Low-dimensional G-bordism and G-modular TQFTs

① Intro part 1

② Intro part 2

③ Intro part 3

④ O-modular case (unoriented)

⑤ Spin-modular case

⑥ Pin_--modular case

⑦ $(Spin \setminus O)$-modular case (unoriented vortices)

These slides available at canyon23.net/slides

Low-dimensional G-bordism and G-modular TQFTs

① Intro part 1
② Intro part 2
③ Intro part 3

[W, 2006]

1612.07792
Barkeshli
Bonderson
Cheng
Jian
W

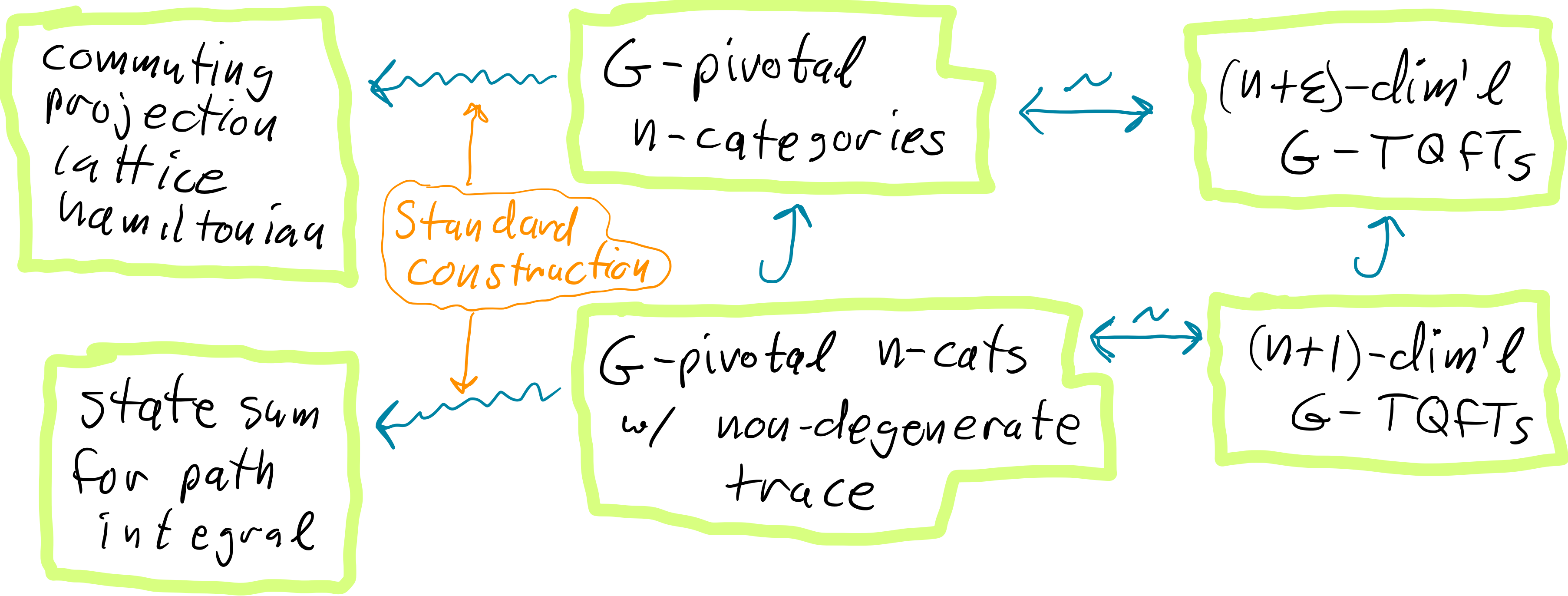
④ O-modular case (unoriented)
⑤ Spin-modular case
⑥ Pin_--modular case

1709.01941
Aasen
Lake
W
(spin, but not modular)

⑦ (Spin\O)-modular case (unoriented vortices)

Intro part 1: Well-behaved TQFTs

G = SO, O, Spin, $\mathrm{Pin}_\pm$, vortex/char



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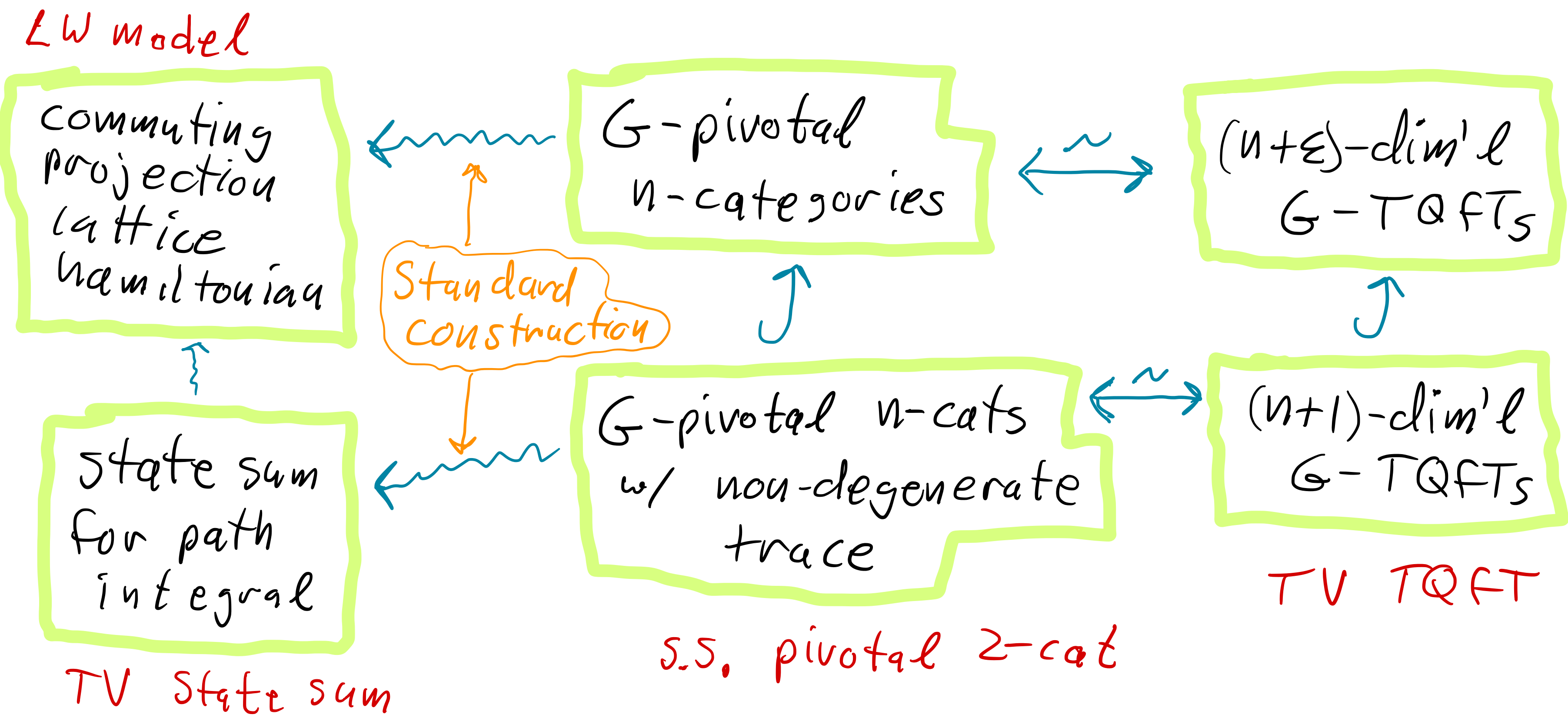


