Invariants of 4-manifolds from Hopf Algebras

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Invariants of 4-manifolds

- Turaev-Viro type construction

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finite group
[Dijkgraaf-Witten, ’90]

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Conjecturally homotopy invariants

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Conjecturally homotopy invariants

Semisimple TQFTs cannot distinguish smooth structures of 4-manifolds
[Reutter, ’20]

- Seiberg-Witten/Donaldson theory.

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dim=3

Turaev-Viro invariants from
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$\Downarrow$ nonsemisimple generalization

Kuperberg invariants from
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dim=3		dim=4
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Kuperberg invariants from (nonsemisimple) Hopf algebras		????

Goal: Construct Kuperberg-type 4-manifold invariants

Hopf Algebras $H(M, i, \Delta, \epsilon, S)$

$M : H \bigotimes H \to H$ $\quad$ $\Delta : H \to H \bigotimes H$

$i : \mathbb{C} \to H$ $\quad$ $\epsilon : H \to \mathbb{C}$ $\quad$ $S : H \to H$

Hopf Algebras $H(M, i, \Delta, \epsilon, S)$

$$M : H \otimes H \to H \qquad \Delta : H \to H \otimes H$$

$$i : \mathbb{C} \to H \qquad \epsilon : H \to \mathbb{C} \qquad S : H \to H$$

- $H(M, i)$ is an associative algebra;
- $H(\Delta, \epsilon)$ is an associative coalgebra;

  $$(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta \qquad (I \otimes \epsilon)\Delta = (\epsilon \otimes I)\Delta = I$$
- Δ is an algebra morphism;

  $$\Delta M = (M \otimes M)(I \otimes \mathrm{Flip} \otimes I)(\Delta \otimes \Delta)$$
- The antipode S is compatible.

  $$M(I \otimes S)\Delta = M(S \otimes I)\Delta = \eta\epsilon$$

$$\mathrm{Flip} : H \otimes H \to H \otimes H, \qquad x \otimes y \mapsto y \otimes x$$

G a finite group

$$\mathbb{C}[G] = \mathbb{C}\{g : g \in G\}$$

$$M(g \otimes g') = gg'$$

$$i(1) = 1_G$$

$$\Delta(g) = g \otimes g$$

$$\epsilon(g) = 1$$

$$S(g) = g^{-1}$$

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$q = e^{\frac{2\pi i}{r}}$

$U_q(sl_2) = \mathbb{C}\langle E, F, K^{\pm 1}\rangle / \sim,$

$\sim$: $KK^{-1} = K^{-1}K = 1,\ KE = q^2 EK,$
$KF = q^{-2} FK,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$
$E^r = F^r = K^r - 1 = 0;$

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$\Delta(E) = 1 \otimes E + E \otimes K,$
$\Delta(F) = K^{-1} \otimes F + F \otimes 1,$
$\Delta(K) = K \otimes K;$

$\epsilon(E) = \epsilon(F) = 0,\ \epsilon(K) = 1;$

$S(E) = -EK^{-1},\ S(F) = -KF,$
$S(K) = K^{-1}$

Semisimple

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Non-semisimple

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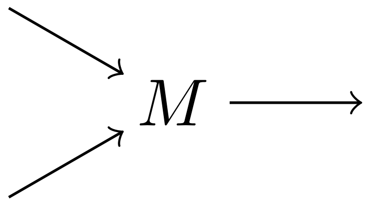
Tensor diagram representations



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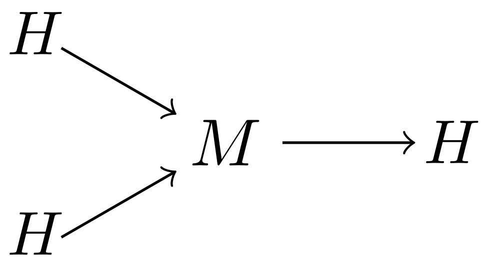

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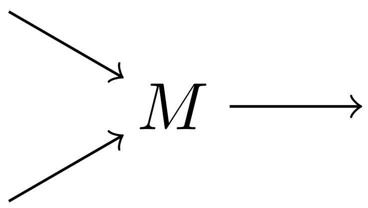

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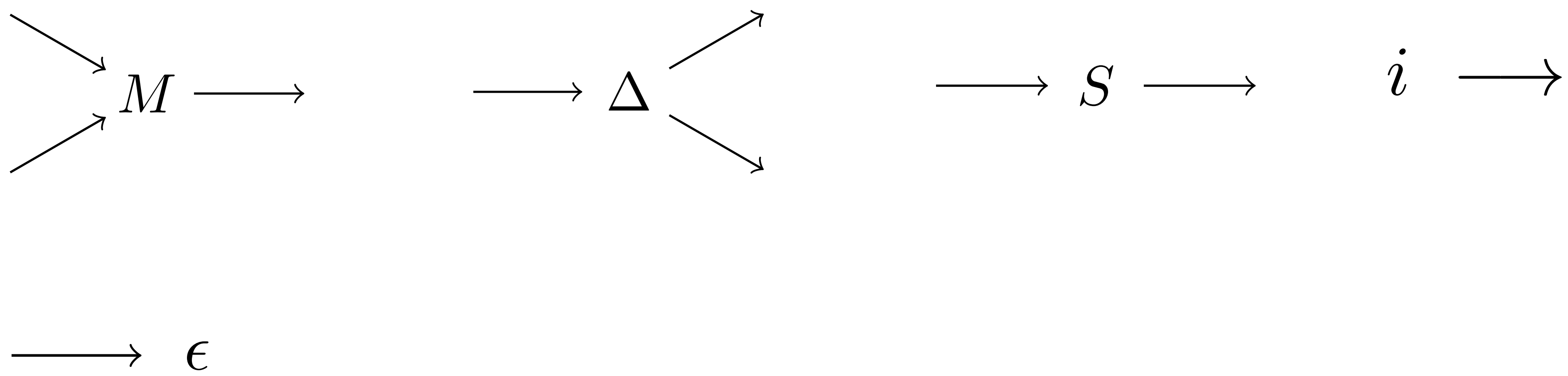

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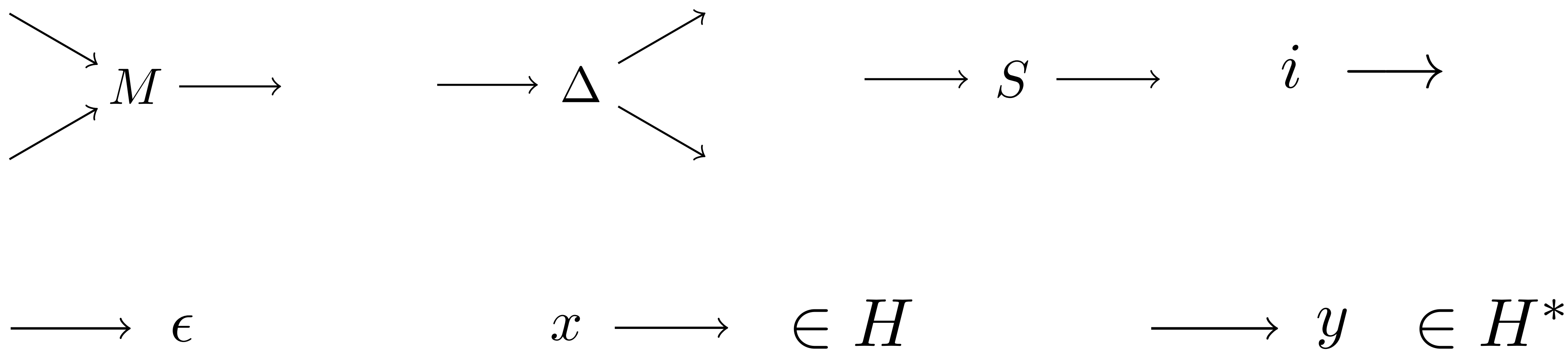

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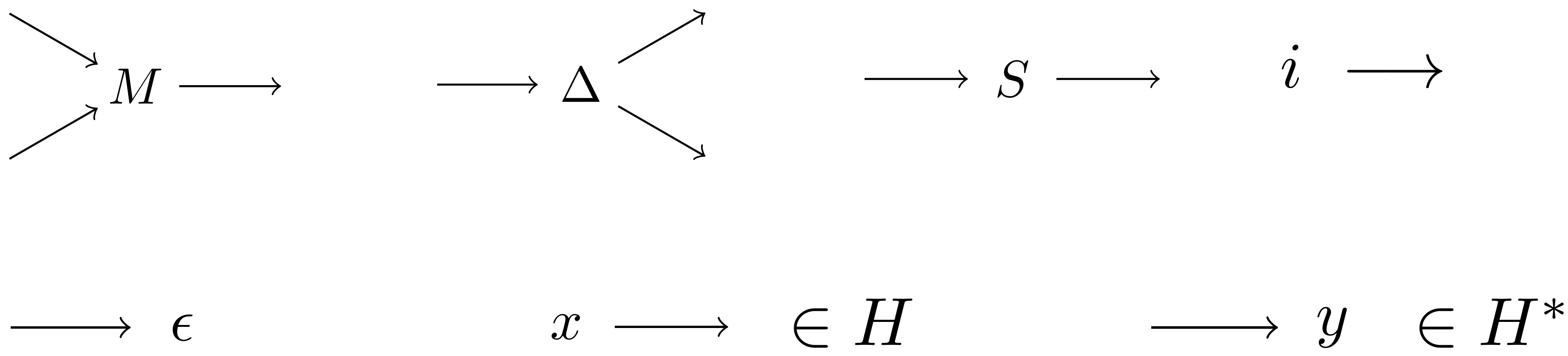
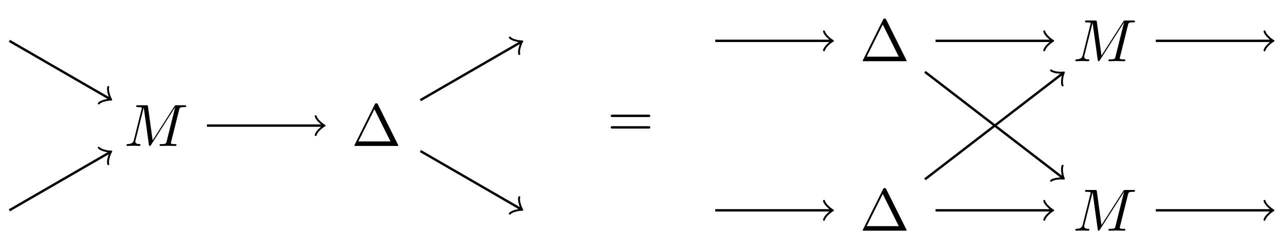


Tensors can be composed (contracted):



$$\Delta M =$$
$$(M \otimes M)(I \otimes \text{Flip} \otimes I)(\Delta \otimes \Delta)$$

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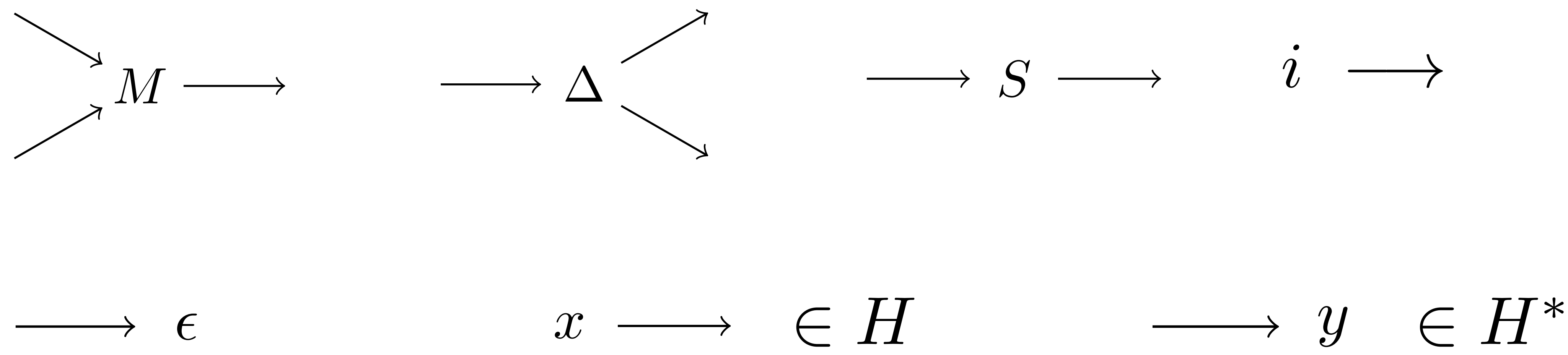

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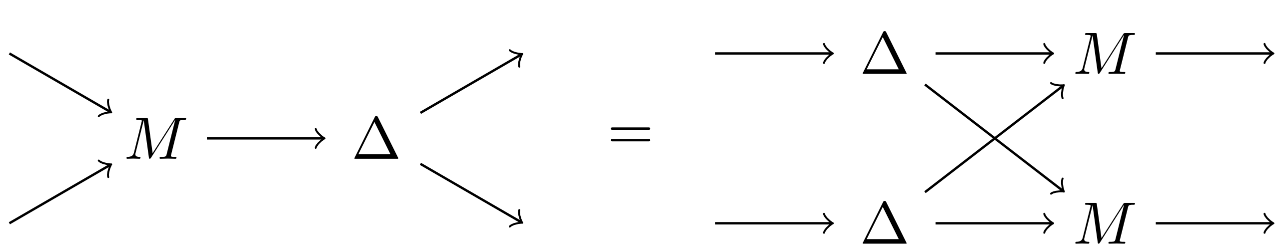

