Tensor 2-categories of Hall modules

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TENSOR 2-CATEGORIES: WHY?

Tensor categories $\leftsquigarrow\!\!\rightsquigarrow$ 3D TQFT

Douglas, Schommer-Pries, Snyder: 2013

Fusion categories: fully dualizable in 3-category $\mathcal{M}\mathrm{or}\,(\mathcal{C}\mathrm{at})$
$\overrightarrow{Cob.Hyp.}$ fully extended 3D TQFT (Turaev-Viro)

Tensor 2-categories: Why?

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Tensor 2-categories $\leftrightsquigarrow$ 4D TQFT

Douglas, Reutter: 2018

Fusion 2-categories: 4D state-sum invariant
Conjecture 1: Extends to fully extended 4D TQFT
Conjecture 2: Fully dualizable objects in 4-category $\mathcal{M}\mathrm{or}\,(\mathcal{C}\mathrm{at}_2)$

Tensor 2-categories: How?

Douglas, Reutter

Fusion 2-category $:= \mathcal{M}\mathrm{od}_{\mathcal{A}}$ + tensor structure + rigidity
$\mathcal{A}$ multifusion category

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$\mathcal{M}\mathrm{od}_{\mathcal{A}}$

2-category of finite semisimple categories with $\mathcal{A}$-action

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$\mathcal{A}$ multifusion category

$\mathcal{M}\mathrm{od}_{\mathcal{A}}$

2-category of finite semisimple categories with $\mathcal{A}$-action

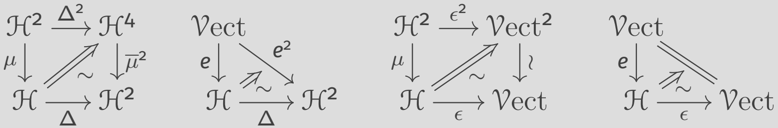
Tensor structure: Bimonoidal categories

Succinct: $\mathcal{H} \in \mathcal{C}\mathrm{oalg}\,(\mathcal{A}\mathrm{lg}(\mathcal{C}\mathrm{at}))$
$\mathcal{M}\mathrm{od}_{\mathcal{H}}$ inherits tensor structure: $\mathcal{M}, \mathcal{N} \in \mathcal{M}\mathrm{od}_{\mathcal{H}}$, $\mathcal{M} \boxtimes \mathcal{N} \in \mathcal{M}\mathrm{od}_{\mathcal{H}}$

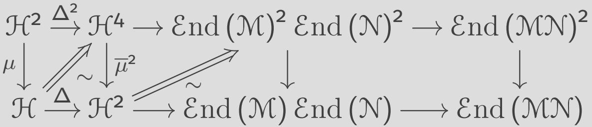
$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H}^2 \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{M}) \boxtimes \mathcal{E}\mathrm{nd}\,(\mathcal{N}) \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{M} \boxtimes \mathcal{N})$$

Bimonoidal categories and their modules in detail

Tensor category $\mathcal{H}$ with compatible comonoidal structure:



Condition for $\mathcal{H}$-action on $\mathcal{M} \boxtimes \mathcal{N}$:



Bimonoidal categories and their modules in detail

Tensor category $\mathcal{H}$ with compatible comonoidal structure:

$$\begin{array}{ccc} \mathcal{H}^2 & \xrightarrow{\Delta^2} & \mathcal{H}^4 \\ {\scriptstyle\mu}\downarrow & \nearrow\!\!\!\nearrow \sim & \downarrow{\scriptstyle\overline{\mu}^2} \\ \mathcal{H} & \xrightarrow[\Delta]{} & \mathcal{H}^2 \end{array} \qquad \begin{array}{ccc} \mathcal{V}\mathrm{ect} & & \\ {\scriptstyle e}\downarrow & \searrow^{e^2} & \\ \mathcal{H} & \xrightarrow[\Delta]{\nearrow\!\!\!\nearrow \sim} & \mathcal{H}^2 \end{array} \qquad \begin{array}{ccc} \mathcal{H}^2 & \xrightarrow{\epsilon^2} & \mathcal{V}\mathrm{ect}^2 \\ {\scriptstyle\mu}\downarrow & \nearrow\!\!\!\nearrow \sim & \downarrow\wr \\ \mathcal{H} & \xrightarrow[\epsilon]{} & \mathcal{V}\mathrm{ect} \end{array} \qquad \begin{array}{ccc} \mathcal{V}\mathrm{ect} & & \\ {\scriptstyle e}\downarrow & \diagdown\!\!\diagdown & \\ \mathcal{H} & \xrightarrow[\epsilon]{\nearrow\!\!\!\nearrow \sim} & \mathcal{V}\mathrm{ect} \end{array}$$