LW model

commuting projection lattice hamiltonian

G-pivotal n-categories

$(n+\varepsilon)$-dim'l G-TQFTs

Standard construction

state sum for path integral

G-pivotal n-cats w/ non-degenerate trace

$(n+1)$-dim'l G-TQFTs

TV State sum

s.s. pivotal 2-cat

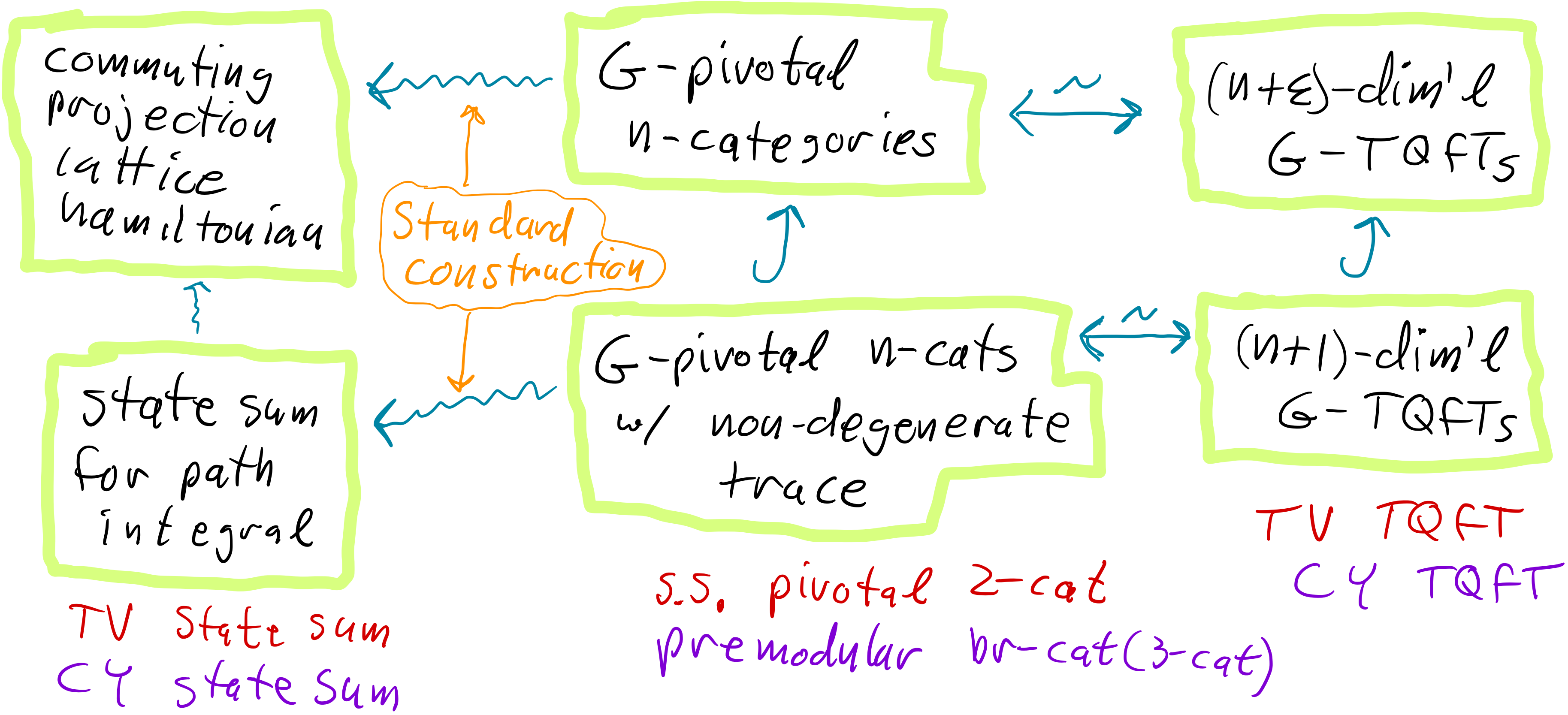
TV TQFT

Intro part 1: Well-behaved TQFTs

G = SO, O, Spin, Pin±, vortex/char

WW model

LW model



commuting projection lattice hamiltonian

G-pivotal n-categories

(n+ε)-dim'l G-TQFTs

Standard construction

state sum for path integral

G-pivotal n-cats w/ non-degenerate trace

(n+1)-dim'l G-TQFTs

TV TQFT

CY TQFT

TV State sum

CY state sum

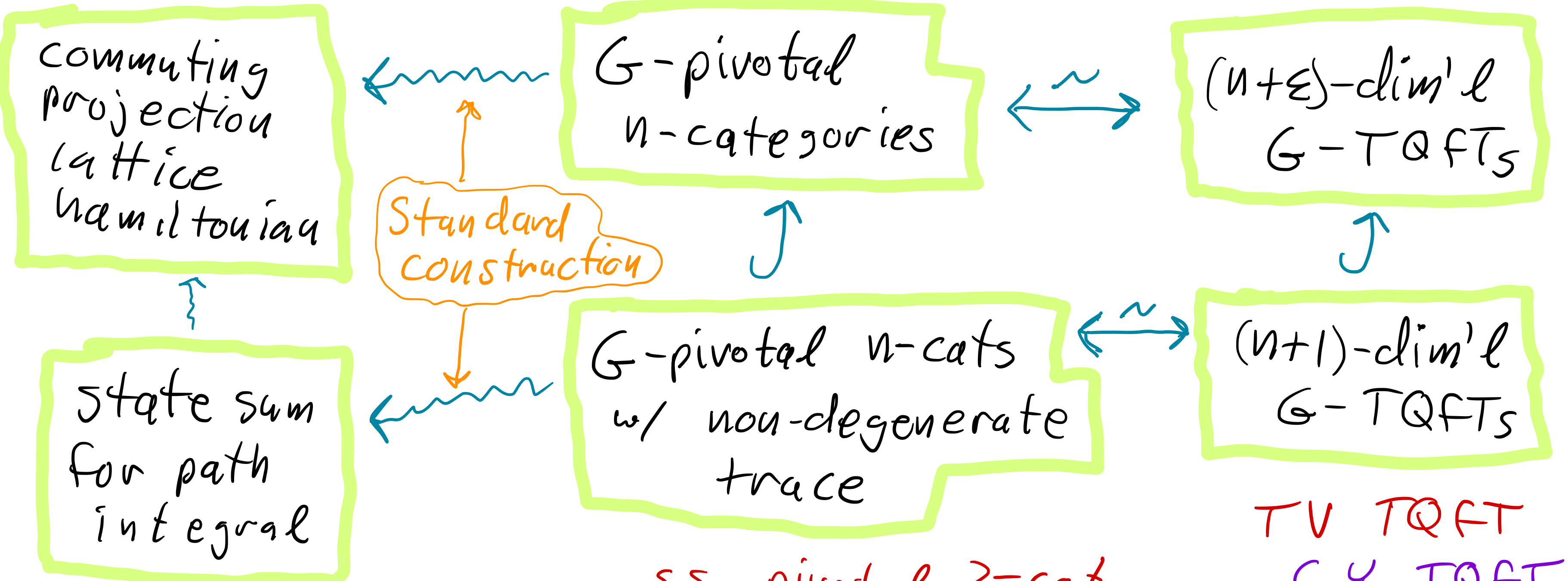
S.S. pivotal 2-cat

premodular br-cat (3-cat)

Intro part 1: Well-behaved TQFTs

G = SO, O, Spin, Pin$_\pm$, vortex/char

Kitaev finite group model
WW model
LW model



commuting projection lattice Hamiltonian

Standard construction

G-pivotal n-categories

~

(n+ε)-dim'l G-TQFTs

state sum for path integral

G-pivotal n-cats w/ non-degenerate trace

~

(n+1)-dim'l G-TQFTs

TV state sum
CY state sum
DW state sum

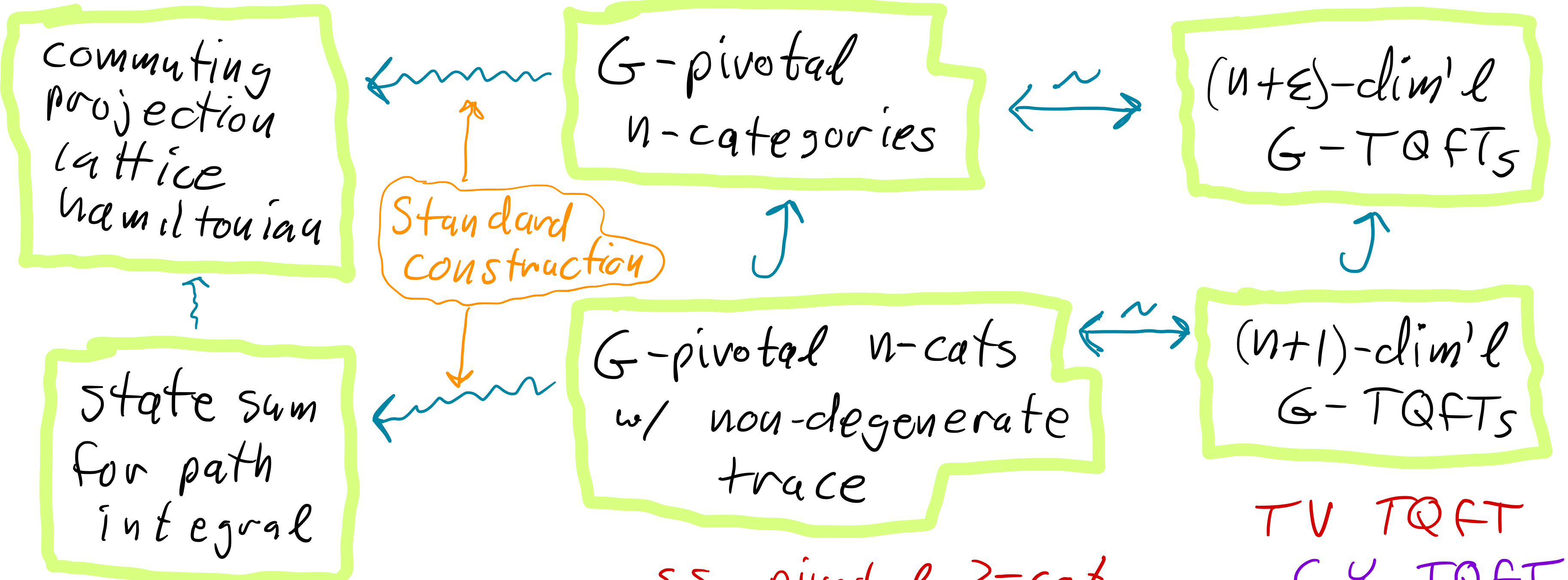
s.s. pivotal 2-cat
premodular br-cat (3-cat)
finite group as n-cat

TV TQFT
CY TQFT
n+1-dim'l DW TQFT

Intro part 1: Well-behaved TQFTs

$G = SO, O, \text{spin}, \text{pin}_\pm$, vortex/char

hamiltonian from [ALW]
Kitaev finite group model
WW model
LW model



TV state sum	s.s. pivotal 2-cat	TV TQFT
CY state sum	premodular br-cat (3-cat)	CY TQFT
DW state sum	finite group as n-cat	n+1-dim'l DW TQFT
super TV state sum	super pivotal 2-cat	super-TV TQFT

Non-example: WRT TQFT (a.k.a. Chern-Simons theory, chiral theories, theories from modular tensor categories (MTCs))

- As is well-known, we can think of a WRT TQFT as living on the boundary of a bordism-invariant 3+1-dim'l oriented (SO) TQFT. (See below.)
- Goal of this talk: Imitate the above construction with SO replaced by O (unoriented), Spin, Pin_+, Pin_-, Spin\O, Pin_-\O, ...