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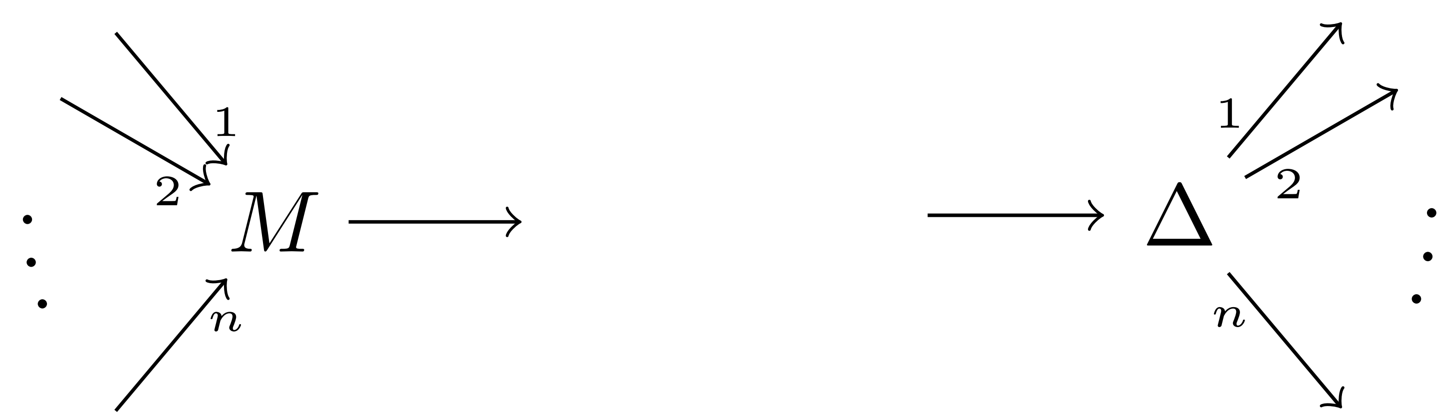
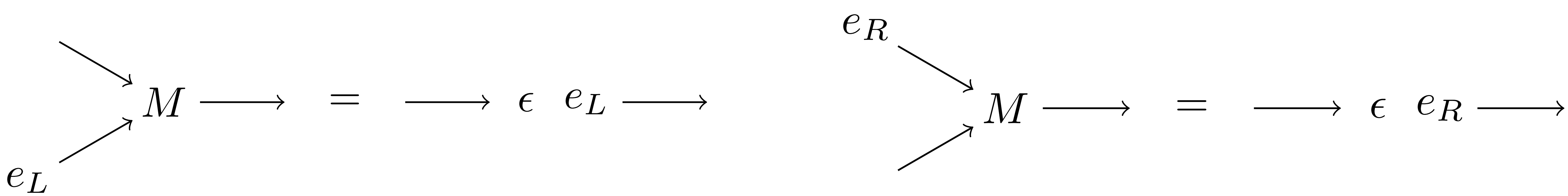
General M, Δ tensors:



$$M(M \otimes I) \cdots (M \otimes I^{\otimes(n-2)}) \qquad (\Delta \otimes I^{\otimes(n-2)}) \cdots (\Delta \otimes I)\Delta$$

$e_L \in H$ (resp. $e_R \in H$) is a left (resp. right) integral if



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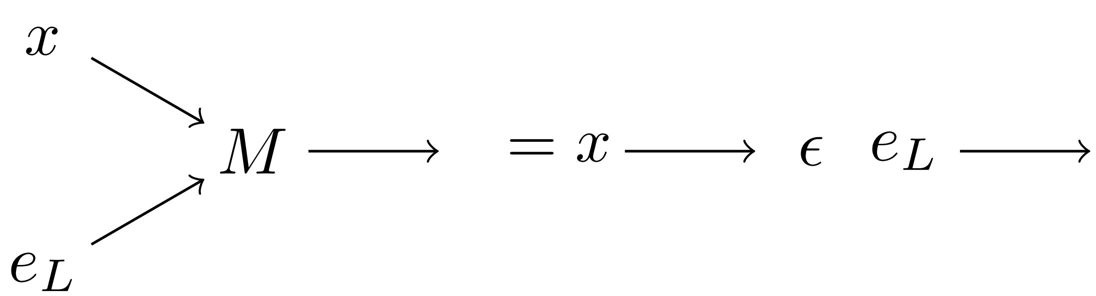

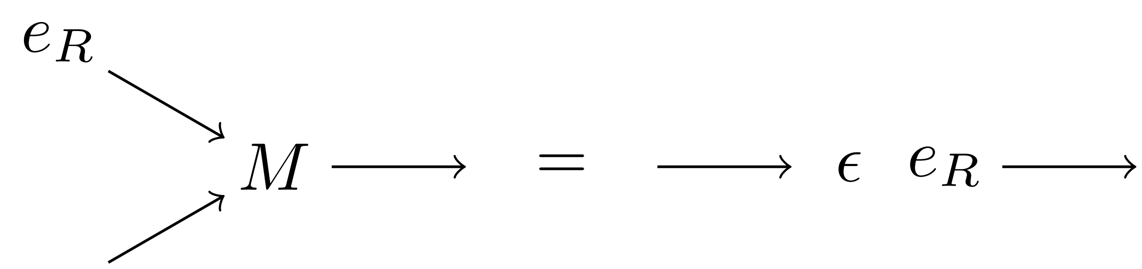

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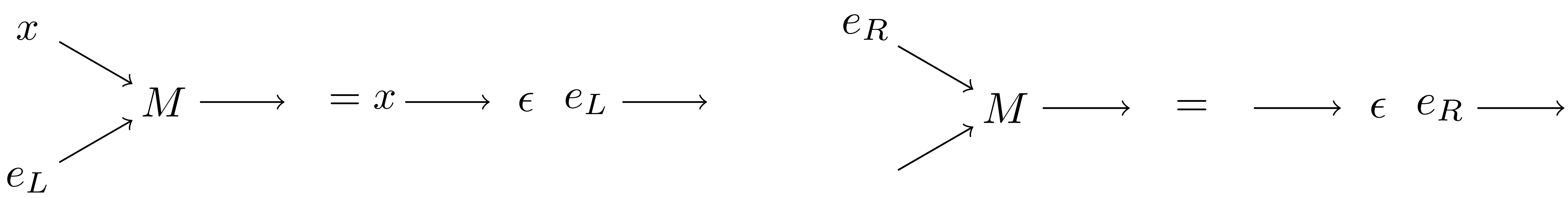

Fact: {left integrals} and {right integral} 1-dimensional.

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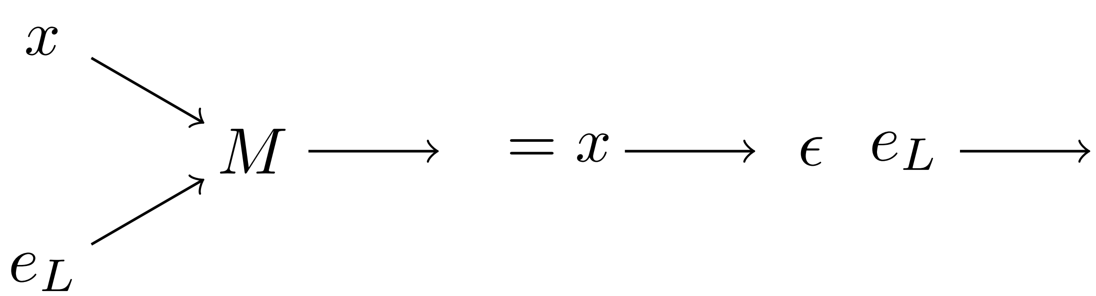

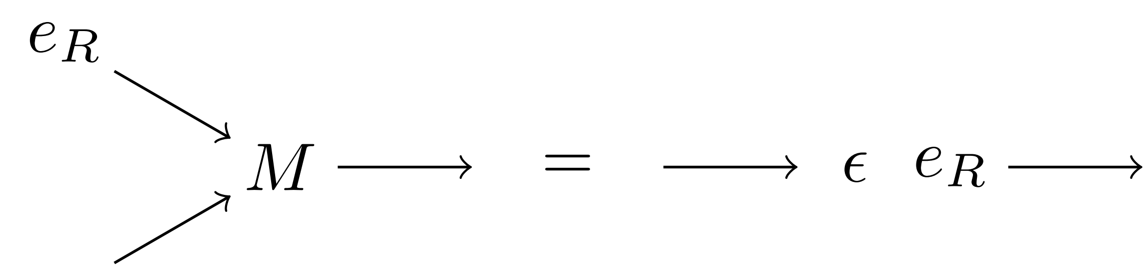

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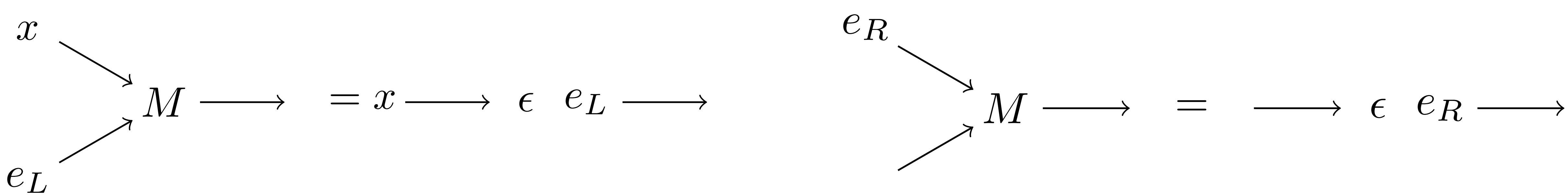

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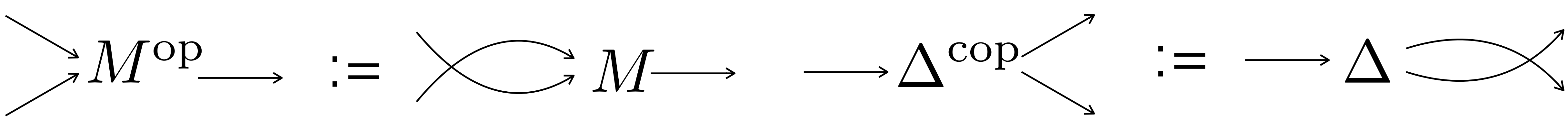

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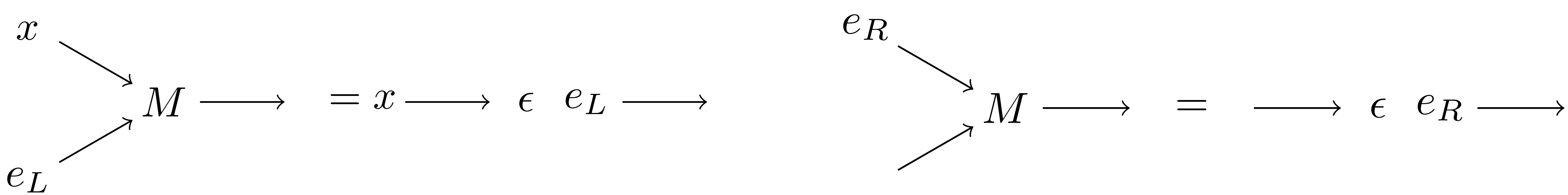

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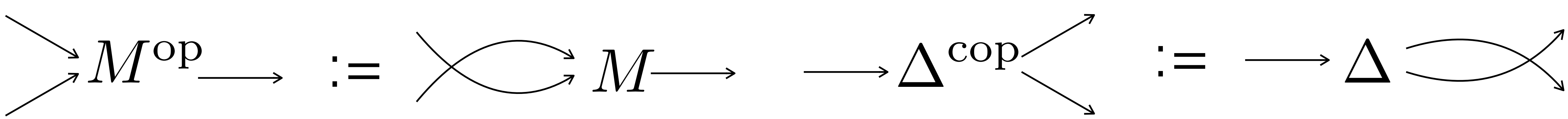

If $H(M, i, \Delta, \epsilon, S)$ is Hopf, so are $H^*(\Delta^*, \epsilon^*, M^*, i^*, S^*)$

$H^{\mathrm{op}}(M^{\mathrm{op}}, i, \Delta, \epsilon, S^{-1}) \quad H^{\mathrm{cop}}(M, i, \Delta^{\mathrm{cop}}, \epsilon, S^{-1})$

$H^{\mathrm{op,cop}}$, $H^{*,\mathrm{cop}} = H^{\mathrm{op},*}$, etc.

The Drinfeld double of H: $D(H) = H^{*,\mathrm{cop}} \otimes H$

$$(M^D, i^D, \Delta^D, \epsilon^D, S^D)$$

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$v \longrightarrow$	$\longrightarrow f$	$f \longleftarrow$ $v \longrightarrow$	$\longleftarrow v$ $\longrightarrow f$
$v \in H$	$f \in H^*$	$f \otimes v \in \mathrm{D}H$	$v \otimes f \in (\mathrm{D}H)^*$

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$$v \in H \qquad f \in H^* \qquad f \otimes v \in \mathrm{D}H \qquad v \otimes f \in (\mathrm{D}H)^*$$

$$i^D \longrightarrow \; := \; \begin{matrix} \epsilon \longleftarrow \\ i \longrightarrow \end{matrix} \qquad \longrightarrow \epsilon^D \; := \; \begin{matrix} \longleftarrow i \\ \longrightarrow \epsilon \end{matrix}$$

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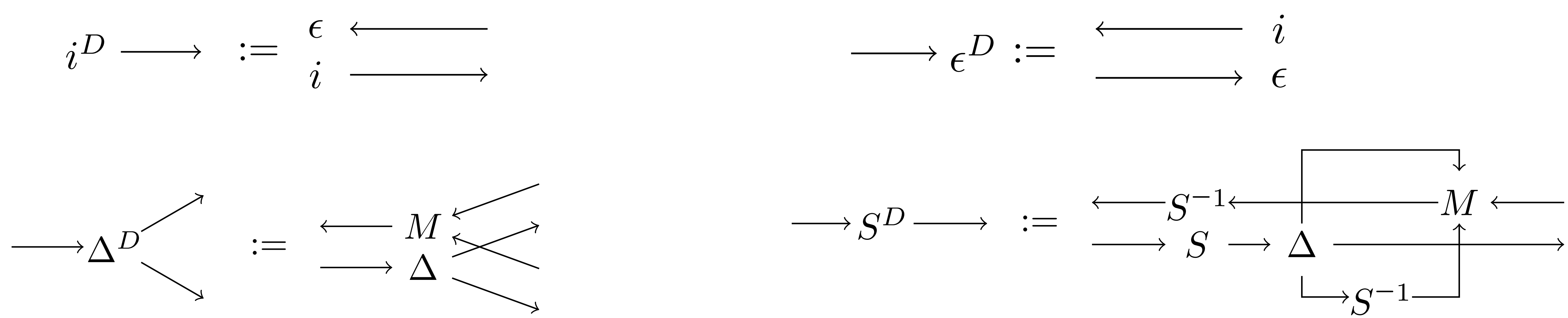