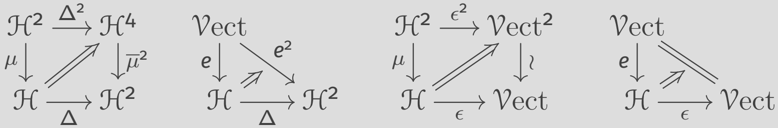
Condition for $\mathcal{H}$-action on $\mathcal{M} \boxtimes \mathcal{N}$:

$$\begin{array}{ccccccc} \mathcal{H}^2 & \xrightarrow{\Delta^2} & \mathcal{H}^4 & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{M})^2\,\mathcal{E}\mathrm{nd}\,(\mathcal{N})^2 & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{M}\mathcal{N})^2 \\ {\scriptstyle\mu}\downarrow & \nearrow\!\!\!\nearrow \sim & \downarrow{\scriptstyle\overline{\mu}^2} & \nearrow\!\!\!\nearrow \sim & \downarrow & & \downarrow \\ \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H}^2 & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{M})\,\mathcal{E}\mathrm{nd}\,(\mathcal{N}) & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{M}\mathcal{N}) \end{array}$$

These 2-morphisms need not be invertible!

Lax bimonoidal categories and lax modules

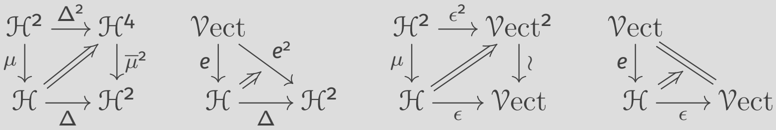
- Tensor category $\mathcal{H}$ with laxly compatible comonoidal structure:



- $\mathcal{H} \to \mathcal{E}\mathrm{nd}(\mathcal{M})$ lax monoidal functor.
  Tensor 2-category of lax modules over $\mathcal{H}$, $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{H}}$.

Lax bimonoidal categories and lax modules

- Tensor category $\mathcal{H}$ with laxly compatible comonoidal structure:



- $\mathcal{H} \to \mathcal{E}\mathrm{nd}\,(\mathcal{M})$ lax monoidal functor.
  Tensor 2-category of lax modules over $\mathcal{H}$, $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{H}}$.

Question: How to construct (lax) bimonoidal categories?

Andruskiewitsch, Natale: 2003

Vacant double category: unique square for each t and r

$$A = l \, \overset{t}{\underset{b}{\square}} \, r$$

$\rightsquigarrow$ Weak bialgebra spanned by squares
Product: Vertical path algebra $(A, B) \mapsto \genfrac{}{}{0pt}{}{A}{B}$
Coproduct: Horizontal decompositions $A \mapsto \sum_{A=BC} B \otimes C$

Hall algebras

Finitary exact category $\mathcal{A}$
$\rightsquigarrow$ (almost) bialgebra spanned by $\{objects\}/iso$
Enumerating extensions
Product: $[a] \cdot [c] = \sum_{[b]} \#(a \to b \to c)\,[b]$
Coproduct: $\Delta([b]) = \sum_{[a],[c]} \#(a \to b \to c)\,[a] \otimes [c]$
Green's Theorem: $\mathcal{A}$ hereditary $\Rightarrow$ braided bialgebra

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Questions: Why are they (co)associative?
Why are they bialgebras?

Every simplicial space $\mathcal{X}_\bullet \in \mathcal{S}_\Delta$ defines algebraic structure:
Product as span: $\mathcal{X}_1^2 \xleftarrow{(d_2,d_0)} \mathcal{X}_2 \xrightarrow{d_1} \mathcal{X}_1$

Associative?