Spin with unoriented vortices

Pin_- with unoriented vortices

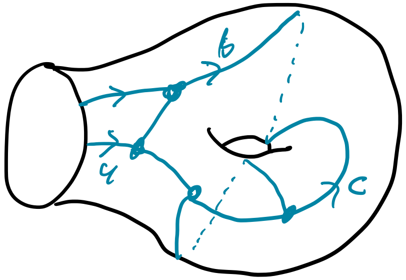
Intro part 2: Quick review of well-behaved (not modular) case

Let C be a G-pivotal n-category.

- For each $0 \leq k \leq n$ and each k-manifold X, we construct an $(n-k)$-category $A(X)$.
- Also define $Z(X) := A(X)^* = \text{mor}\left[A(X) \to \mathbb{1}\right]$
- $Z(\cdots)$ satisfies fully extended Atiyah-Segal axioms

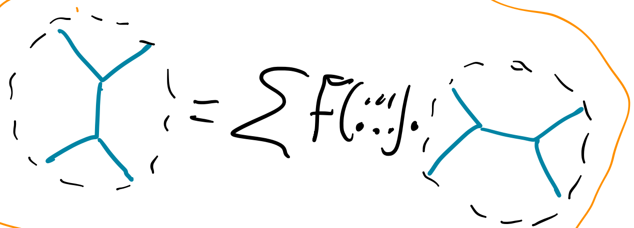
$k=4$

$$A(X^4) := \mathbb{C}\left[\left\{ \quad \right\}\right] \Big/ \langle \text{local relations} \rangle$$



C-string-nets on X

e.g.

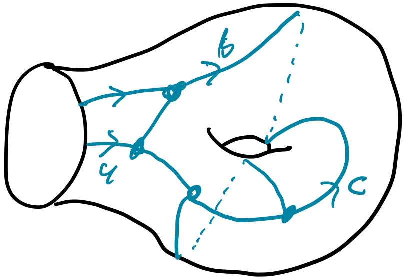


Hilbert space

$$Z(X) := A(X)^*$$

$k=4$

If $\partial X \neq \emptyset$, then must also specify a (fixed) boundary condition on X (which I will suppress from notation)

$A(X^4) := \mathbb{C}\Big[\Big\{$ 

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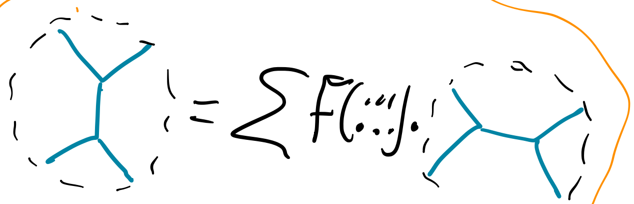


Hilbert space

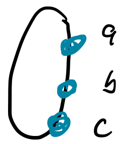
$Z(X) := A(X)^*$

Depending on C, "string nets" might look more like foams or soap bubbles (e.g. for DW TQFT)

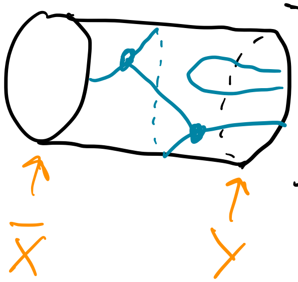
$K = n-1$

$$A(X^{n-1}) =$$

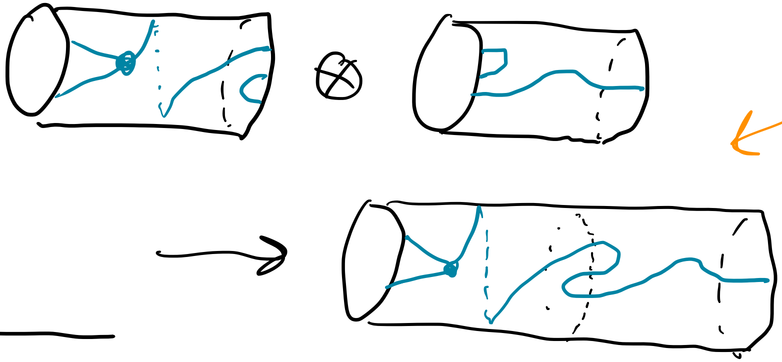
1-category

- objects: $\{$  $\}$ ← C-string-nets on X

- $\mathrm{mor}(x \to y) = A(X \times I; \bar{x}, y)$

$= \mathbb{C}\big[\{$  $\}\big] / \langle \text{loc. rel} \rangle$

- composition



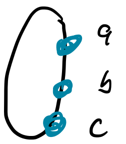
glue together

$$Z(X) = A(X)^*$$
$$= \mathrm{Rep}(A(X))$$
$$= \mathrm{functors}(X \to \mathrm{Vect})$$

$K = n-1$

$A(X^{n-1}) =$

1-category

- objects: $\Big\{$  $\Big\}$ ← C-string-nets on X

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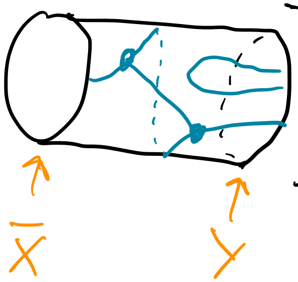
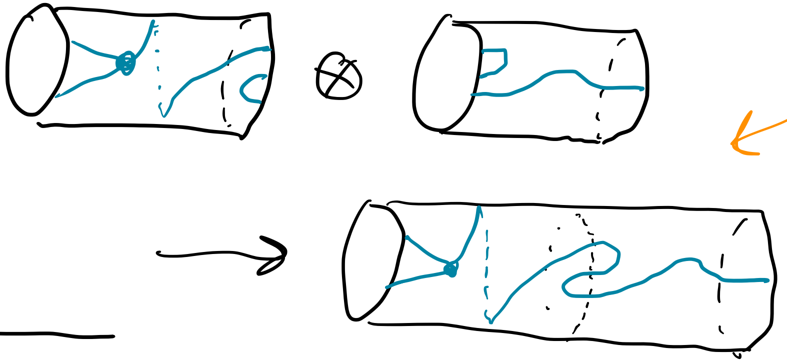
$= \mathbb{C}\Big[\Big\{$  $\Big\}\Big] \Big/ \langle \text{loc. rel} \rangle$

- Composition



glue together

$$Z(X) = A(X)^* = \text{Rep}(A(X)) = \text{functors}(X \to \text{Vect})$$

$\text{Rep}(A(X)) \longleftrightarrow$ possible particles/excitations of shape X

And so on, all the way down to $k=0$ (points)

Note that we never had to choose a cell decomposition (or triangulation) for X when defining $A(X)$

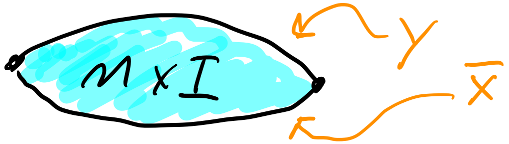
$k=n+1$ (path integrals)

What we want:

⓪ $Z(W^{n+1}) : A(\partial W) \to \mathbb{C}$ (i.e. $Z(W^{n+1}) \in Z(\partial W)$)

① Inner product on $A(M^n)$ given by

$$\langle x, y \rangle = Z(M \times I)(\bar{x} \cup y)$$



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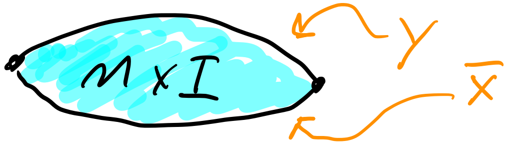
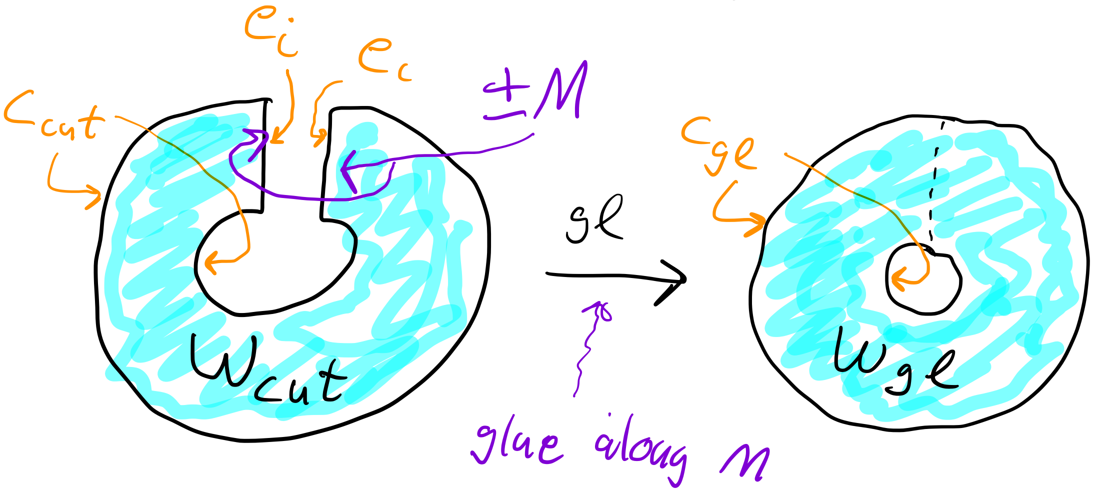


② Gluing relation

$$Z(W_{gl})(c_{gl}) = \sum_i Z(W_{cut})(c_{cut} \cup \bar{e}_i \cup e_i)\cdot \frac{1}{\langle e_i, e_i\rangle}$$



$\{e_i\}$ is orthogonal basis of $A(M)$

Thm [W, 2006]. Let $tr \in A(S^n)^*$. Suppose

(a) $\dim(A(M^n)) < \infty \quad \forall M$

(b) tr induces a non-degenerate inner product on $A(B^n; c)$ $\forall$ boundary conditions c

(c) this inner product is positive-definite

or, more generally

(c') $A(Y^{n-1})$ is semisimple $\forall Y$

Then $\exists!$ path integral satisfying 0-2 above, with $Z(B^{n+1}) = tr$.