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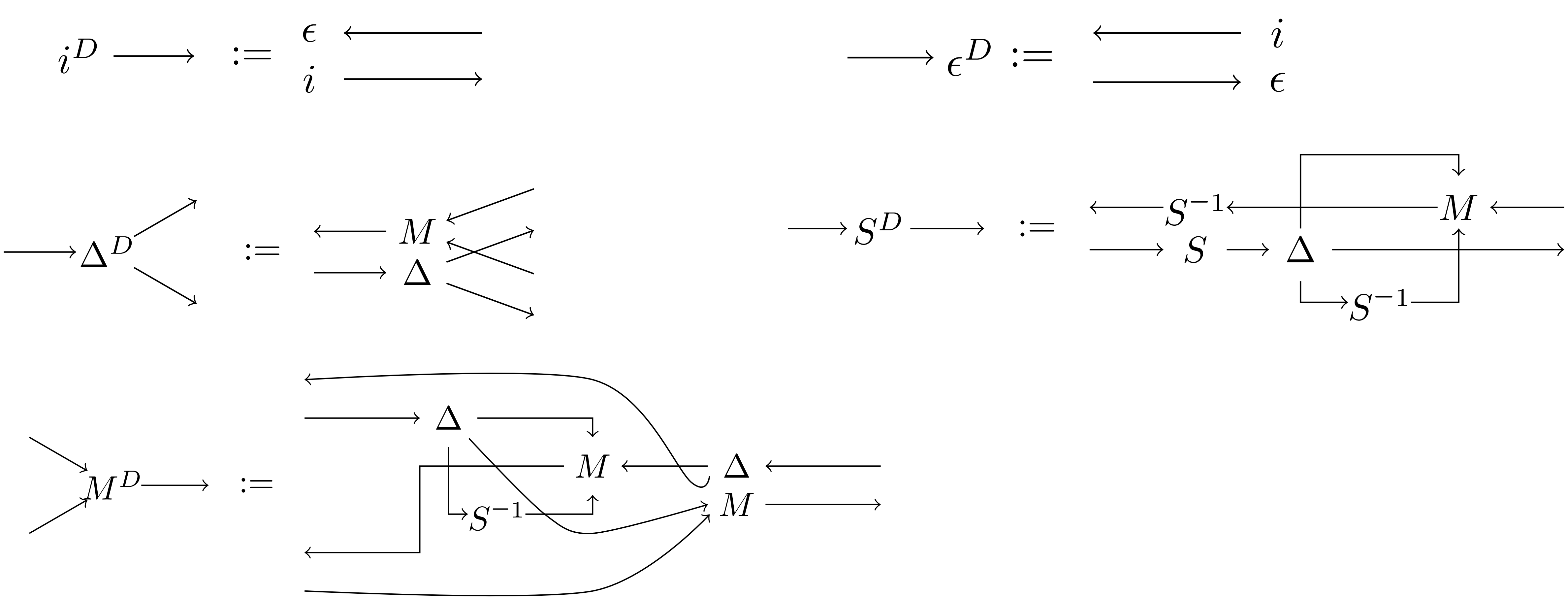

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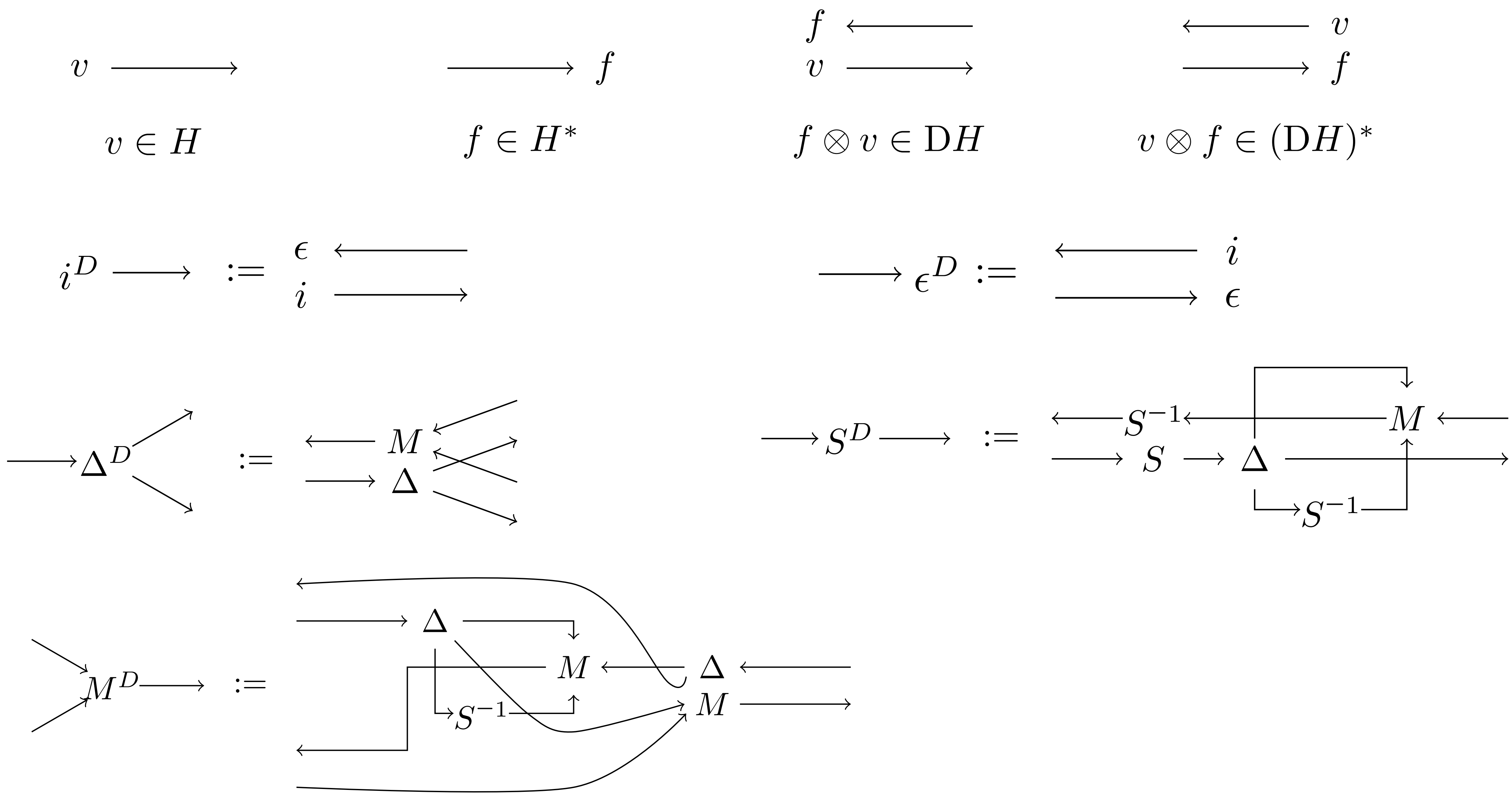

Generalized Drinfeld double : $D(H_1, H_2; \phi) = H_1 \otimes H_2$
given $\phi : H_1 \to H_2^{*,\mathrm{cop}}$

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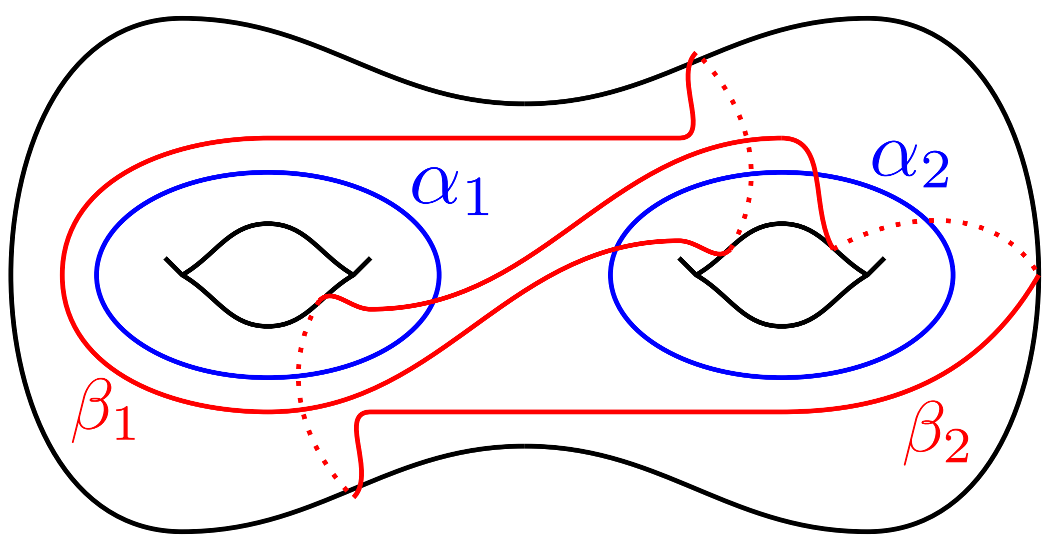
For a 3-manifold M^3, choose a Heegaard diagram

$(\Sigma_g, \alpha = \{\alpha_1, \cdots, \alpha_g\}, \beta = \{\beta_1, \cdots, \beta_g\})$. The $\alpha_i\,'\mathrm{s}$ are disjoint, $\Sigma_g \setminus (\cup \alpha_i)$ is a punctured sphere; so are the $\beta_j\,'\mathrm{s}$.



Kuperberg Invariant $Z^H_{Kup}(M^3; f)$

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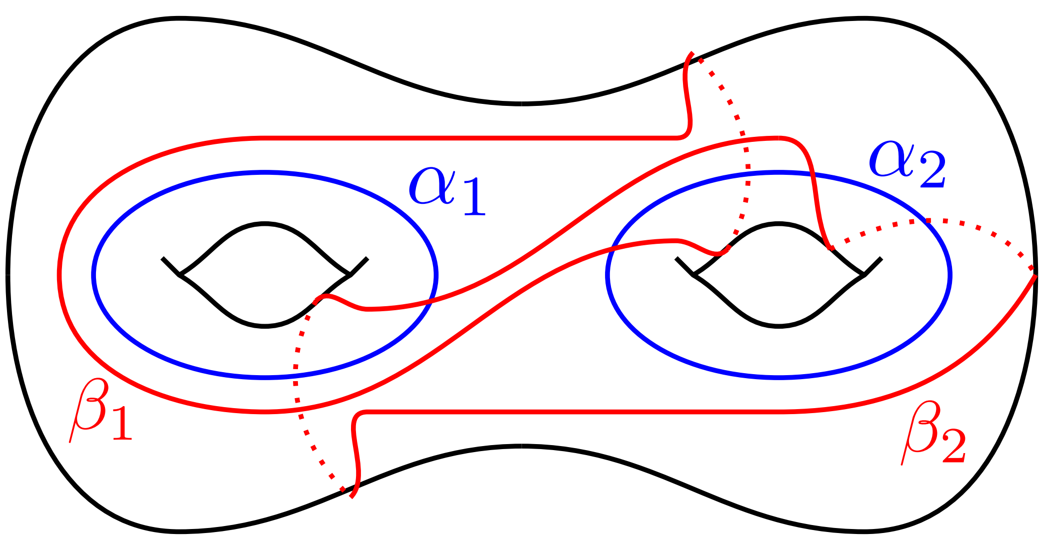
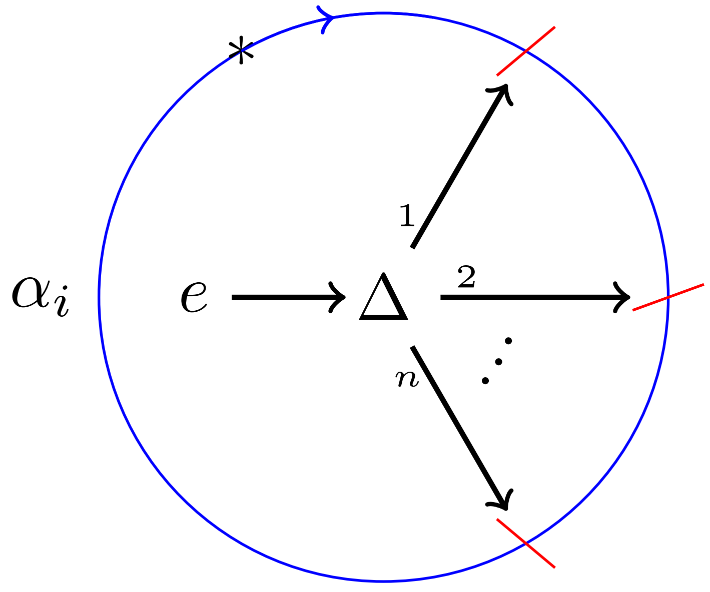
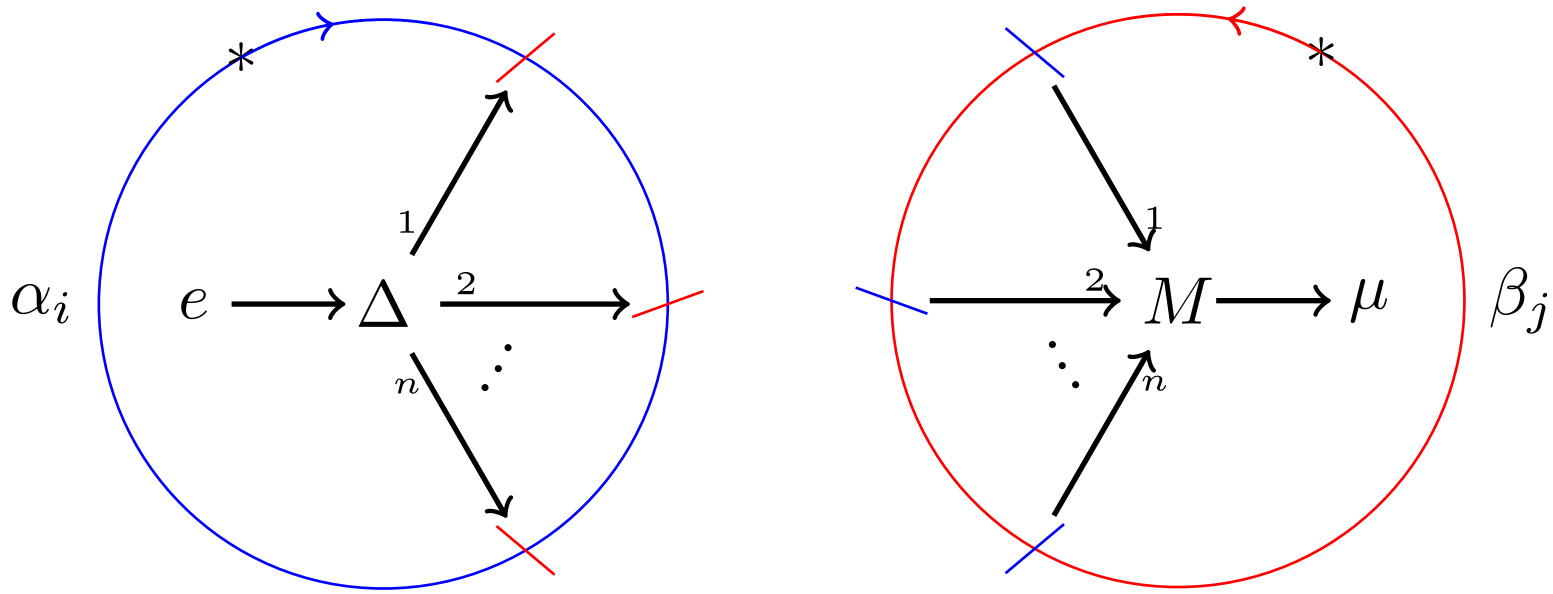
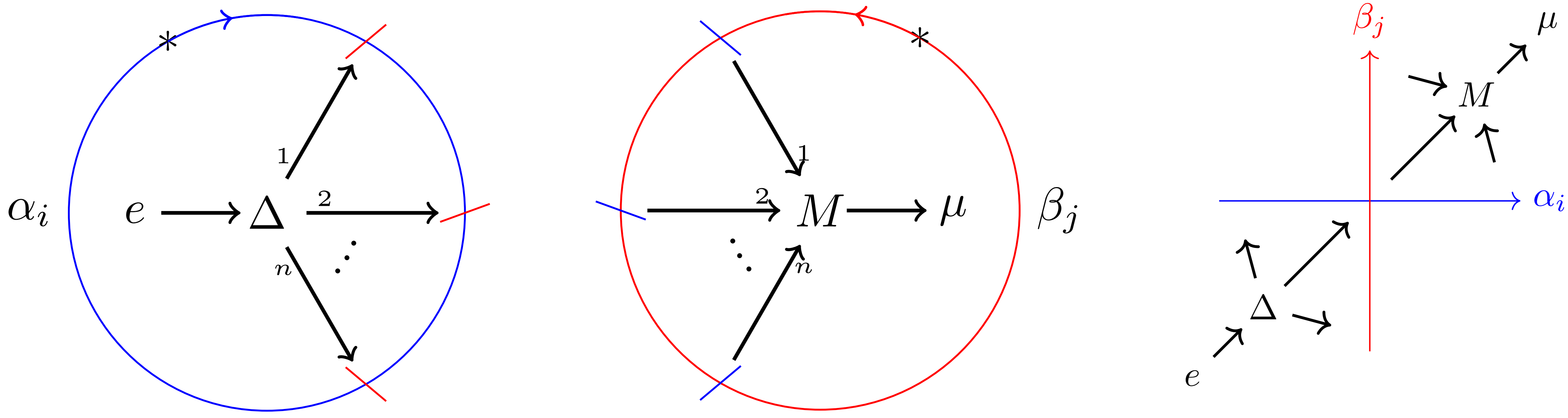