[1]Dyckerhoff–Kapranov, Galvéz-Carrillo–Kock–Tonks, Stern

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Associative? Generally no!

$$\begin{array}{ccccc} & & \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 & & \\ & \swarrow & \uparrow & \searrow & \\ \mathcal{X}_1^3 & \longleftarrow & \mathcal{X}_3 & \longrightarrow & \mathcal{X}_1 \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathcal{X}_2 \tilde{\times}_{\mathcal{X}_1} \mathcal{X}_2 & & \end{array}$$

Theorem (P, 2017)

$\{$*Simplicial spaces*$\} \simeq \{$*Lax algebras in* $\mathcal{S}\mathrm{pan}_2\,\mathcal{S}\}$
$\mathcal{X} \mapsto \alpha_{\mathcal{X}}$ *"Universal Hall algebra"*

[1] Dyckerhoff–Kapranov, Galvéz-Carrillo–Kock–Tonks, Stern

2-Segal spaces

Every simplicial space $\mathcal{X}_\bullet \in \mathcal{S}_\Delta$ defines algebraic structure:

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Theorem (P, 2017)

$\{$*Simplicial spaces*$\} \simeq \{$*Lax algebras in* $\mathrm{Span}_2\, \mathcal{S}\}$

$\mathcal{X} \mapsto \alpha_{\mathcal{X}}$ *"Universal Hall algebra"*

Associative if and only if $\mathcal{X}$ *satisfies 2-Segal conditions:*[1]

$\mathcal{X}_n \simeq \mathcal{X}_2 \times_{\mathcal{X}_1} \cdots \times_{\mathcal{X}_1} \mathcal{X}_2$

[1]Dyckerhoff–Kapranov, Galvéz-Carrillo–Kock–Tonks, Stern

Bisimplicial spaces and bialgebras

Every bisimplicial space $\mathcal{X}_{\bullet,\bullet}$ defines bialgebraic structure:

Product μ as span: $\mathcal{X}_{1,1}^2 \xleftarrow[(d_2^v, d_0^v)]{} \mathcal{X}_{2,1} \xrightarrow[d_1^v]{} \mathcal{X}_{1,1}$

Coproduct Δ as span: $\mathcal{X}_{1,1} \xleftarrow[d_1^h]{} \mathcal{X}_{1,2} \xrightarrow[(d_2^h, d_0^h)]{} \mathcal{X}_{1,1}^2$

By above, (co)associative $\Leftrightarrow$ $\mathcal{X}$ is a double 2-Segal space

Theorem (P, 2017)

$\{$*Double 2-Segal spaces*$\} \to \{$*Lax bialgebras in* $\mathcal{S}\mathrm{pan}_2\,\mathcal{S}\}$

$\mathcal{X} \mapsto \beta_{\mathcal{X}}$ *"Universal Hall bialgebra"*

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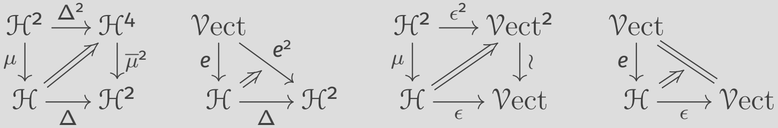
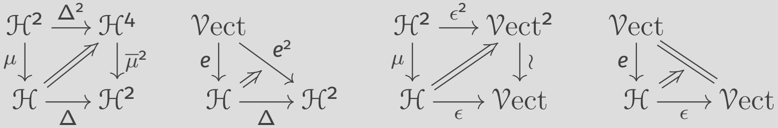
$\{$*Double 2-Segal spaces*$\} \to \{$*Lax bialgebras in* $\mathcal{S}\mathrm{pan}_2\,\mathcal{S}\}$
$\mathcal{X} \mapsto \beta_{\mathcal{X}}$ *"Universal Hall bialgebra"*
A bialgebra if and only if $\mathcal{X}_{\bullet,\bullet}$ *satisfies series of pullback conditions*