Proof: Calculate $Z(W^{n+1})$ in terms of a handle decomposition. Show that the answer is invariant under handle slides and handle reorderings.

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Proof: Calculate $Z(W^{n+1})$ in terms of a handle decomposition. Show that the answer is invariant under handle slides and handle reorderings.

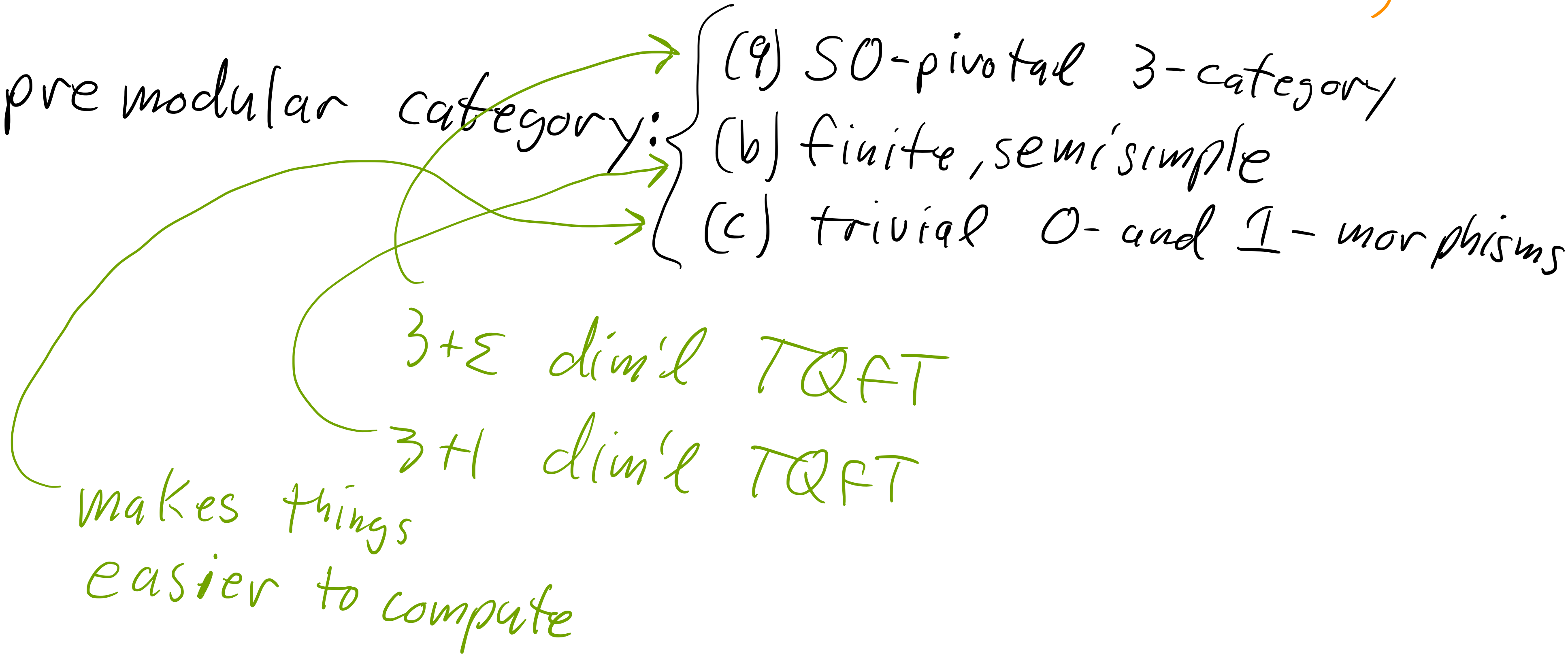
Note: proof works without change for $G = O$, Spin, $\mathrm{Pin}_\pm$

Intro part 3: SO-MTC (usual oriented MTC)

premodular category:
- (a) SO-pivotal 3-category
- (b) finite, semisimple
- (c) trivial 0- and 1-morphisms

Intro part 3: SO-MTC

(usual oriented MTC)



premodular category:
- (a) SO-pivotal 3-category
- (b) finite, semisimple
- (c) trivial 0- and 1-morphisms

3+ε dim'l TQFT

3+1 dim'l TQFT

makes things easier to compute

Intro part 3: SO-MTC (usual oriented MTC)

premodular category: $\begin{cases} \text{(a) SO-pivotal 3-category} \\ \text{(b) finite, semisimple} \\ \text{(c) trivial 0- and 1-morphisms} \end{cases}$

- $A(S^3)$ is 1-dim'l. Can choose $Z(B^4) = \lambda \cdot [\text{std eval}]$ ($Z(B^4)(\emptyset) = \lambda$) for any $\lambda \in \mathbb{C}^\times$
- Thm $\rightsquigarrow$ Crane-Yetter TQFT

When is this TQFT bordism-invariant?

When is CY TQFT bordism invariant?

In 4d, need:

(a) $Z(S^4) = Z(\emptyset)$

(b) $Z(S^3 \times I) = Z(B^4 \times S^0)$

(c) $Z(S^2 \times B^2) = Z(B^3 \times S^1)$

When is CY TQFT bordism invariant?

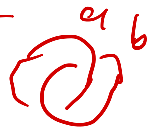
In 4d, need:

(a) $Z(S^4) = Z(\emptyset) \rightsquigarrow \lambda^2 \cdot \sum_a d_a^2 = 1$

(b) $Z(S^3 \times I) = Z(B^4 \times S^0) \rightsquigarrow (\sum d_a^2)^{-1} = \lambda^2$

(c) $Z(S^2 \times B^2) = Z(B^3 \times S^1) \rightsquigarrow \lambda^2 \cdot \sum d_a^2 = 1$ and

no nontrivial transparent objects $\Longleftrightarrow$ $A(S^2) \simeq \text{triv}$ $\Longleftrightarrow$

det [


] $\neq 0$

"modular" condition

Goal: start with well-behaved and bordism-invariant (3+1)-dim'l CY TQFT Z_{3+1} and derive a less well-behaved (2+1)-dim'l TQFT Z_{2+1} ← (WRT TQFT)

Slogan: $Z_{2+1}(X) := Z_{3+1}(\partial^{-1}(X))(\emptyset)$

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$$\Omega_*^{SO} = \overset{0}{\mathbb{Z}}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{\mathbb{Z}}$$

Could be empty.
Could be multi-valued

$$\boxed{\text{Slogan: } Z_{2+1}(X) := Z_{3+1}(\partial^{-1}(X))(\emptyset)}$$

$$\boxed{\Omega_*^{SO} = \overset{0}{\mathbb{Z}}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{\mathbb{Z}}}$$

Could be empty.
Could be multi-valued

- signed # of points
- generated by •+

- detected by σ (signature)
- generated by $\mathbb{C}P^2$

1st attempt at implementing slogan:

- $Z_{2+1}(M^3_{\text{closed}}) = Z_{3+1}(W)(\emptyset)$, ambiguous up to factors of
  $\partial W = M$

$$Z_{3+1}(\mathbb{C}P^2) = \lambda \sum \theta_a d_a^2$$

exponentiated central charge

(1st attempt cont.)

$$\Omega_*^{SO} = \underset{0}{\mathbb{Z}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{4}{\mathbb{Z}}$$

$$Z_{2+1}(Y^2_{\text{closed}}) = Z_{3+1}(M^3), \quad \partial M = Y$$

- $\Omega_3^{SO} = 0 \Rightarrow$ well-defined up to isomorphism
- $\Omega_4^{SO} = \mathbb{Z} \Rightarrow$ ambiguous up to factors of $Z(\mathbb{CP}^2)$

(Diff(Y) acts only projectively)

(1st attempt cont.)

$$\Omega^{SO}_* = \underset{0}{\mathbb{Z}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{4}{\mathbb{Z}}$$

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- $\Omega^{SO}_3 = 0 \Rightarrow$ well-defined up to isomorphism
- $\Omega^{SO}_4 = \mathbb{Z} \Rightarrow$ ambiguous up to factors of $Z(\mathbb{CP}^2)$

(Diff(Y) acts only projectively)

$$Z_{2+1}(S^1) = Z_{3+1}(D^2)$$

$$Z_{2+1}(\bullet^+) = ??$$

$$Z_{2+1}(\bullet^+ \, \bullet^-) = Z_{3+1}(I) \cong \mathcal{C} \text{ as a } \otimes\text{-category}$$