Glue a disk along each α_i on one side of Σ_g and glue a disk along each β_j on the other side, and fill the rest with balls.

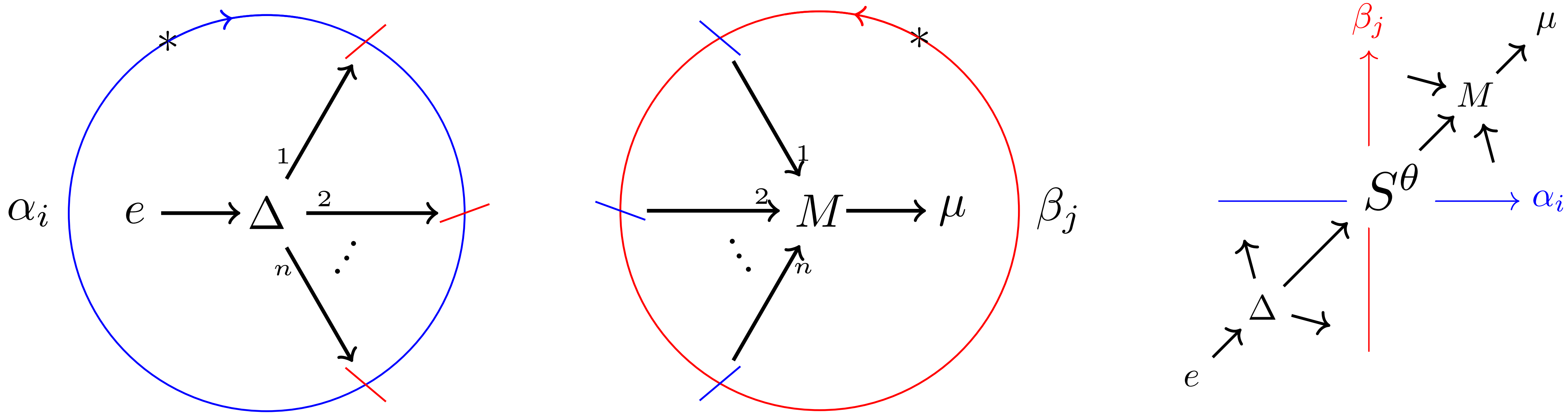

α_i
e
Δ
1
2
n
*



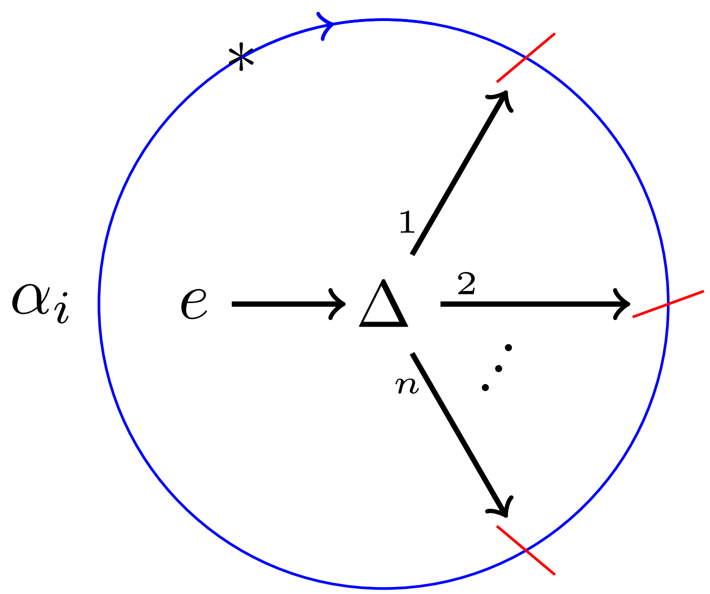
α_i
e
Δ
1
2
n
M
μ
1
2
n
β_j
*
*



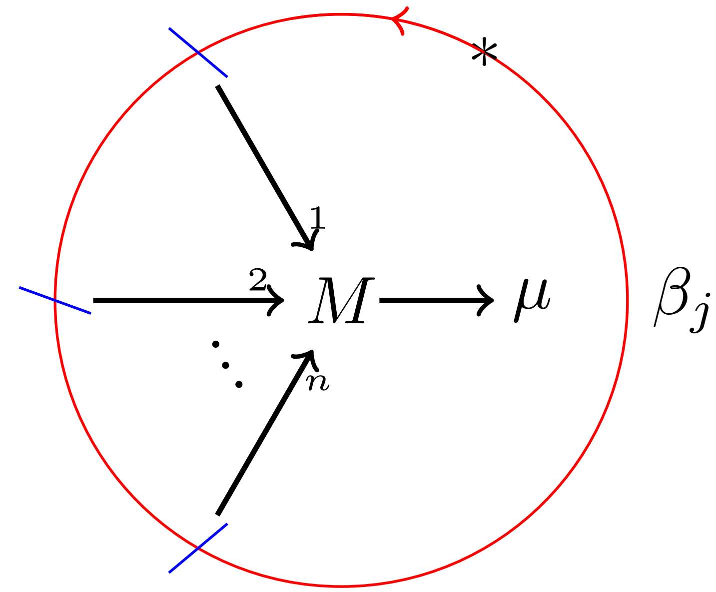
α_i
e
Δ
1
2
n
M
μ
β_j
β_j
μ
M
α_i
Δ
e



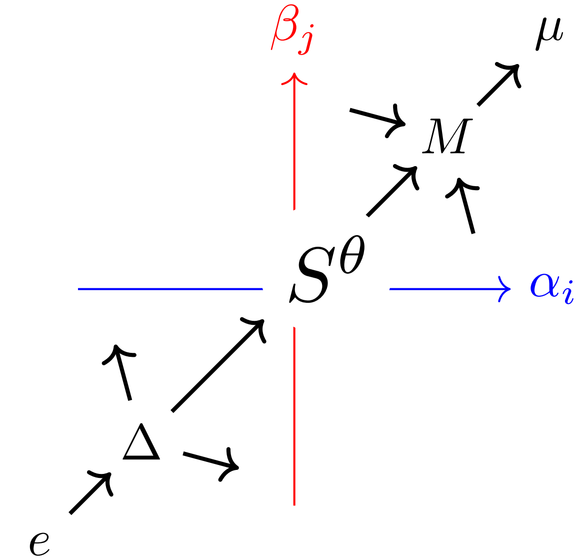
α_i
e
Δ
1
2
n
M
μ
β_j
β_j
μ
M
S^θ
α_i
Δ
e