Recall: lax bimonoidal category



Recall: lax bimonoidal category



Lax bialgebra from double 2-Segal space

$$
\begin{array}{ccccc}
\mathcal{X}_{11}^2 & \leftarrow & \mathcal{X}_{12}^2 & \rightarrow & \mathcal{X}_{11}^4 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{22} & \rightarrow & \mathcal{X}_{21}^2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}_{11}^2
\end{array}
\qquad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{02} & \rightarrow & \mathcal{X}_{01}^2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}_{11}^2
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{X}_{11}^2 & \leftarrow & \mathcal{X}_{10}^2 & \rightarrow & * \\
\uparrow & & \uparrow & & \| \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{20} & \rightarrow & * \\
\downarrow & & \downarrow & & \| \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
\qquad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \| \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{00} & \rightarrow & * \\
\downarrow & & \downarrow & & \| \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
$$

Bialgebra $\Leftrightarrow$ Top right and bottom left squares are pullbacks

$$
\begin{array}{ccccc}
\mathcal{X}_{11}^{2} & \leftarrow & \mathcal{X}_{12}^{2} & \rightarrow & \mathcal{X}_{11}^{4} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{22} & \rightarrow & \mathcal{X}_{21}^{2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}_{11}^{2}
\end{array}
\quad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{02} & \rightarrow & \mathcal{X}_{01}^{2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}_{11}^{2}
\end{array}
\quad
\begin{array}{ccccc}
\mathcal{X}_{11}^{2} & \leftarrow & \mathcal{X}_{10}^{2} & \rightarrow & * \\
\uparrow & & \uparrow & & \| \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{20} & \rightarrow & * \\
\downarrow & & \downarrow & & \| \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
\quad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \| \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{00} & \rightarrow & * \\
\downarrow & & \downarrow & & \| \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
$$

Pullback conditions in detail

1. $\mathcal{X}_{00} \simeq *$
2. $\mathcal{X}_{0n} \simeq \mathcal{X}_{01}^{n}$ and $\mathcal{X}_{n0} \sim \mathcal{X}_{10}^{n}$

$$
\begin{array}{ccccc}
\mathcal{X}^2_{11} & \leftarrow & \mathcal{X}^2_{12} & \rightarrow & \mathcal{X}^4_{11} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{22} & \rightarrow & \mathcal{X}^2_{21} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}^2_{11}
\end{array}
\quad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{02} & \rightarrow & \mathcal{X}^2_{01} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{12} & \rightarrow & \mathcal{X}^2_{11}
\end{array}
\quad
\begin{array}{ccccc}
\mathcal{X}^2_{11} & \leftarrow & \mathcal{X}^2_{10} & \rightarrow & * \\
\uparrow & & \uparrow & & \Vert \\
\mathcal{X}_{21} & \leftarrow & \mathcal{X}_{20} & \rightarrow & * \\
\downarrow & & \downarrow & & \Vert \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
\quad
\begin{array}{ccccc}
* & = & * & = & * \\
\uparrow & & \uparrow & & \Vert \\
\mathcal{X}_{01} & \leftarrow & \mathcal{X}_{00} & \rightarrow & * \\
\downarrow & & \downarrow & & \Vert \\
\mathcal{X}_{11} & \leftarrow & \mathcal{X}_{10} & \rightarrow & *
\end{array}
$$

Pullback conditions in detail

1. $\mathcal{X}_{00} \simeq *$
2. $\mathcal{X}_{0n} \simeq \mathcal{X}^n_{01}$ and $\mathcal{X}_{n0} \sim \mathcal{X}^n_{10}$
3. Three other independent squares:

$$
\begin{array}{ccc}
\mathcal{X}_{22} & \rightarrow & \mathcal{X}_{21} \\
\downarrow & & \downarrow \\
\mathcal{X}_{12} & \rightarrow & \mathcal{X}_{11}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{X}_{20} & \rightarrow & \mathcal{X}_{21} \\
\downarrow & & \downarrow \\
\mathcal{X}_{10} & \rightarrow & \mathcal{X}_{11}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{X}_{22} & \rightarrow & \mathcal{X}^2_{21} \\
\downarrow & & \downarrow \\
\mathcal{X}^2_{12} & \rightarrow & \mathcal{X}^4_{11}
\end{array}
$$