(1st attempt cont.)

$$\Omega^{SO}_{*} = \underset{0}{\mathbb{Z}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{4}{\mathbb{Z}}$$

$Z_{2+1}(Y^2_{\text{Closed}}) = Z_{3+1}(M^3)$, $\partial M = Y$

- $\Omega^{SO}_3 = 0 \Rightarrow$ well-defined up to isomorphism
- $\Omega^{SO}_4 = \mathbb{Z} \Rightarrow$ ambiguous up to factors of $Z(\mathbb{CP}^2)$
  (Diff(Y) acts only projectively)

$Z_{2+1}(S^1) = Z_{3+1}(D^2)$, well-defined by special properties of D^2

$Z_{2+1}(\bullet^+) = ??$

$Z_{2+1}(\bullet^+ \bullet^-) = Z_{3+1}(I) \cong \mathcal{C}$ as a $\otimes$-category

If Z_{2+1} is fully extended, then

$\mathcal{C} \cong Z(\bullet^+ \bullet^-) \cong Z(\bullet^+) \otimes Z(\bullet^-)$, but most MTCs do not split like this

2nd attempt at implementing slogan

"extended" manifold $X \rightsquigarrow (X, W)$ $\quad W \in \partial^{-1}(X)$

(replace ordinary manifold by pair)

- $(M^3, W^4) \rightsquigarrow (M^3, n)$ $\quad n = \sigma(W)$
- $(Y^2, M^3) \rightsquigarrow (Y^2, L)$ $\quad L = \ker(H_1(Y) \to H_1(M))$

Summary: $\Omega_4^{SO} \neq 0 \rightsquigarrow$ central charge, extension of $\mathrm{Diff}(Y^2)$

$\Omega_0^{SO} \neq 0 \rightsquigarrow$ can't define Z_{2+1} on points

In general:

non-zero bordism groups here prevent us from defining Z_{d+1} on all manifolds

$$\Omega_*^G = \Omega_0^G, \Omega_1^G, \Omega_2^G, \Omega_3^G, \Omega_4^G$$

non-zero bordism groups here cause anomalies

	0	1	2	3	4
SO	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
O	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \times \mathbb{Z}/2$
Spin	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
Pin$_-$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0
Pin$_+$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$
Spin\O	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \times \mathbb{Z}$
Pin$_-$\O	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \times \mathbb{Z}/4 \times \mathbb{Z}/2$
Spin\SO	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$

Spin-c

$G = O$ (unoriented manifolds)

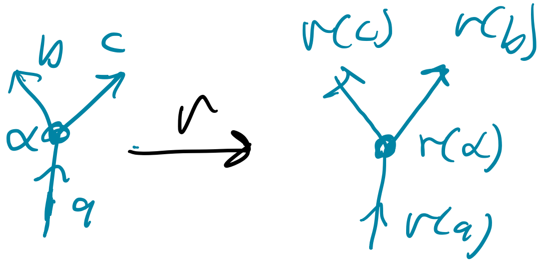
First, must define an O-premodular category

- SO-premod. cat. $\mathcal{C}$
- with antiautomorphism r, $r^2 = id$

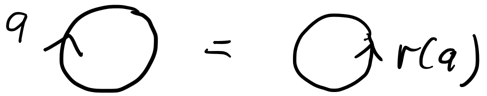
  $$r : a \longrightarrow r(a)$$

  $$r : V_a^{bc} \longrightarrow V_{r(a)}^{r(c)\, r(b)}$$

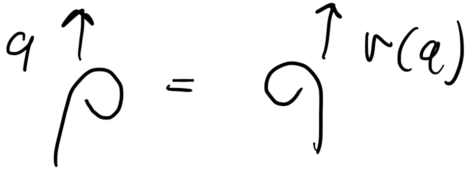
  swapped



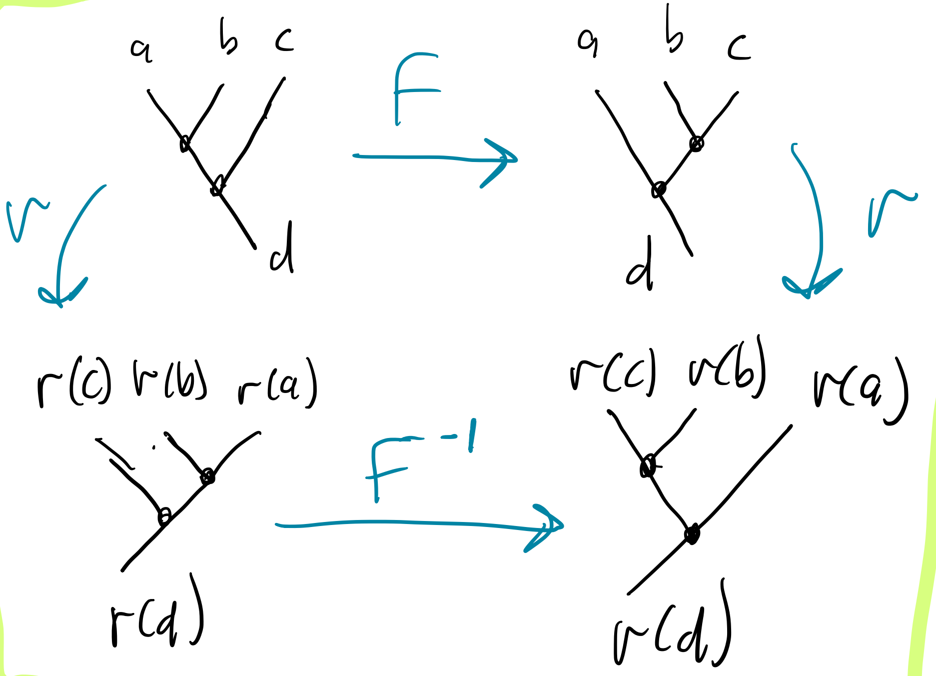
- Satisfying ....

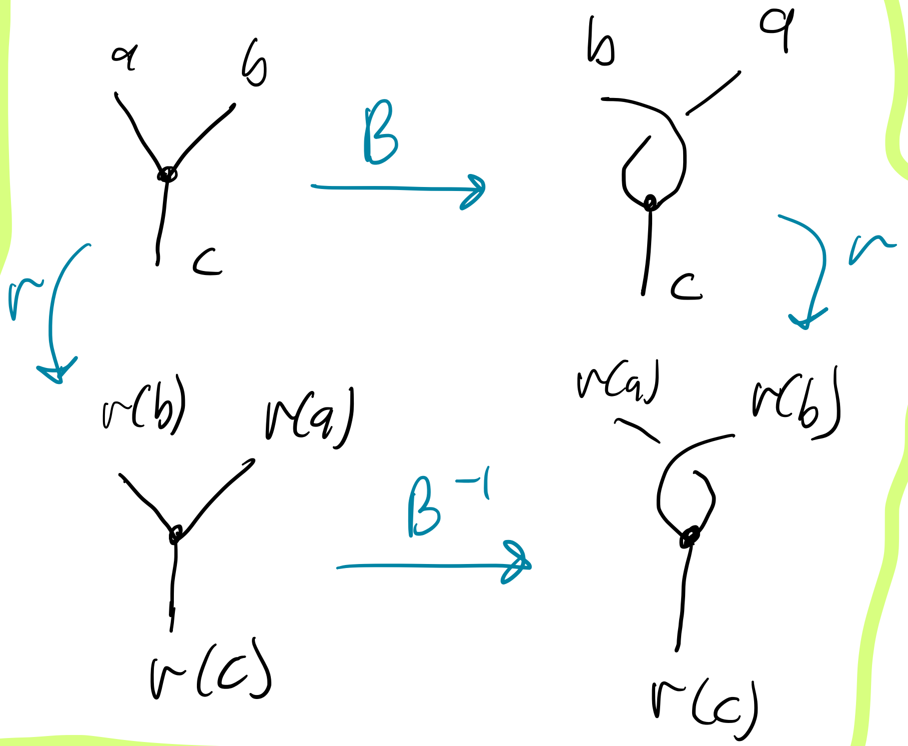


$d_a = d_{r(a)}$



$\theta_a = \theta^{-1}_{r(a)}$





r intertwines with B and F

O-MTC

$$\Omega^{O}_{*} = \overset{0}{\mathbb{Z}/2}, \overset{1}{0}, \overset{2}{\mathbb{Z}/2}, \overset{3}{0}, \overset{4}{\mathbb{Z}/2 \times \mathbb{Z}/2}$$

- Z_{3+1} is bordism invariant if

① $\lambda^2 \sum d_a^2 = 1$

② no transparent objects

$\left.\right\}$ SO-MTC conditions

③ $r: A(S^3) \to A(S^3)$ is id

needed for unoriented 1-handles.

automatically true ∵ $r(\emptyset_{S^3}) = \emptyset_{S^3}$

Question/Problem

Find

O-MTCs which are
① not group-like, and
② not a Drinfeld center
of a SO-MTC
?

$$\overset{0}{}\quad\overset{1}{}\quad\overset{2}{}\quad\overset{3}{}\quad\overset{4}{}$$

$$\Omega^O_* = \mathbb{Z}/2,\ 0,\ \mathbb{Z}/2,\ 0,\ \mathbb{Z}/2\times\mathbb{Z}/2$$