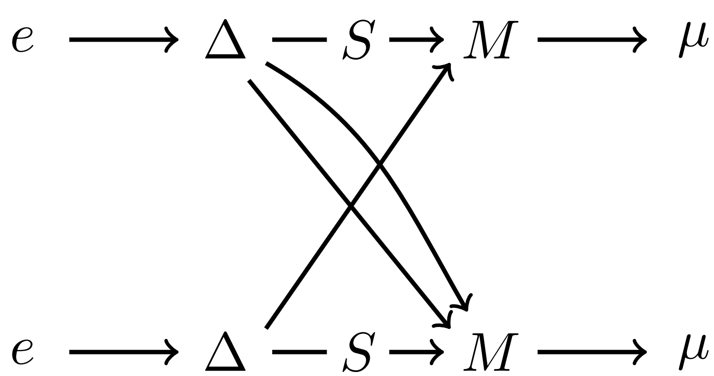
*
α_i
e
Δ
1
2
n



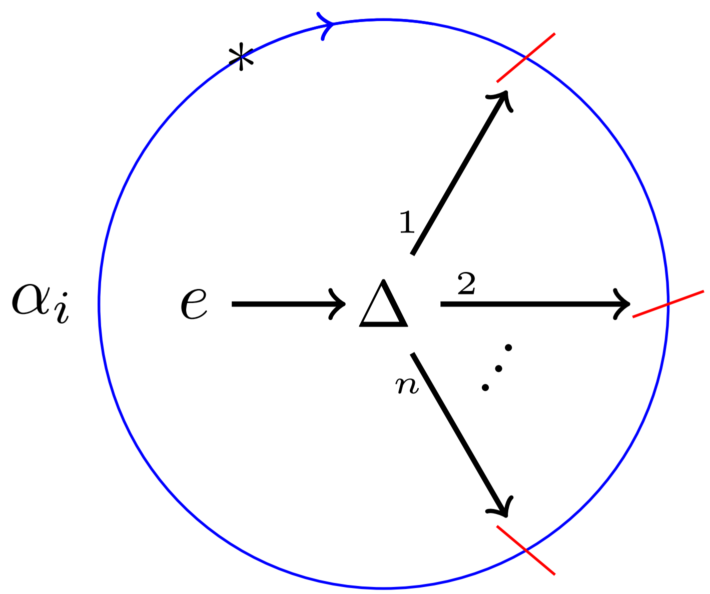
*
1
2
M
μ
n
β_j

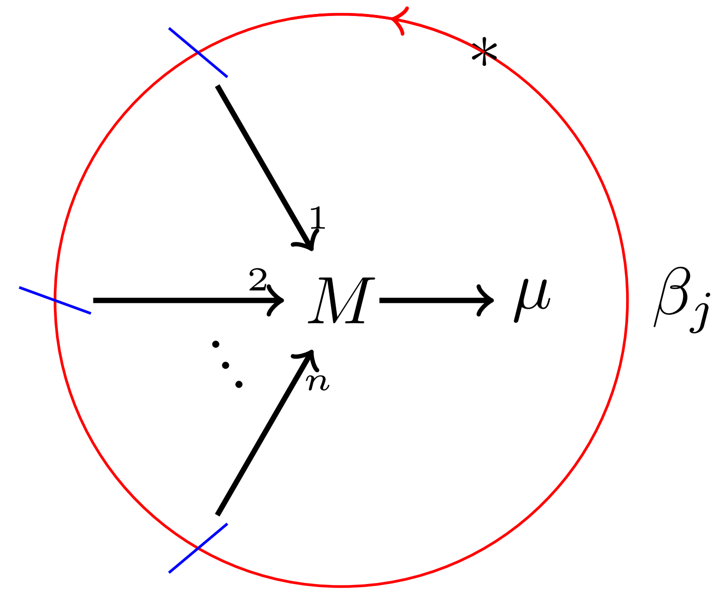


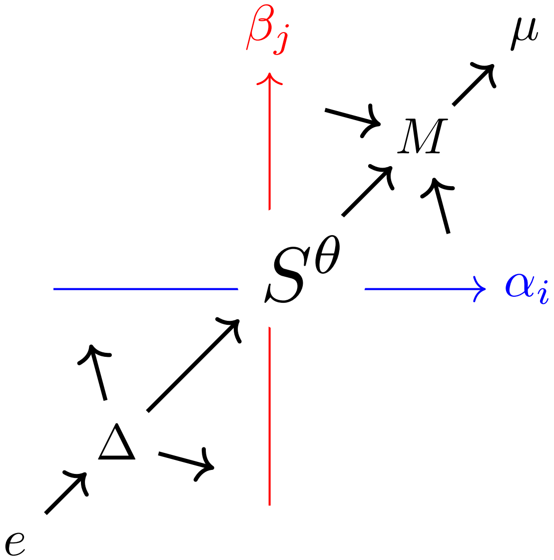
β_j
μ
M
S^θ
α_i
Δ
e

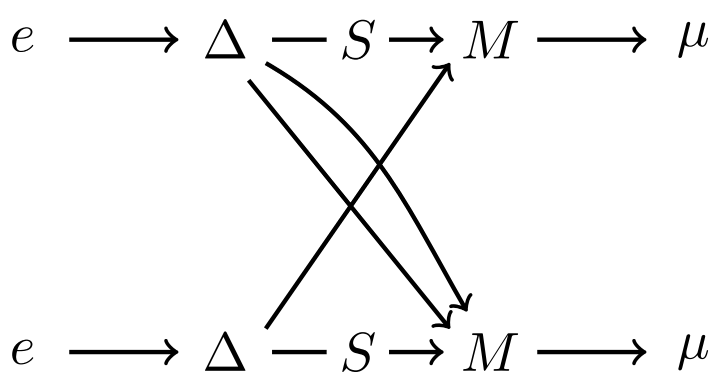


e
Δ
S
M
μ
e
Δ
S
M
μ

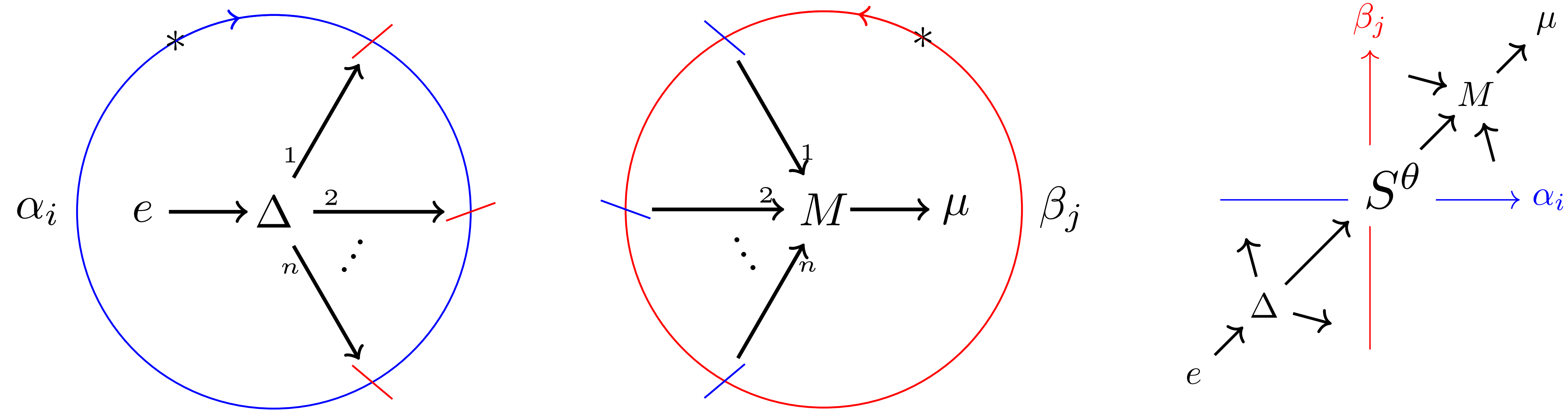


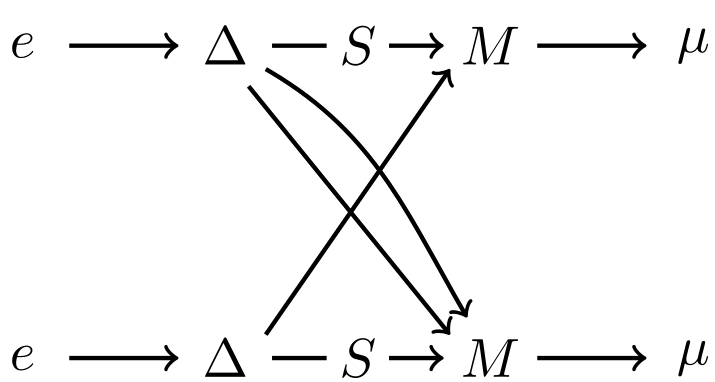






$Z^H_{\mathrm{Kup}}(M^3)$ is the contraction of the tensor diagrams





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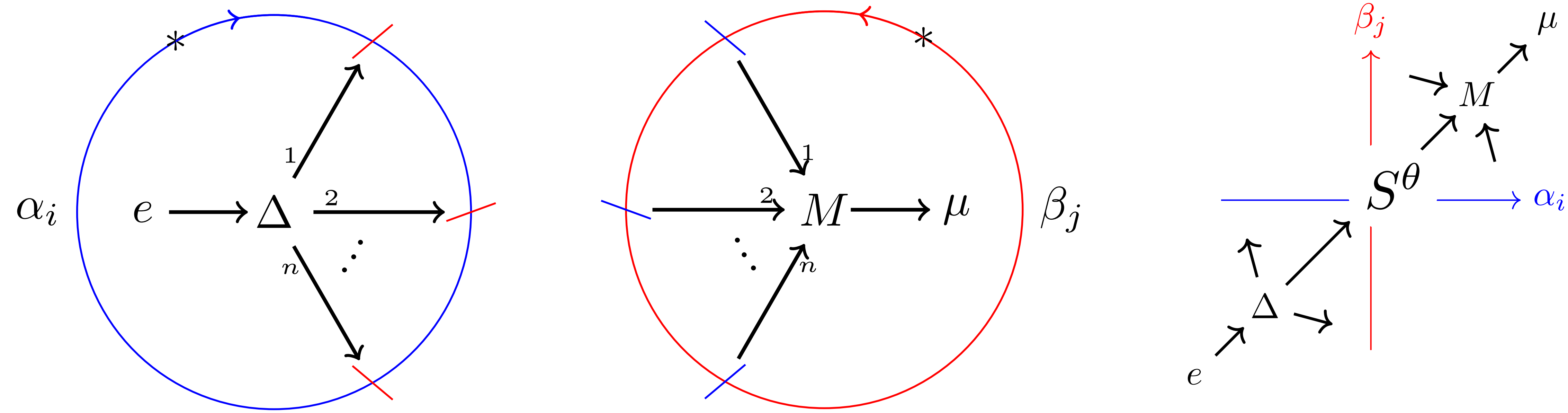

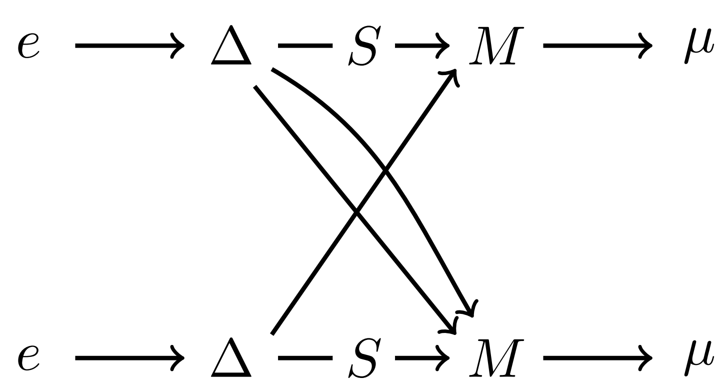

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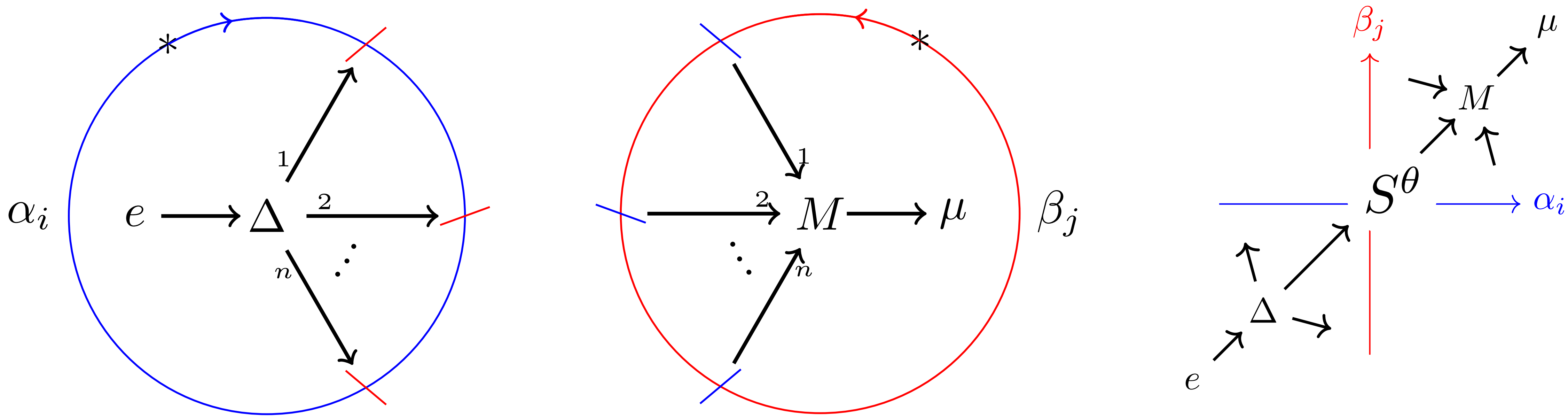

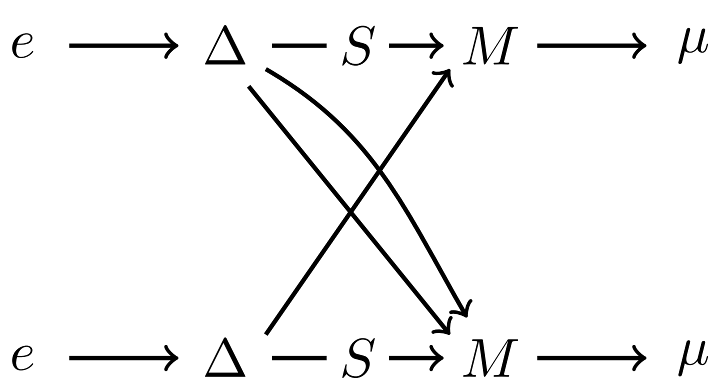

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- Z^H_{Kup} recovers Seiberg-Witten invariant for certain H in $s\mathcal{V}ec$ [Lopez-Neumann, '19]

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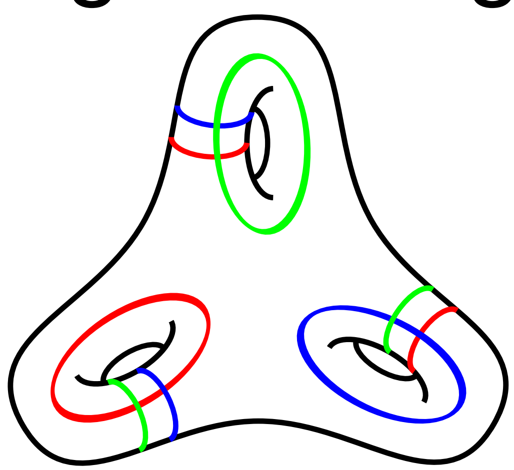

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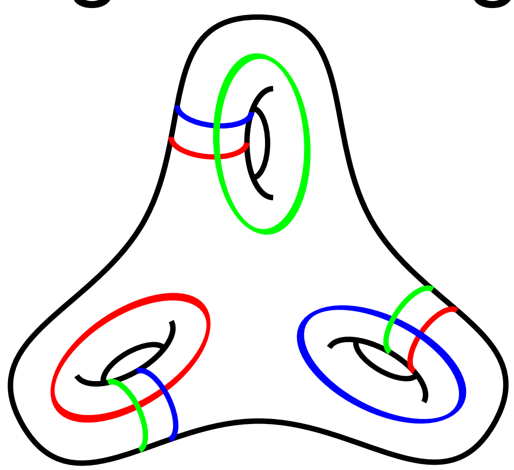

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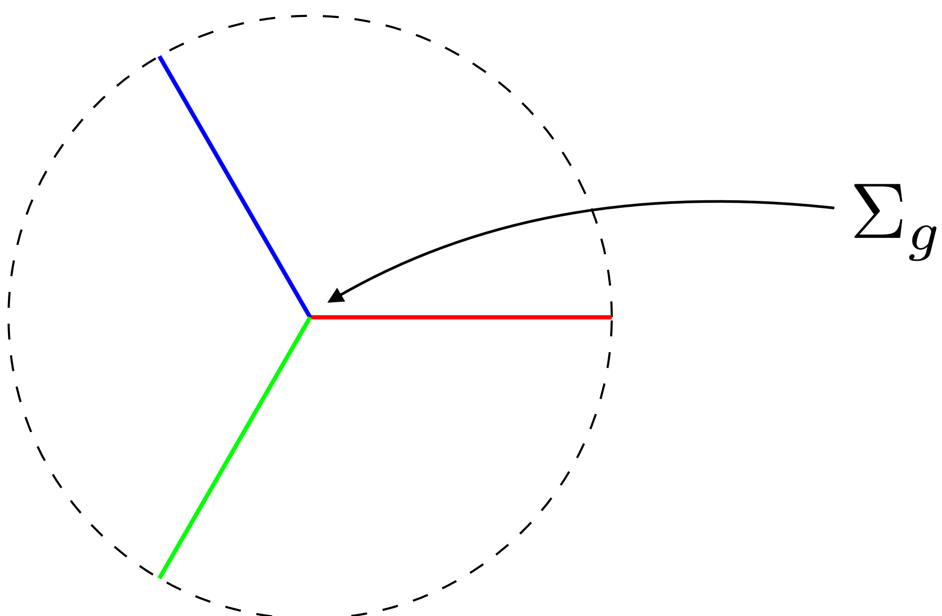

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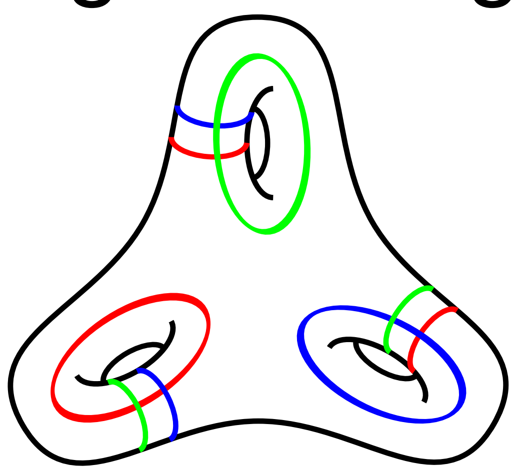

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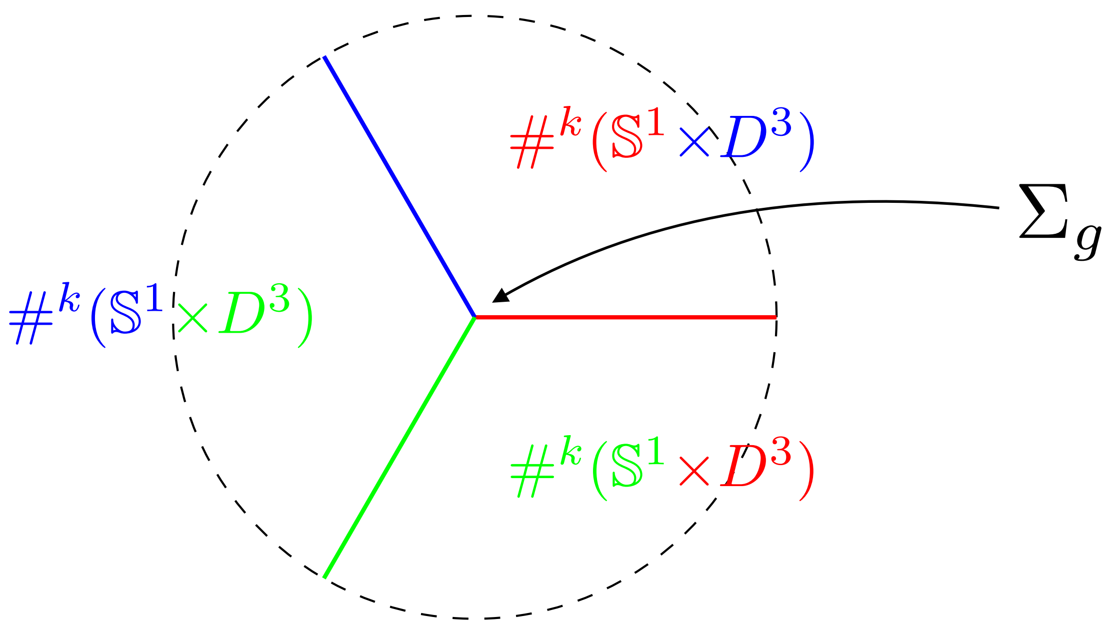