Number these 3(i), 3(ii), 3(iii)

$$\begin{array}{ccc} \mathcal{X}_{22} \to \mathcal{X}_{21} & \mathcal{X}_{20} \to \mathcal{X}_{21} & \mathcal{X}_{22} \to \mathcal{X}_{21}^{2} \\ \downarrow \quad\quad \downarrow & \downarrow \quad\quad \downarrow & \downarrow \quad\quad \downarrow \\ \mathcal{X}_{12} \to \mathcal{X}_{11} & \mathcal{X}_{10} \to \mathcal{X}_{11} & \mathcal{X}_{12}^{2} \to \mathcal{X}_{11}^{4} \end{array}$$

Vacant double category

$\mathcal{X}_{\bullet,\bullet} \in \mathcal{S}\mathrm{et}_{\Delta}$ is bisimplicial nerve of double category

- Theorem:[2] Every Segal space is 2-Segal
  $\Rightarrow$ (co)associativity
- Condition 3(iii) $\Leftrightarrow$ interchange law in double category
  Conditions 3(i) and 3(ii) follow from vacancy condition
  $\Rightarrow$ Δ is multiplicative
- Conditions 1 and 2 hold $\Leftrightarrow$ single object
  $\Rightarrow$ weak bialgebra unless only has a single object

[2] Dyckerhoff, Kapranov 2012; Galvéz-Carrillo, Kock, Tonks 2014

Hall algebra

$\mathcal{X}_{\bullet,\bullet} = S_{\bullet,\bullet}(\mathcal{A})$ is the Iterated Waldhausen S-construction:
$S_{n,k} =$ "Space of length n flags of length k flags"
$S_{n,k} = S_{k,n}$, $S_{0,k} \simeq * \simeq S_{k,0}$, $S_{1,1} = \mathcal{O}\mathrm{bj}\,(\mathcal{A})$, $S_{2,1} = \mathcal{S}\mathrm{ES}\,(\mathcal{A})$

- Theorem:[3] $S_{\bullet,\bullet}$ is double 2-Segal
  $\Rightarrow$ (co)associativity
- Conditions 1 and 2 immediate
- Conditions 3(i) and 3(ii) follow from universal properties
- Condition 3(iii) holds $\Leftrightarrow \mathcal{A} = 0$
  $\Rightarrow$ braided bialgebra

[3] Dyckerhoff, Kapranov 2012; Galvéz-Carrillo, Kock, Tonks 2014

Transfer to categories

Bimonoidal categories: Must transfer examples from spans to categories
Have symmetric monoidal functor[4] $\Lambda : \mathcal{S}\mathrm{pan}_2\,\mathcal{S} \to \mathcal{C}\mathrm{at}$
$X \mapsto \mathcal{L}\mathrm{oc}\,(X)$. Pull-push on morphisms

Theorem

- *Every double 2-Segal space $\mathcal{X}$ defines a lax bimonoidal category $\Lambda\,(\mathcal{X})$*
  $\Rightarrow$ *defines a tensor 2-category* $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}} := \mathcal{L}\mathcal{M}\mathrm{od}_{\Lambda(\mathcal{X})}$
- *If $\mathcal{X}$ satisfies the pullback conditions then $\Lambda\,(\mathcal{X})$ is bimonoidal*
  $\Rightarrow$ *defines a tensor 2-category* $\mathcal{M}\mathrm{od}_{\mathcal{X}} := \mathcal{M}\mathrm{od}_{\Lambda(\mathcal{X})}$

[4]Morton 2010; Walde 2016. Finiteness assumptions. Special case of ambidexterity à la Hopkins–Lurie

Hall algebra: $\mathcal{X}_{\bullet,\bullet} = S_{\bullet,\bullet}(\mathcal{A})$

$\mathcal{M} \in \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}}$ consists of
$a \in \mathcal{A} \rightsquigarrow \rho_a : \mathcal{M} \to \mathcal{M}$ and $a \to b \to c \rightsquigarrow \rho_c \circ \rho_a \Rightarrow \rho_b$