χ mod 2

$\mathbb{RP}^2$

$[\chi \bmod 2,\ w_2^2]$

$\mathbb{CP}^2 = (1,1)$

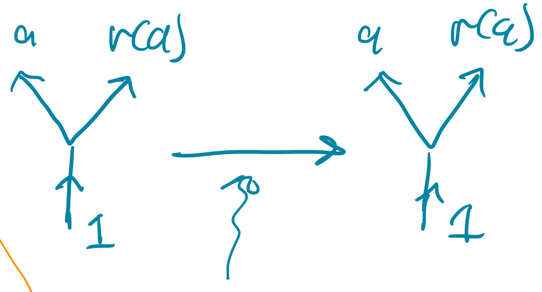
$\mathbb{RP}^4 = (1,0)$

dim 3

- $Z_{2+1}(M^3)$ ambiguous up to factors of $\left\{\begin{array}{l} Z_{3+1}(\mathbb{CP}^2) = \lambda\cdot\sum_a \theta_a d_a^2 = \pm 1 \\ Z_{3+1}(\mathbb{RP}^4) = \lambda\cdot\sum_a n_a\theta_a d_a = \pm 1 \end{array}\right.$

unoriented central charges

- Can resolve ambiguity by equipping M with an element of $\mathbb{Z}/2\times\mathbb{Z}/2$-torsor determined by (w_4, w_2^2)



$r = n_a = \pm 1$ or 0

"reflection Frobenius-Schur"

dim 2

Y^2_{closed}

$$\Omega^O_* = \overset{0}{\mathbb{Z}/2},\ \overset{1}{0},\ \overset{2}{\mathbb{Z}/2},\ \overset{3}{0},\ \overset{4}{\mathbb{Z}/2\times\mathbb{Z}/2}$$

Case 0: $\chi(Y)$ even → $Y = \partial M^3$ → similar to SO case

⇝ w_2^2-gerbe for Y, central extension by $\mathbb{Z}/2\times\mathbb{Z}/2$

Case 1: $\chi(Y)$ odd → Y does not bound

→ slogan does not give an answer ☆

dim 2

Y^2_{closed}

$$\Omega^{O}_{*} = \overset{0}{\mathbb{Z}/2},\ \overset{1}{0},\ \overset{2}{\mathbb{Z}/2},\ \overset{3}{0},\ \overset{4}{\mathbb{Z}/2\times\mathbb{Z}/2}$$

Case 0: $\chi(Y)$ even $\to Y = \partial M^3 \to$ similar to SO case

$\rightsquigarrow w_2^2$-gerbe for Y, central extension by $\mathbb{Z}/2\times\mathbb{Z}/2$

Case 1: $\chi(Y)$ odd $\to Y$ does not bound

$\to$ slogan does not give an answer

Conjecture: If $Z_{3+1}(\mathbb{RP}^2\times\mathbb{RP}^2) = Z_{3+1}(\mathbb{CP}^2) = -1$, then cannot extend Z_{2+1} to $\mathbb{RP}^2$.

(Proof??: unoriented Moore-Seiberg thm)

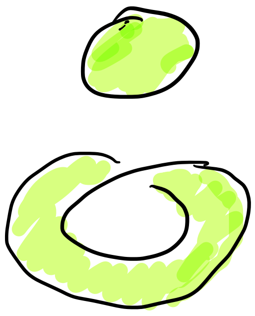
Problem: State and prove Moore-Seiberg type thm for $G = O, Spin, Spin \ltimes O$, etc.

dim 1

$$\begin{array}{cccccc} & 0 & 1 & 2 & 3 & 4 \\ \Omega^O_* = & \mathbb{Z}/2, & 0, & \mathbb{Z}/2, & 0, & \mathbb{Z}/2\times\mathbb{Z}/2 \end{array}$$

- All 1-manifolds bound ($\Omega^O_1 = 0$), but they bound in two non-cobordant ways ($\Omega^O_2 = \mathbb{Z}/2$).

$$S^1 = \begin{cases} \partial D^2 & (\text{or } \partial[\chi \text{ odd}]) \\ \partial MB & (\text{or } \partial[\chi \text{ even}]) \end{cases}$$



- So

$$Z_{2+1}(S^1) = \begin{cases} Z_{3+1}(D^2) \cong \mathbb{C} \\ Z_{3+1}(MB) \cong ?? \end{cases}$$

Two different classes of unoriented anyons

$$\Omega_*^O = \overset{0}{\mathbb{Z}/2},\ \overset{1}{0},\ \overset{2}{\mathbb{Z}/2},\ \overset{3}{0},\ \overset{4}{\mathbb{Z}/2\times\mathbb{Z}/2}$$

- $MB \underset{\text{cob}}{\simeq} D^2 \amalg \mathbb{R}P^2$
- $\Rightarrow Z_{3+1}(MB) \simeq Z_{3+1}(D^2)\times Z_{3+1}(\mathbb{R}P^2) \simeq \mathcal{C}\times Z_{3+1}(\mathbb{R}P^2)$
- $Z_{3+1}(\mathbb{R}P^2\times S^1)$ is 1-dim'l $\Rightarrow$ $Z_{3+1}(\mathbb{R}P^2)$ has only one simple object $\Rightarrow$ trivial as a plain 1-cat (SO 1-cat)

$$\Omega^{O}_{*} = \overset{0}{\mathbb{Z}/2},\ \overset{1}{0},\ \overset{2}{\mathbb{Z}/2},\ \overset{3}{0},\ \overset{4}{\mathbb{Z}/2\times\mathbb{Z}/2}$$

- $MB \underset{\text{cob}}{\simeq} D^2 \amalg \mathbb{RP}^2$

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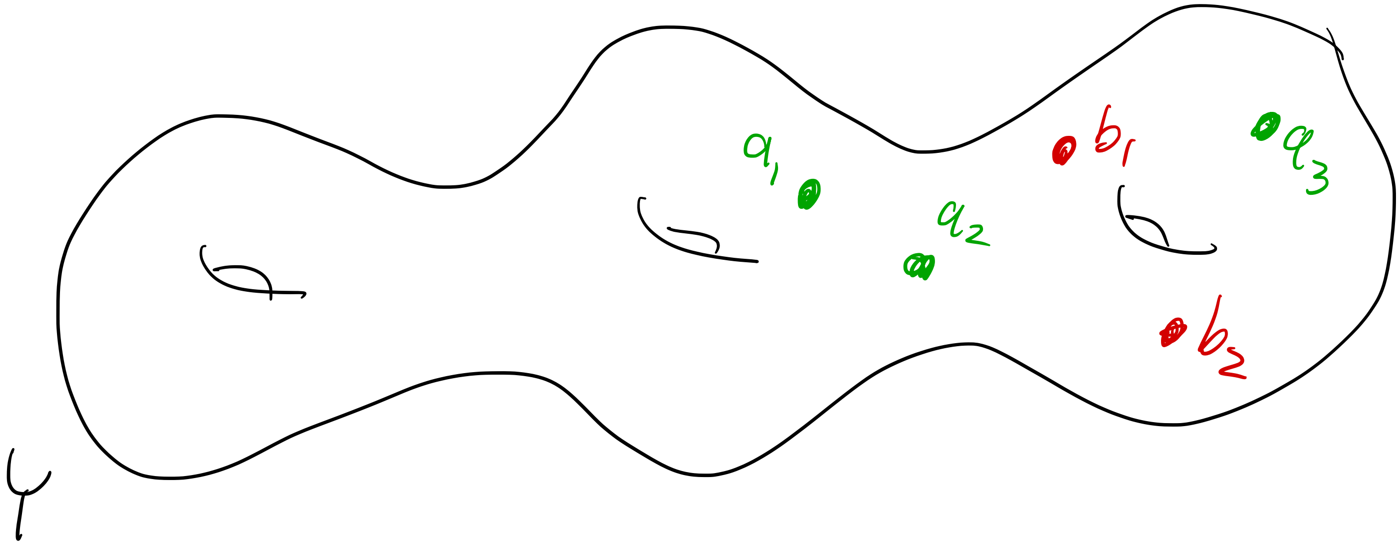
- $Z_{3+1}(\mathbb{RP}^2\times S^1)$ is 1-dim'l $\Rightarrow Z_{3+1}(\mathbb{RP}^2)$ has only one simple object $\Rightarrow$ trivial as a plain 1-cat (SO 1-cat)

- But not necessarily trivial as an unoriented 1-cat with trace. Let α be the simple object of $Z_{3+1}(\mathbb{RP}^2)$.

- If $Z_{3+1}(\mathbb{RP}^2\times\mathbb{RP}^2) = -1$, then $\underset{\text{q-dim}}{d_\alpha} \cdot \underset{\text{reflection Frob-Schur}}{h_\alpha} = -1$

🔴 – χ-even (MB) anyons

🟢 – χ-odd (ordinary) anyons

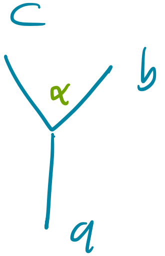


$Z_{2+1}(Y; a_i, b_i)$ defined $\iff$ $\chi(Y) + \# b_i$ is even.

$G = \mathrm{Spin}$

Spin-premodular category

- Simple objects of two types: $\begin{cases} \mathrm{End}(a) \cong \mathbb{C} & \text{"m-type"} \\ \mathrm{End}(a) \cong \mathbb{C}l_1 & \text{"q-type"} \end{cases}$
- V_a^{bc} is a super vector space and a module for $\mathrm{End}(a) \otimes \mathrm{End}(b) \otimes \mathrm{End}(c)$
- 

$= (-1)^F \cdot$


- 

$= \theta_a \cdot$

?