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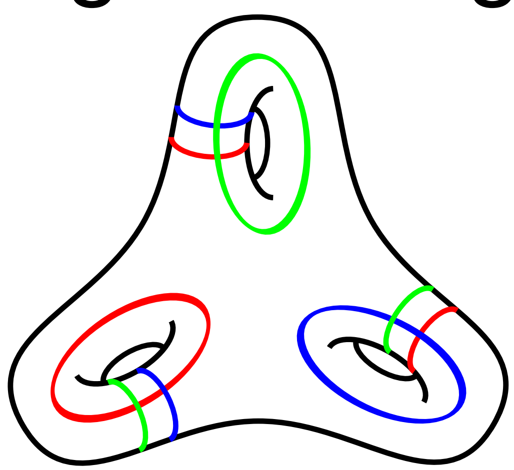

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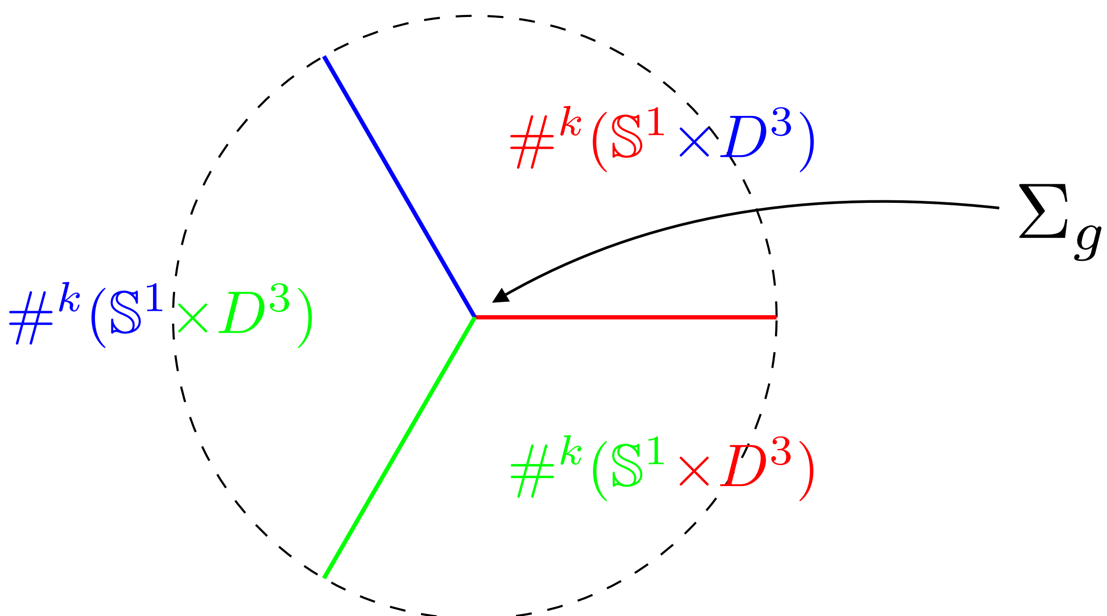

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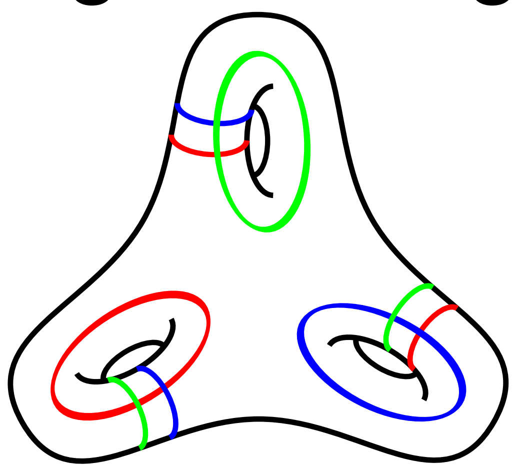

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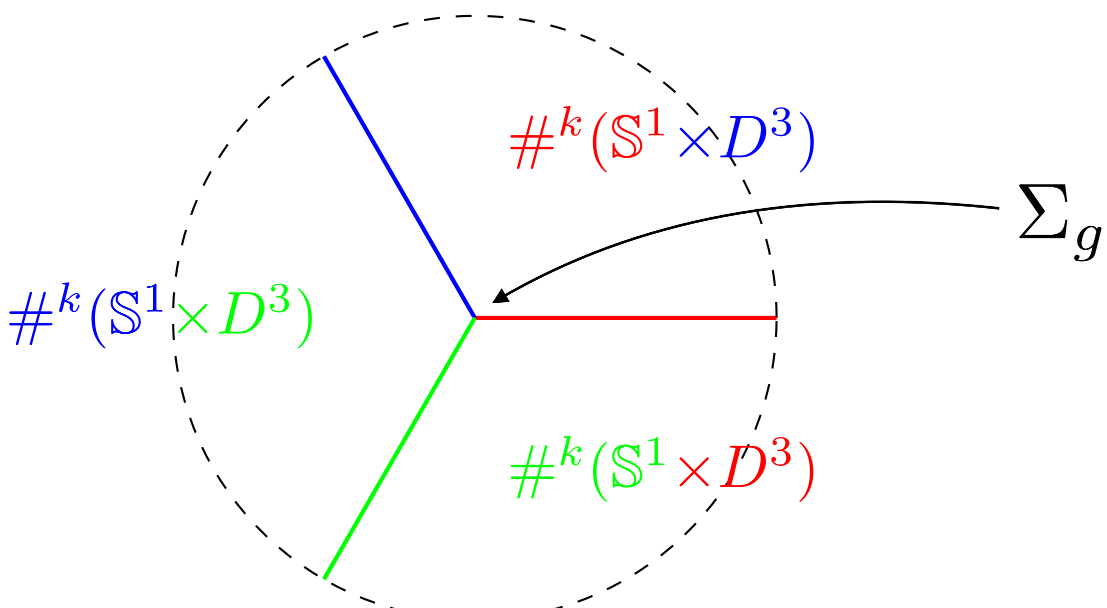

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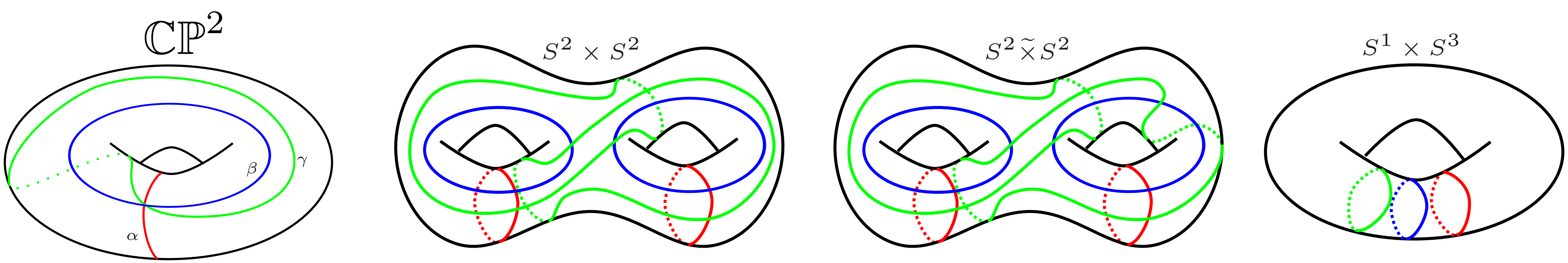

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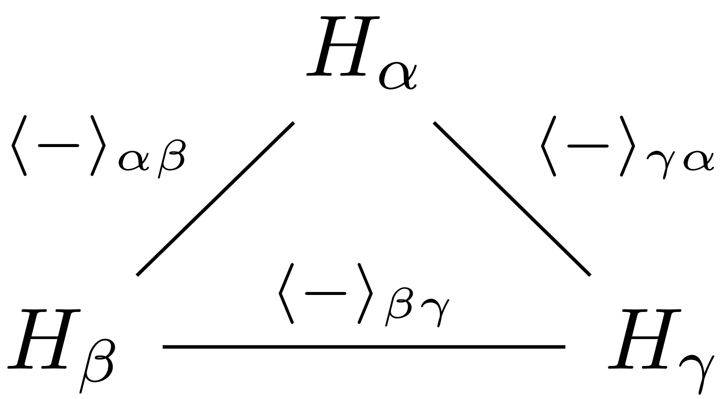
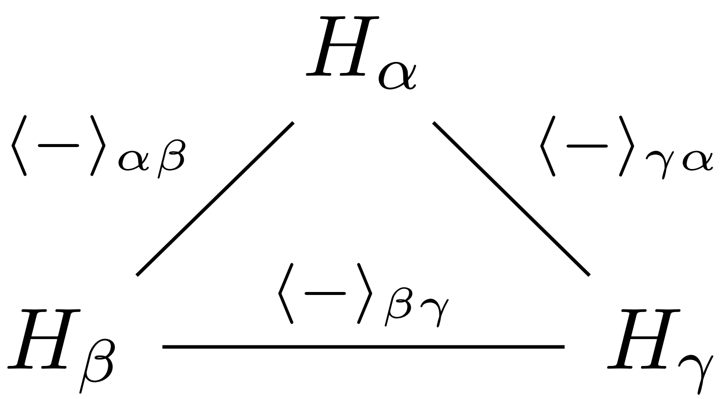
  $(= H_\alpha^{\mathrm{op}} \otimes H_\beta^{\mathrm{cop}})$

Generalized Drinfeld double


H_α
$\langle - \rangle_{\alpha\beta}$
$\langle - \rangle_{\gamma\alpha}$
$\langle - \rangle_{\beta\gamma}$
H_β
H_γ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H_β
$\langle - \rangle_{\beta\gamma}$
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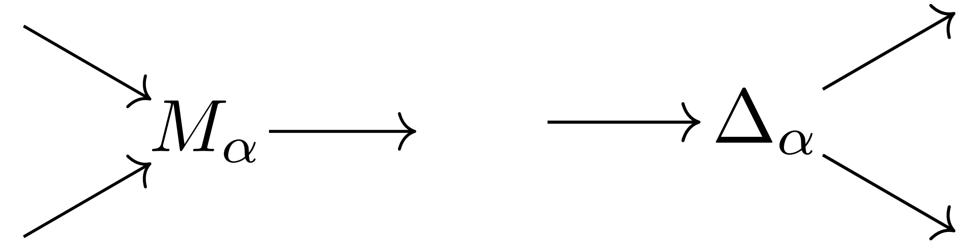
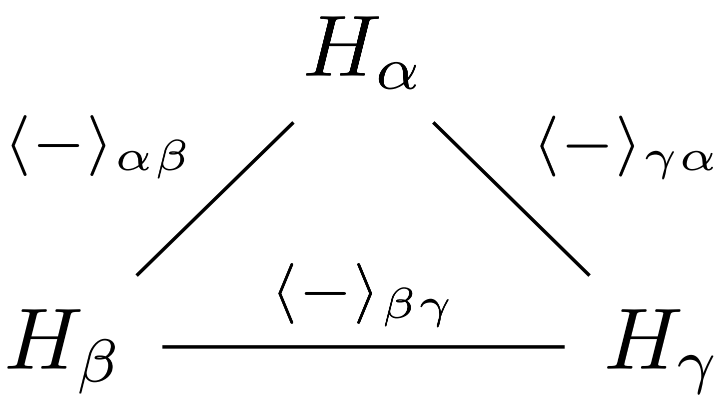
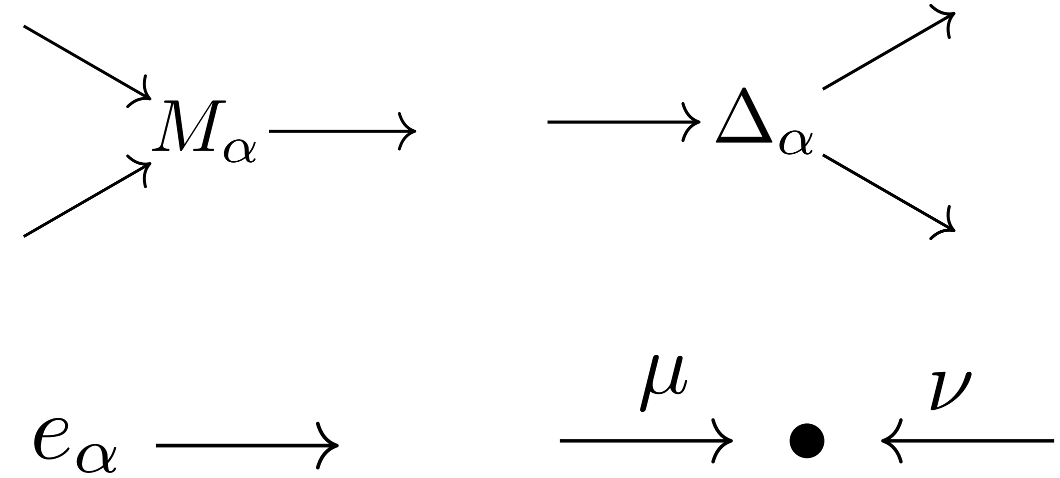