Our two bialgebras a third time

Hall algebra: $\mathcal{X}_{\bullet,\bullet} = S_{\bullet,\bullet}(\mathcal{A})$

$\mathcal{M} \in \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}}$ consists of

$a \in \mathcal{A} \rightsquigarrow \rho_a : \mathcal{M} \to \mathcal{M}$ and $a \to b \to c \rightsquigarrow \rho_c \circ \rho_a \Rightarrow \rho_b$

+ action of automorphisms in $\mathcal{A}$ on ρ_a's

Compatibility with automorphisms and exact sequences

Our two bialgebras a third time

Hall algebra: $\mathcal{X}_{\bullet,\bullet} = S_{\bullet,\bullet}(\mathcal{A})$

$\mathcal{M} \in \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}}$ consists of
$a \in \mathcal{A} \rightsquigarrow \rho_a : \mathcal{M} \to \mathcal{M}$ and $a \to b \to c \rightsquigarrow \rho_c \circ \rho_a \Rightarrow \rho_b$
+ action of automorphisms in $\mathcal{A}$ on ρ_a's
Compatibility with automorphisms and exact sequences
Example: $\mathcal{A} = \mathcal{V}\mathrm{ect}_{\mathbb{F}_q}$ and $\mathcal{M} = \mathcal{V}\mathrm{ect}$

Hall algebra: $\mathcal{X}_{\bullet,\bullet} = S_{\bullet,\bullet}(\mathcal{A})$

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+ action of automorphisms in $\mathcal{A}$ on ρ_a's
Compatibility with automorphisms and exact sequences
Example: $\mathcal{A} = \mathcal{V}\mathrm{ect}_{\mathbb{F}_q}$ and $\mathcal{M} = \mathcal{V}\mathrm{ect}$
Data consists of $\mathrm{GL}_n(\mathbb{F}_q)$-rep V_n $\forall n$ and
$V_m \otimes V_n \to V_{n+m}$ ($P_{n,m}$-equivariant)

Double Segal groupoids

Generalize double category to double Segal groupoid $\mathcal{X}_{\bullet,\bullet} \in \mathcal{G}\mathrm{rpd}$

[5] Actually, modulo subtleties in definition of sphericality

Double Segal groupoids

Generalize double category to double Segal groupoid $\mathcal{X}_{\bullet,\bullet} \in \mathcal{G}\mathrm{rpd}$
Then $\mathcal{X}_{\bullet,1}$ is the vertical $(2,1)$-category
Module 2-categories: $\mathcal{M}\mathrm{od}_{\mathcal{X}} = \mathcal{F}\mathrm{un}\,(\mathcal{X}_{\bullet,1}, \mathcal{C}\mathrm{at})$ and
$\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}} = \mathcal{F}\mathrm{un}_{\mathrm{lax}}\,(\mathcal{X}_{\bullet,1}, \mathcal{C}\mathrm{at})$
Tensor structure from horizontal decomposition

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Double Segal groupoids

Generalize double category to double Segal groupoid $\mathcal{X}_{\bullet,\bullet} \in \mathcal{G}\mathrm{rpd}$
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Module 2-categories: $\mathcal{M}\mathrm{od}_{\mathcal{X}} = \mathcal{F}\mathrm{un}\left(\mathcal{X}_{\bullet,1}, \mathcal{C}\mathrm{at}\right)$ and
$\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{X}} = \mathcal{F}\mathrm{un}_{\mathrm{lax}}\left(\mathcal{X}_{\bullet,1}, \mathcal{C}\mathrm{at}\right)$
Tensor structure from horizontal decomposition

Theorem (P, 2020[5])

Let $\mathcal{X}$ double Segal groupoid satisfying:
1) The bialgebra conditions;
2) Each $\mathcal{X}_{n,k}$ is a finite groupoid; and,
3) Each row and column is a groupoid object à la Lurie.
Then $\mathcal{M}\mathrm{od}_{\mathcal{X}}$ is a spherical fusion 2-category

[5]Actually, modulo subtleties in definition of sphericality