- Simple objects of two types: $\begin{cases} \mathrm{End}(a) \cong \mathbb{C} & \text{"m-type"} \\ \mathrm{End}(a) \cong \mathbb{C}\ell_1 & \text{"q-type"} \end{cases}$
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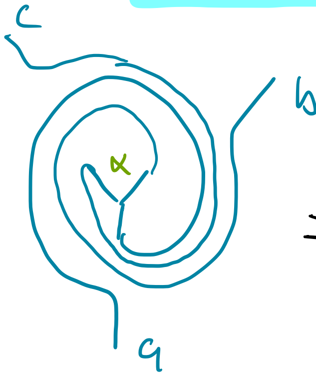
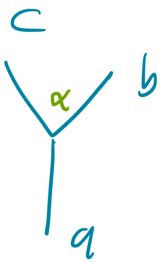
- 

$= (-1)^F \cdot$


- 

~~$= \theta_a \cdot$~~

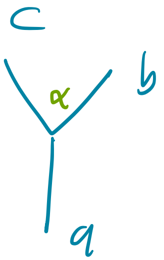
No!

↑ and  have different relative spin structures

- Simple objects of two types: $\begin{cases} \mathrm{End}(a) \cong \mathbb{C} & \text{"m-type"} \\ \mathrm{End}(a) \cong \mathbb{C}\ell_1 & \text{"q-type"} \end{cases}$

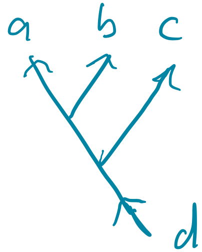
- V_a^{bc} is a super vector space and a module for $\mathrm{End}(a) \otimes \mathrm{End}(b) \otimes \mathrm{End}(c)$

- 

$= (-1)^F \cdot$


- 

$= \theta_a^2$


- 

$\rightsquigarrow \bigoplus_{x} V_x^{ab} \otimes_{\mathrm{End}(x)} V_d^{xc}$

Spin-MTC

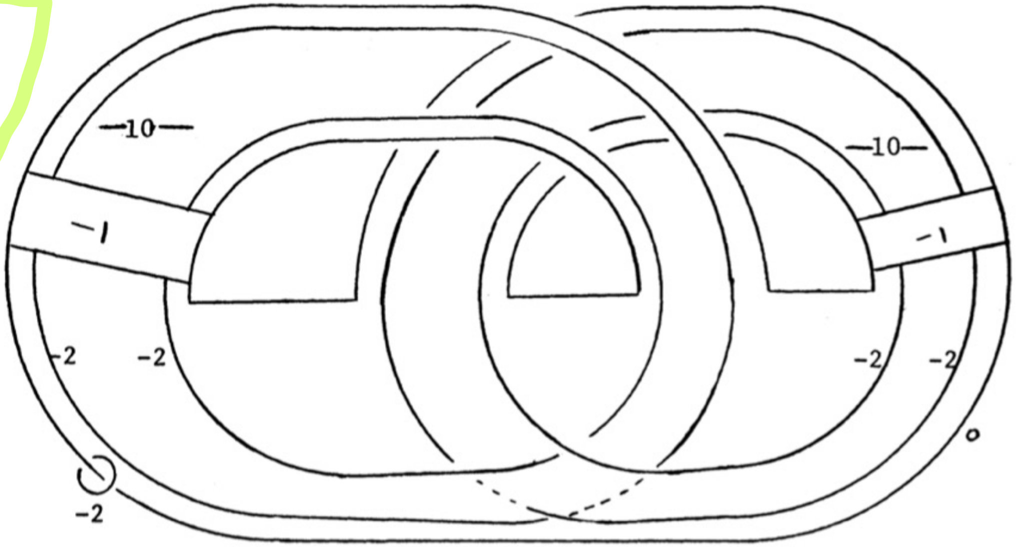
$$\Omega_*^{Spin} = \overset{0}{\mathbb{Z}}, \underset{\uparrow\, S^1_N}{\overset{1}{\mathbb{Z}/2}}, \underset{\uparrow\, T^2_{NN}}{\overset{2}{\mathbb{Z}/2}}, \overset{3}{0}, \underset{\uparrow\, K3}{\overset{4}{\mathbb{Z}}}$$

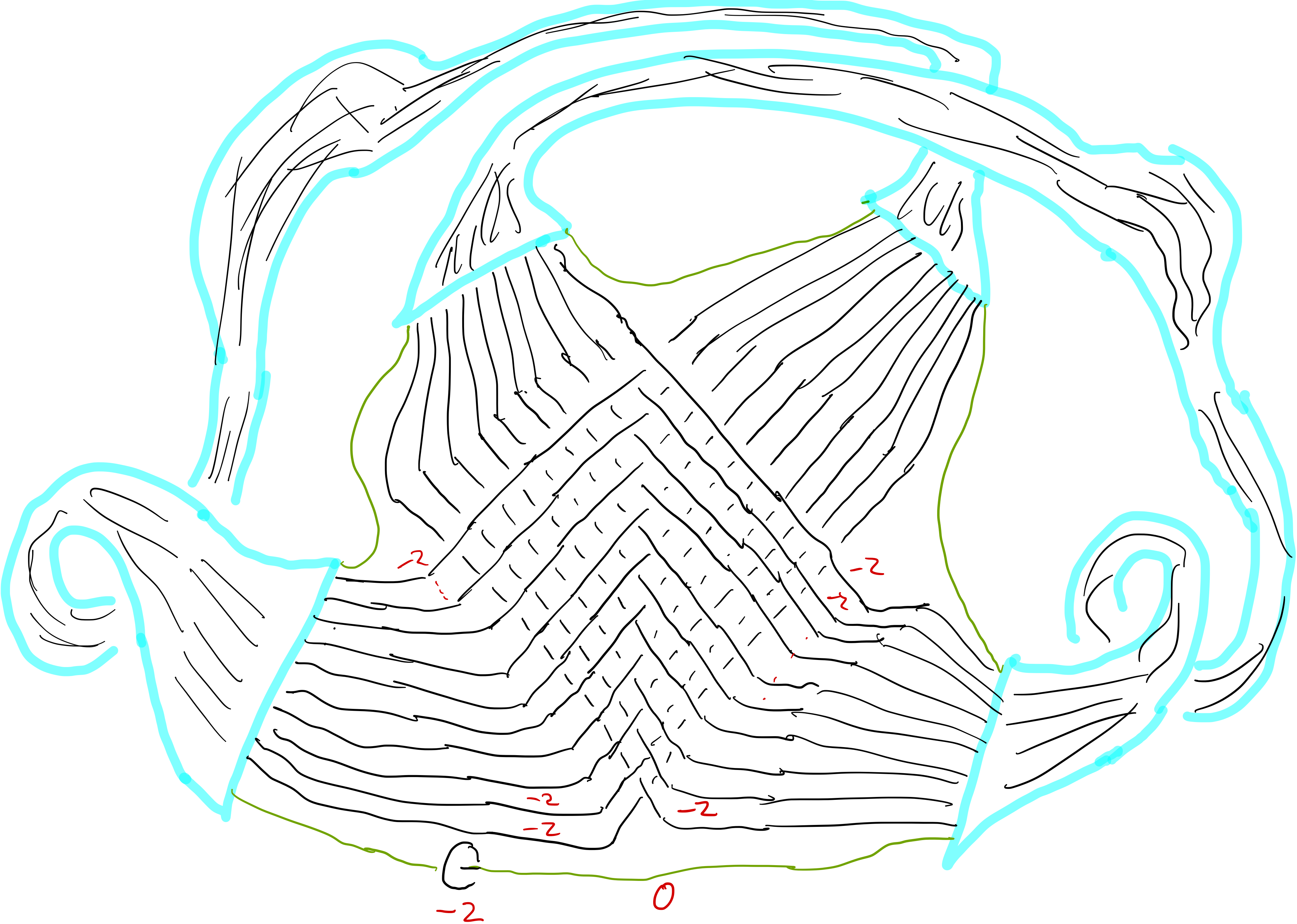
- $\lambda^2 \cdot \sum_a \frac{d_a^2}{|End(a)|} = 1$
- No transparent objects

- Central charge $= Z_{3+1}(K3)$ (complicated!)