M_α
Δ_α

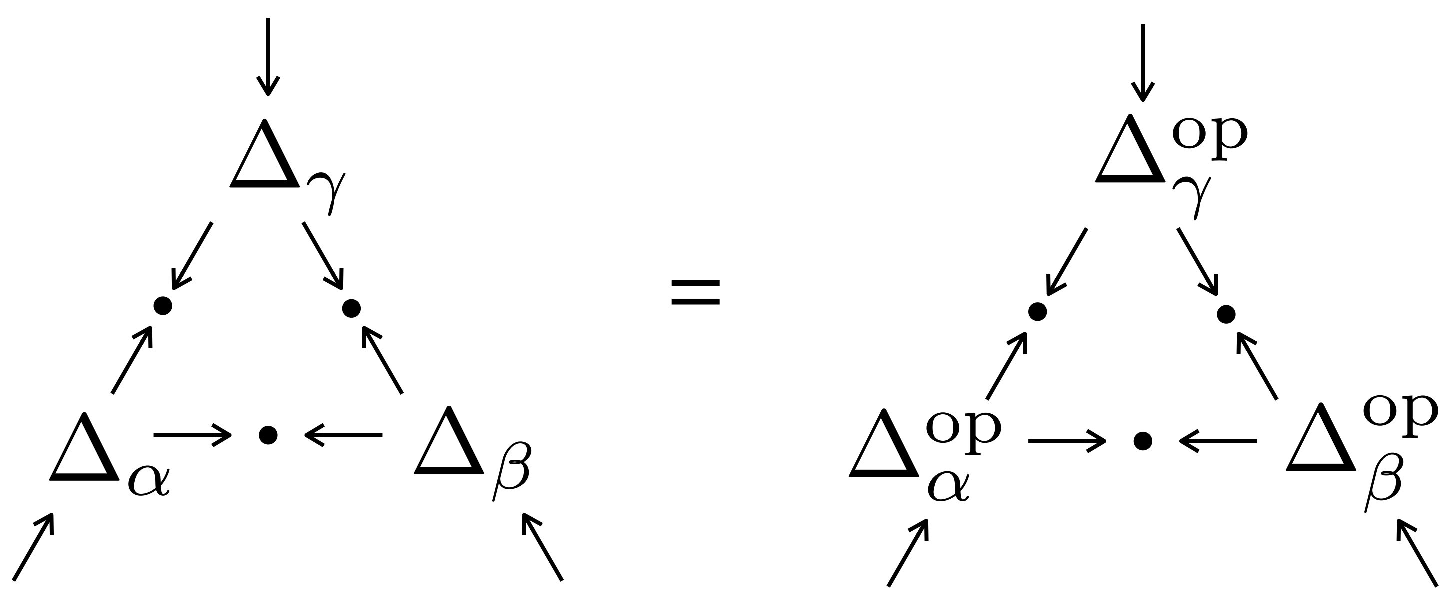

e_α
μ
ν





Lemma [Chaidez, Cotler, C-]

The third condition in the definition is equivalent to



Examples of Hopf triplets

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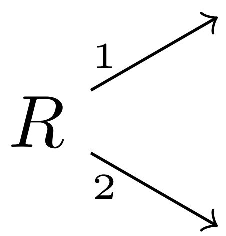
- Let (H, R) be a semisimple quasi-triangular Hopf algebra



$\in H \otimes H$

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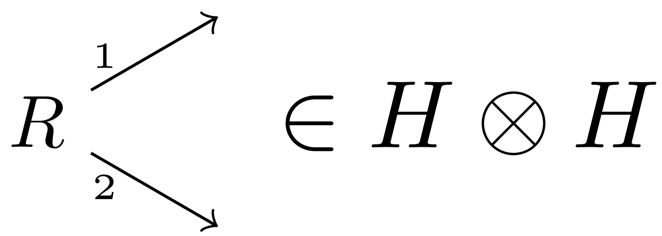

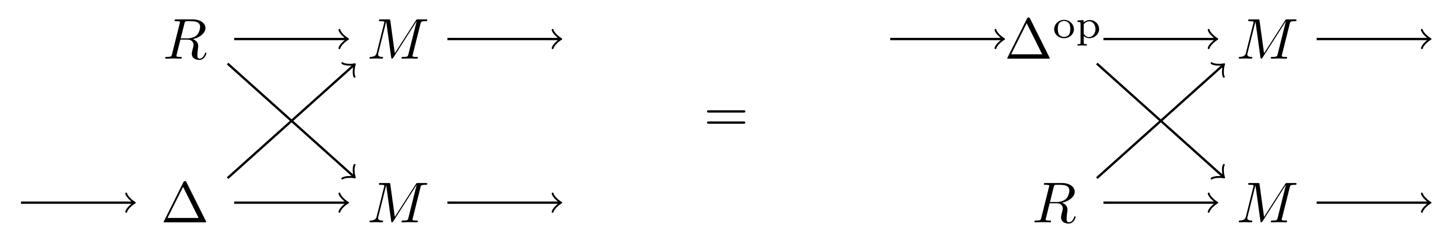

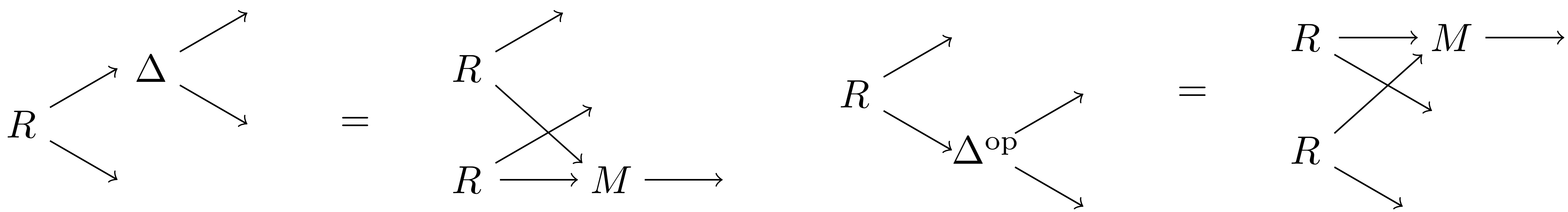

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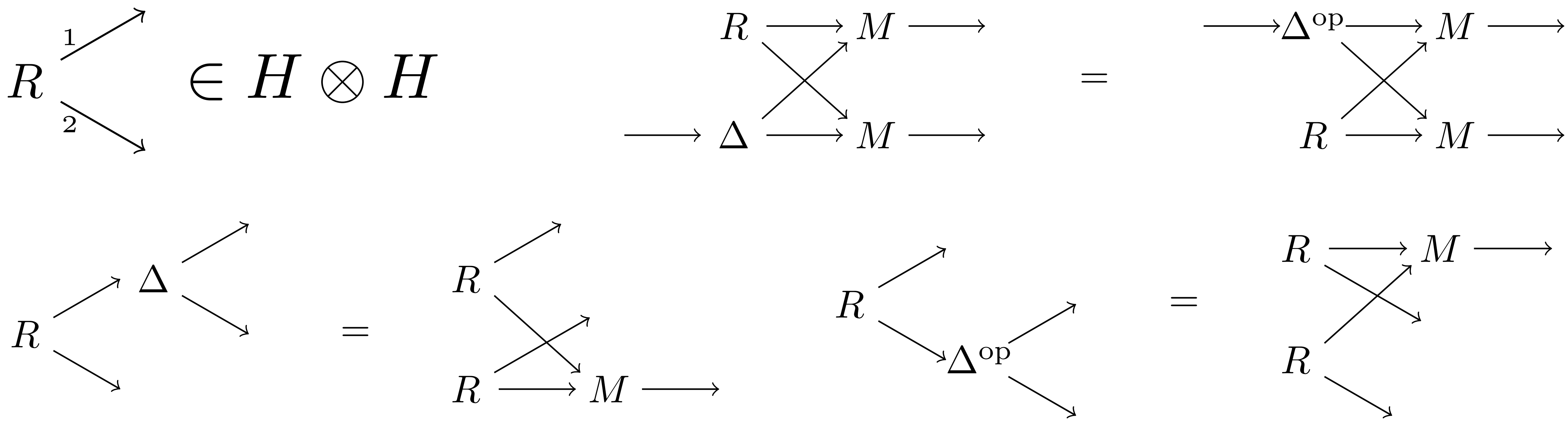

$$\mathcal{H}_H = (H^{*,\mathrm{cop},\mathrm{op}}, H^{\mathrm{cop}}, H^*)$$

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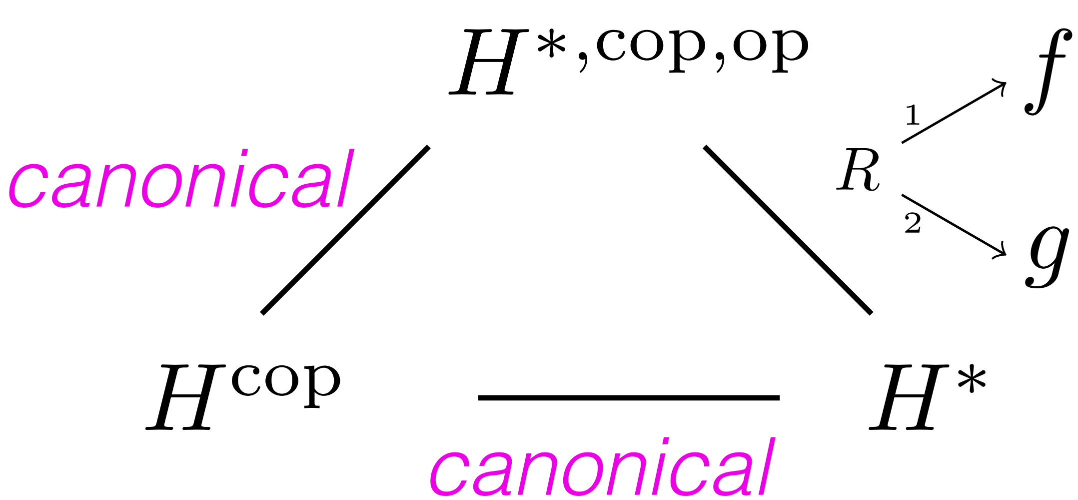

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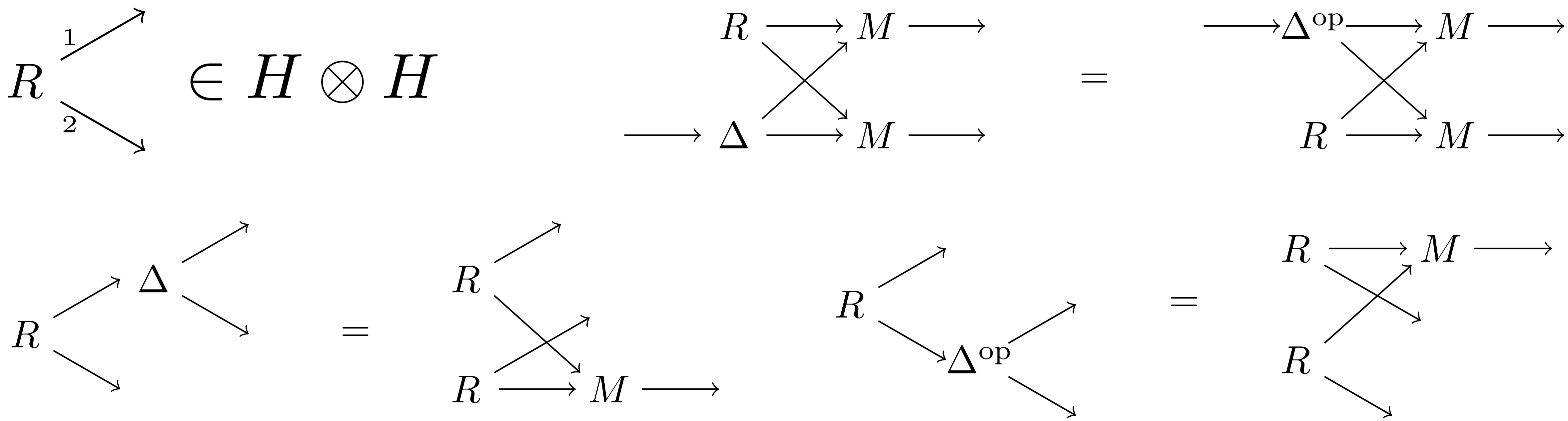

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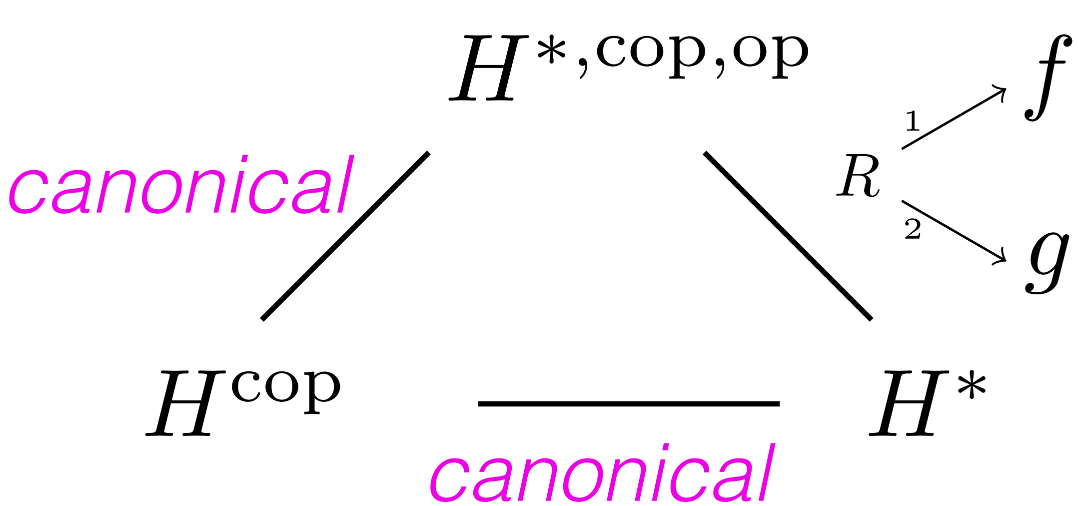

$\mathbb{C}[\mathbb{Z}_N] = \mathbb{C}\langle a \rangle / (a^N - 1)$

is quasi-triangular with

$$R = \frac{1}{N} \sum_{i,j=0}^{N-1} e^{-\frac{2\pi\sqrt{-1}ij}{N}} a^i \otimes a^j$$

- Let H_8 be the unique Hopf algebra in dim 8 that is neither commutative nor cocommutative: $H_8 = \mathbb{C}[x, y, z]/I$

$$I = \langle xy - yx, xz - zy, yz - zx, x^2 - 1, y^2 - 1,\ z^2 - \frac{1}{2}(1 + x + y - xy) \rangle$$

