trivializing the quadruple dual bimodule.
- The fact that $x \to {}^{**}(x^{**}) \to ({}^{**}x)^{**} \to x$ is trivial.
- A new theorem giving an identity satisfied by the distinguished invertible object in Radford's theorem.

Calculating in further detail

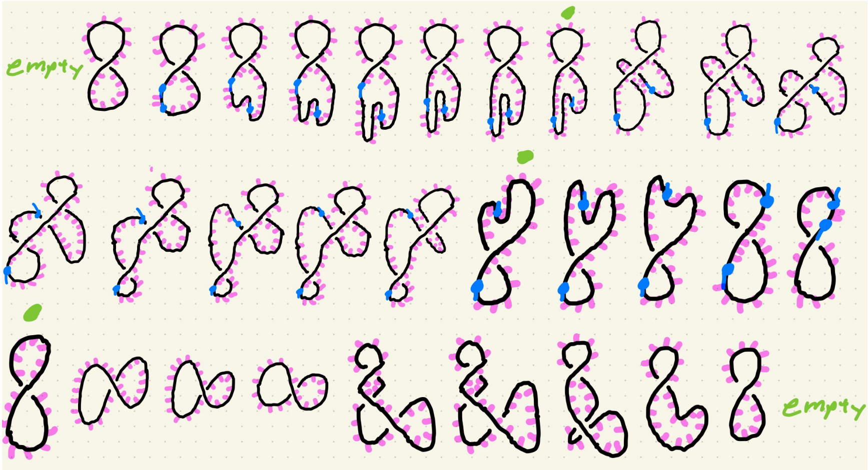
Really we need to do two things:

1. Say what naturality means in detail.
2. Describe constructions like q and $(\eta q)/2$ in terms of defining morphisms.

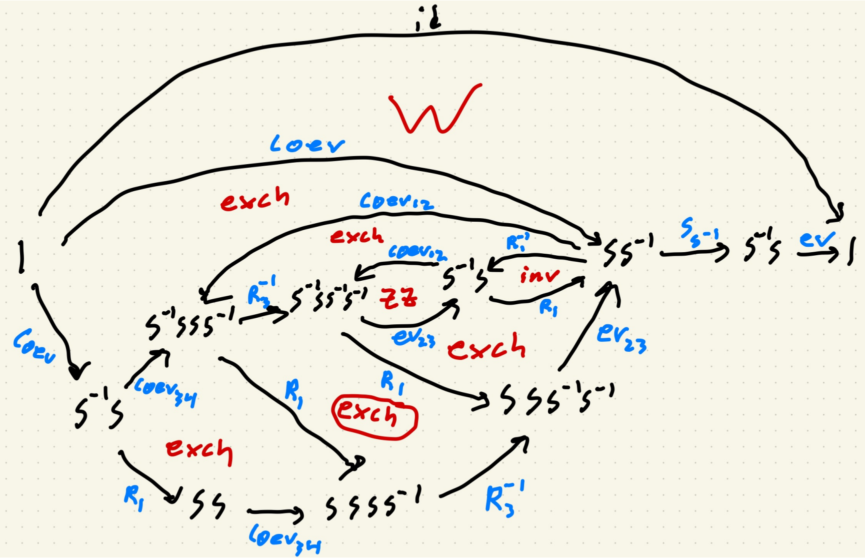
I'll end by demonstrating how the latter is done

"Morse" version of $(\eta q)/2$

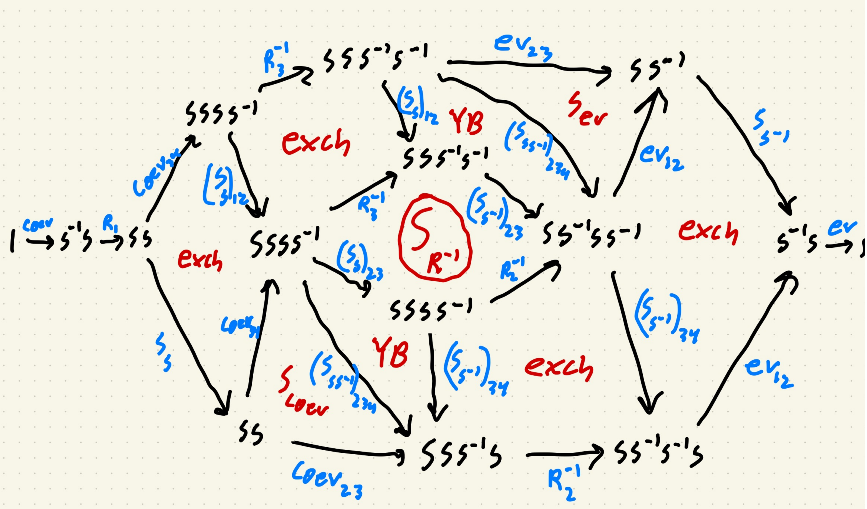
Each step is a basic morphism



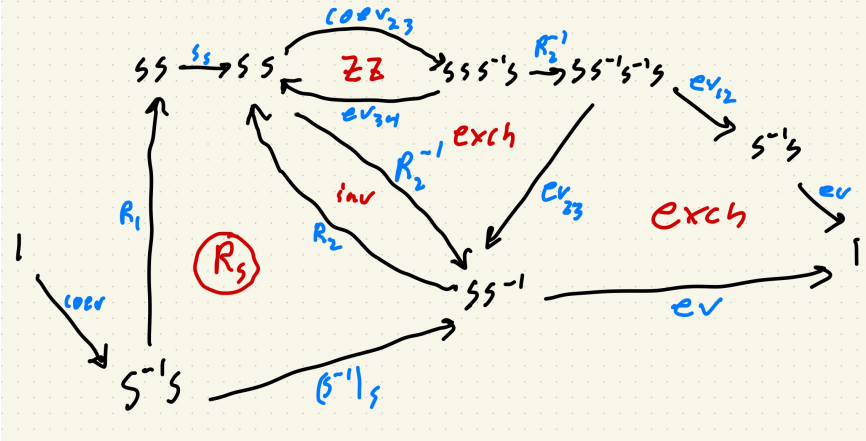
Commutative diagram, Part I



Commutative diagram, Part II



Commutative diagram, Part III



Commutative diagram, Part IV