All framings are even

22 1-handles




-2
-2
-2
-2
-2
-2
-2
-2
0

Spin-MTC

$$\Omega_*^{\mathrm{Spin}} = \overset{0}{\mathbb{Z}},\ \underset{\overset{\uparrow}{S^1_N}}{\overset{1}{\mathbb{Z}/2}},\ \underset{\overset{\uparrow}{T^2_{NN}}}{\overset{2}{\mathbb{Z}/2}},\ \overset{3}{0},\ \underset{\overset{\uparrow}{K3}}{\overset{4}{\mathbb{Z}}}$$

- $\lambda^2 \cdot \sum_q \frac{d_q^2}{|\mathrm{End}(q)|} = 1$
- No transparent objects

- Central charge $= Z_{3+1}(K3)$ (complicated!)
- Can't define $Z_{2+1}(Y)$ if $\mathrm{arf}(Y) = 1$
  (e.g. $Y = T^2_{NN}$)
- Can't define $Z_{2+1}(S^1_N)$

Spin-MTC

$$\Omega_*^{Spin} = \overset{0}{\mathbb{Z}},\ \overset{1}{\underset{\uparrow\, S^1_N}{\mathbb{Z}/2}},\ \overset{2}{\underset{\uparrow\, T^2_{NN}}{\mathbb{Z}/2}},\ \overset{3}{0},\ \overset{4}{\underset{\uparrow\, K3}{\mathbb{Z}}}$$

- $\lambda^2 \cdot \sum_q \frac{d_q^2}{|End(q)|} = 1$
- No transparent objects

- Central charge $= Z_{3+1}(K3)$ (complicated!)
- Can't define $Z_{2+1}(Y)$ if $arf(Y) = 1$ (e.g. $Y = T^2_{NN}$)
- Can't define $Z_{2+1}(S^1_N)$
- $Z_{2+1}(S^1_B) = \begin{cases} Z_{3+1}(D^2) \cong C \\ Z_{3+1}(D^2 \amalg T^2_{NN}) \cong C \times Z_{3+1}(T^2_{NN}) \end{cases}$
- $Z_{2+1}(\bullet^+ \bullet^-) = \begin{cases} Z_{3+1}(I) \cong C \\ Z_{3+1}(I \amalg S^1_N) \cong C \times Z_{3+1}(S^1_N) \end{cases}$

	0	1	2	3	4
SO	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
O	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \times \mathbb{Z}/2$
Spin	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
Pin_-	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0
Pin_+	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$
Spin \ O	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \times \mathbb{Z}$
Pin_- \ O	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \times \mathbb{Z}/4$ $\times \mathbb{Z}/2$
Spin \ SO	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$

Spin-c

$G = \mathrm{Pin}_-$

Pin_--premodular cat:

$$r : V^{ab}_{c} \to V^{r(b)\, r(a)}_{r(c)}$$

$$r^2 = (-1)^F \,, \quad r^4 = id$$

Pin_--MTC

- No central charge $(\Omega_4^{\mathrm{Pin}_-} = 0)$

$$\boxed{\Omega_*^{\mathrm{Pin}_-} = \overset{0}{\mathbb{Z}/2}, \overset{1}{\mathbb{Z}/2}, \overset{2}{\mathbb{Z}/8}, \overset{3}{0}, \overset{4}{0}}$$

- $Z_{2+1}(Y^2)$ only defined if $\beta(Y) = 0 \in \mathbb{Z}/8$

  Brown-Arf invt.

- $Z_{2+1}(S^1_B) \rightsquigarrow \mathbb{Z}/8$-extension of $Z_{3+1}(D^2) \cong C$

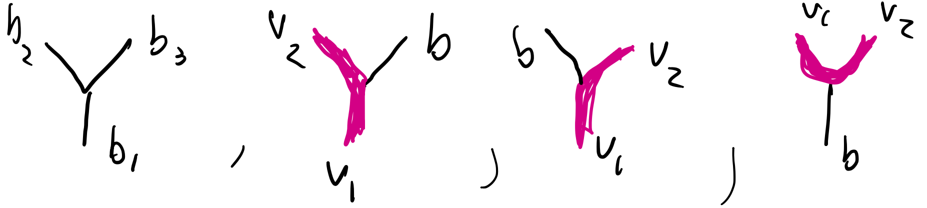
$G = \mathrm{Spin} \setminus O$ (Spin with unoriented vortices)

Spin defects

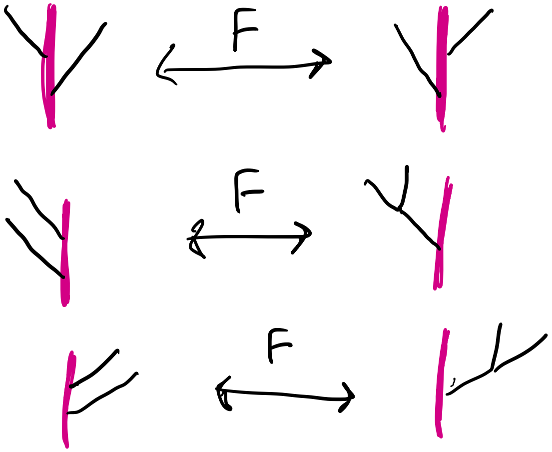
Premodular vortex category

☆ Vortices are treated as part of the manifold, not fluctuating string nets. ☆

Data:

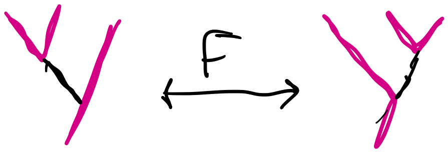


$$\bigcirc^{b} = d_b \cdot \emptyset$$

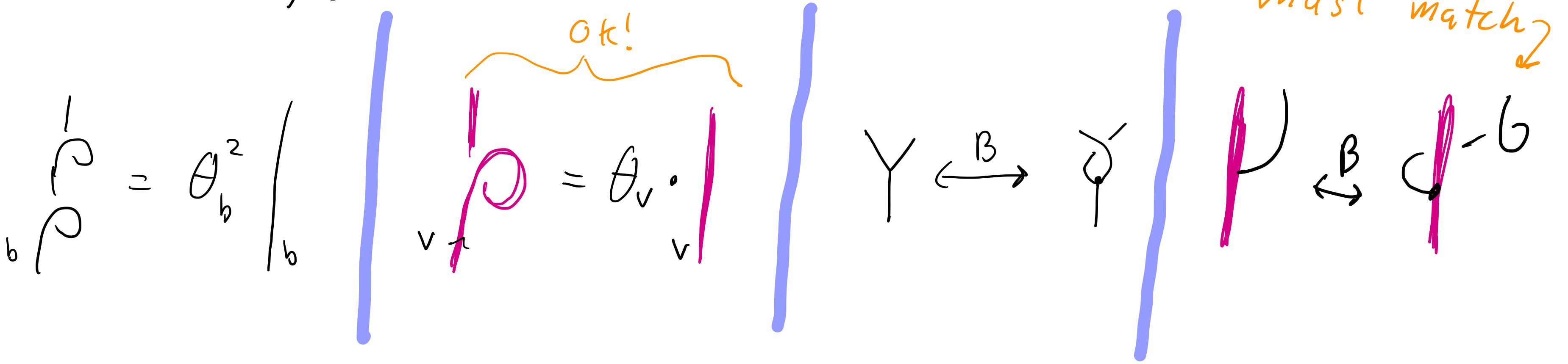


NOT THESE:

$$\bigcirc^{v} \overset{?}{=} d_v \cdot \emptyset$$

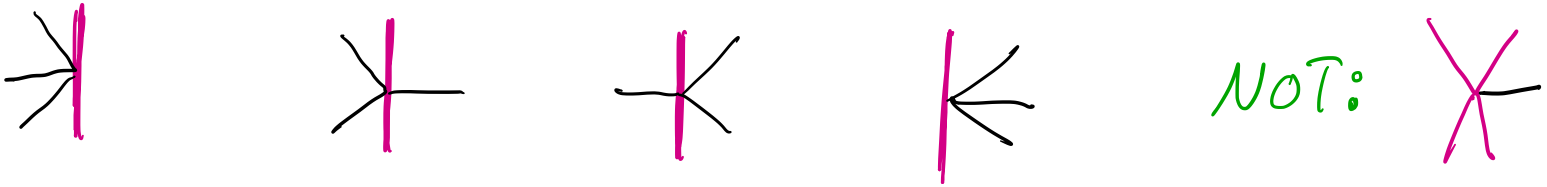


Data (cont.):



Coherence Relations:

- Only some instances of pentagon eqn:



etc.

- Only some instances of hexagon eqn.

Examples of V-premodular cats:

- SO-MTC/ψ
- Tube category of Spin 2-cat

To define 3+1-dim'l TQFT, must specify both

$$Z(B^4) \in Z(S^3, \emptyset) \quad \text{and} \quad Z(B^4, B^2) \in Z(S^3, S^1)$$

vortex

$\lambda \cdot$ std-eval

$\mu \cdot$ random-eval

Z_{3+1} is bordism-invariant iff:

- $\lambda^2 \sum_b d_b^2 / |\mathrm{End}(b)| = 1$
- $\mu^2 = \lambda^2 \cdot \dfrac{\sum_b N_{vvb} \cdot d_b}{d_v^2}$
- No transparent bounding simple objects

$$\Omega_*^{\mathrm{Spin}\backslash O} = \overset{0}{\mathbb{Z}}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{\mathbb{Z}\times\mathbb{Z}}$$

Ω_4 generators $\begin{cases} (\mathbb{CP}^2, \mathbb{CP}^1) \\ (S^4, \mathbb{RP}^2) \end{cases}$

$$Z_{3+1}(\mathbb{CP}^2, \mathbb{CP}^1) = \frac{u^2}{\lambda} \sum_v \theta_v d_v^2 \cdot \frac{1}{|\mathrm{End}(v)|}$$

$$Z_{3+1}(S^4, \mathbb{RP}^2) = \pm \frac{\lambda \theta_v^{-2}}{u\, d_v} \cdot \sum_{b,\alpha} R^{v b,\alpha}_{v,\alpha} \cdot d_b$$

$$K3 \simeq 16 \cdot (\mathbb{CP}^2, \mathbb{CP}^1) - 8 \cdot (S^4, \mathbb{RP}^2)$$

Problem: Find examples of (Spin\0)-MTCs such that:

- "forbidden pentagons" are not satisfied
- $Z_{3+1}(S^4, \mathbb{R}P^2) = \pm \frac{\lambda \theta_v^{-2}}{\mu d_v} \cdot \sum_{b,\alpha} R_{v,\alpha}^{vb,\alpha} \cdot d_b \neq 1$