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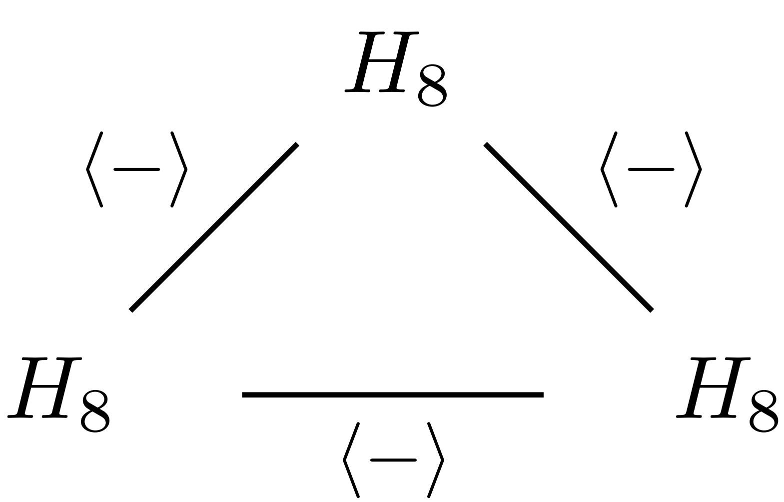
Consider triplet of the form

$(H_8, H_8, H_8; \langle - \rangle)$



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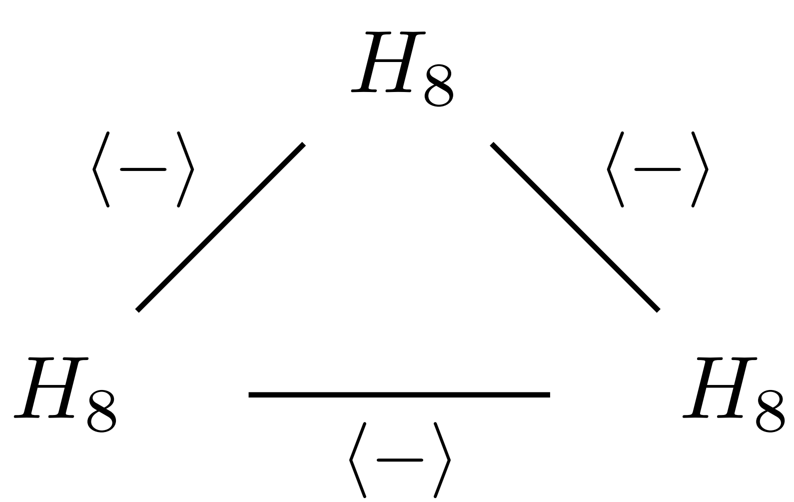

There are MANY choices for $\langle - \rangle$, e.g.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -i & i & i & -i \\ 1 & -1 & -1 & 1 & i & -i & -i & i \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & i & -i & -1 & -\sqrt{2} & 0 & 0 & \sqrt{2} \\ 1 & -i & i & -1 & 0 & -i\sqrt{2} & i\sqrt{2} & 0 \\ 1 & -i & i & -1 & 0 & i\sqrt{2} & -i\sqrt{2} & 0 \\ 1 & i & -i & -1 & \sqrt{2} & 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Invariant of 4-manifolds from Hopf triplets

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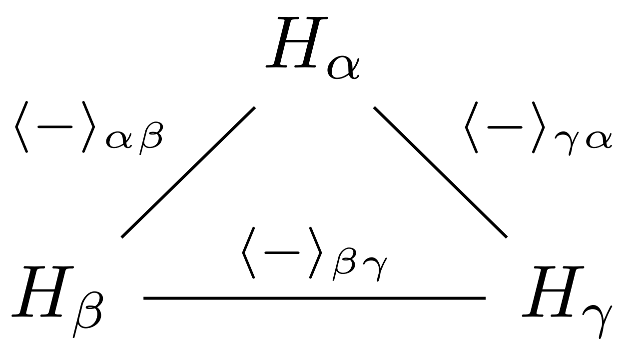

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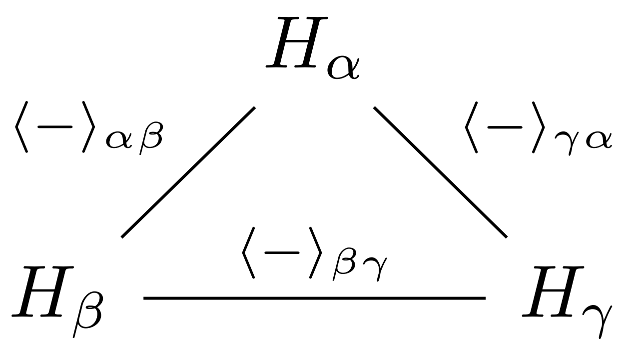

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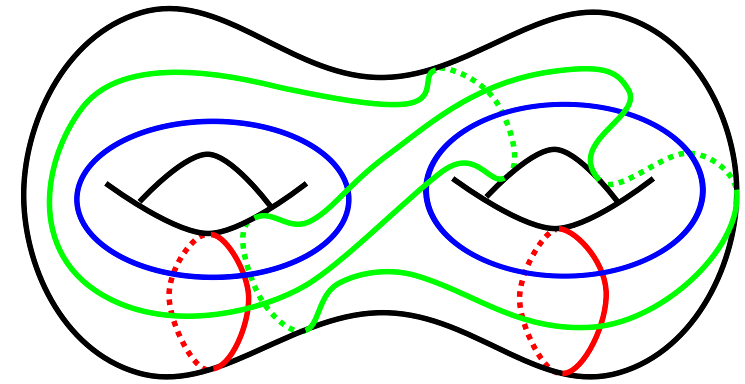

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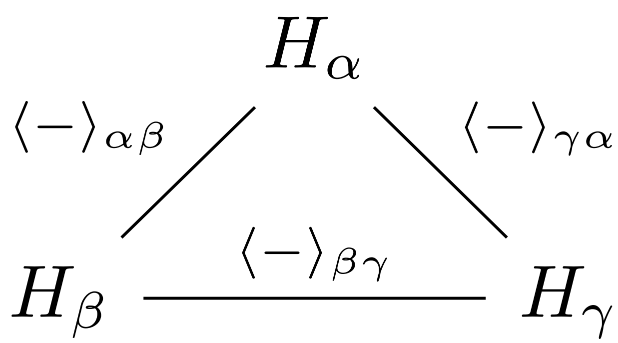

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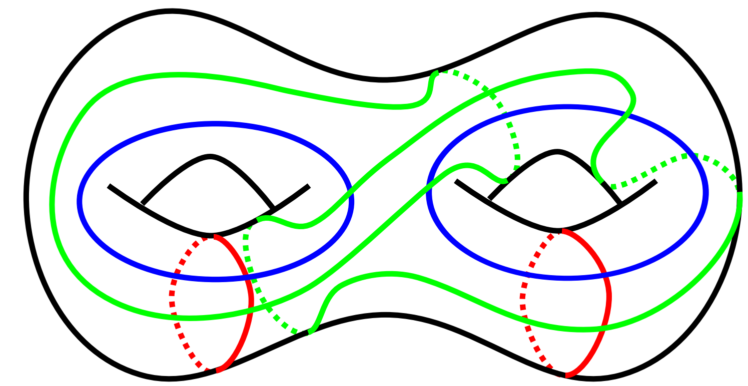

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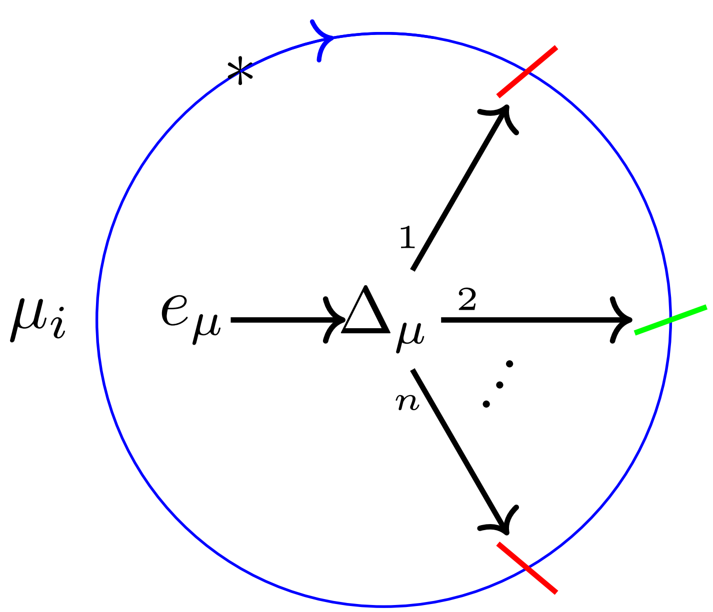

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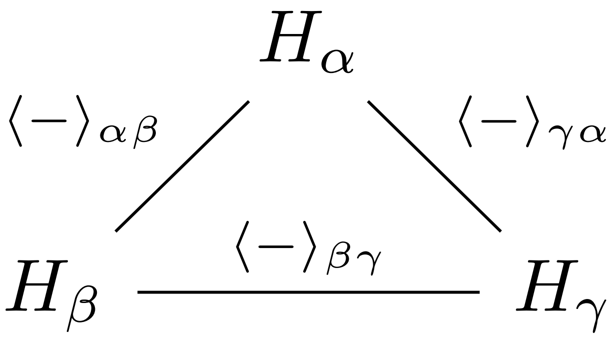

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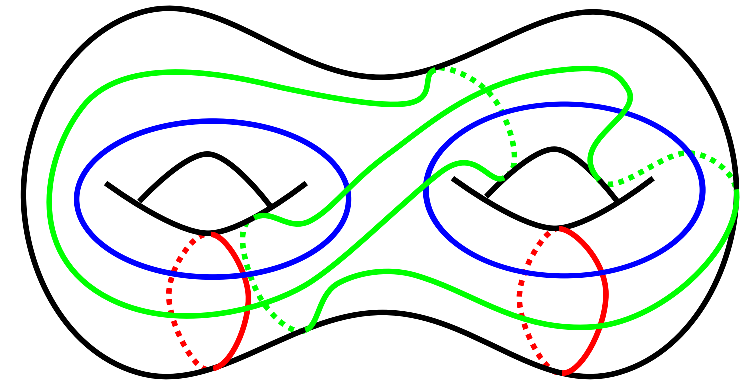

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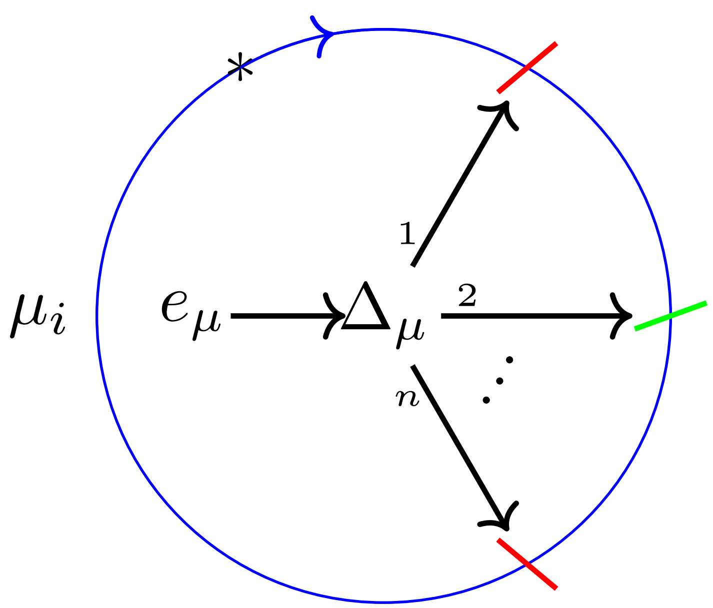
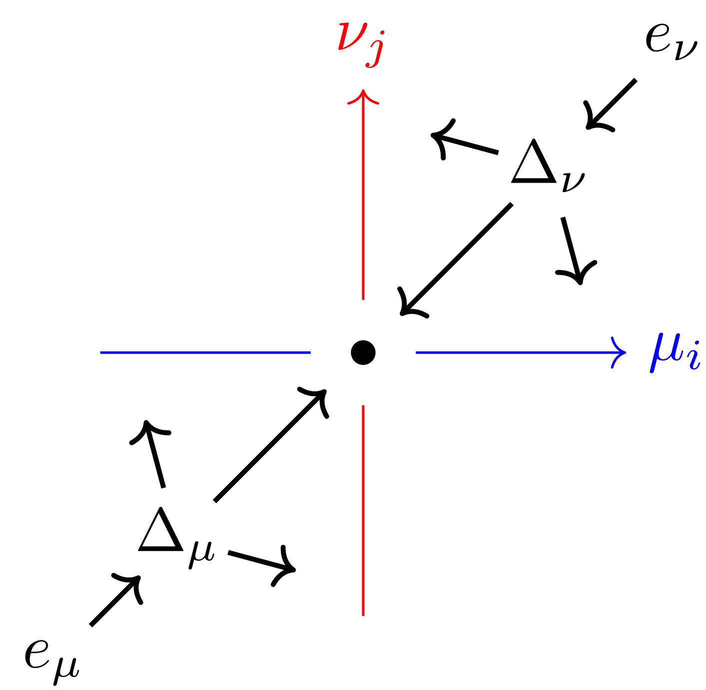


$$\mu\nu \in \{\alpha\beta, \beta\gamma, \gamma\alpha\}$$



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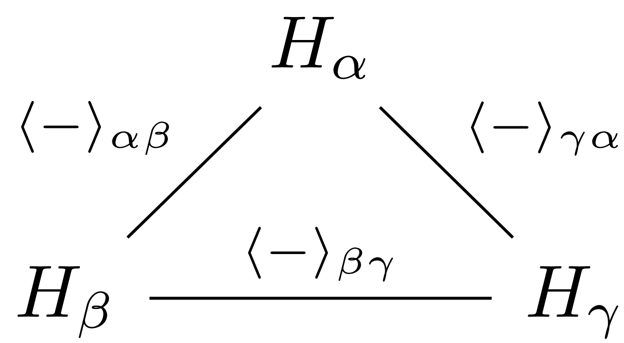

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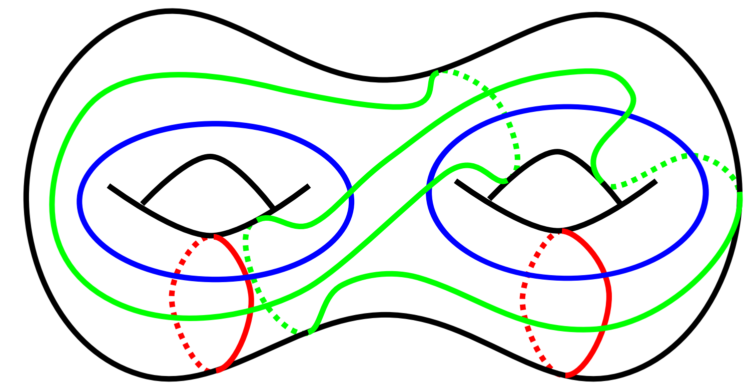

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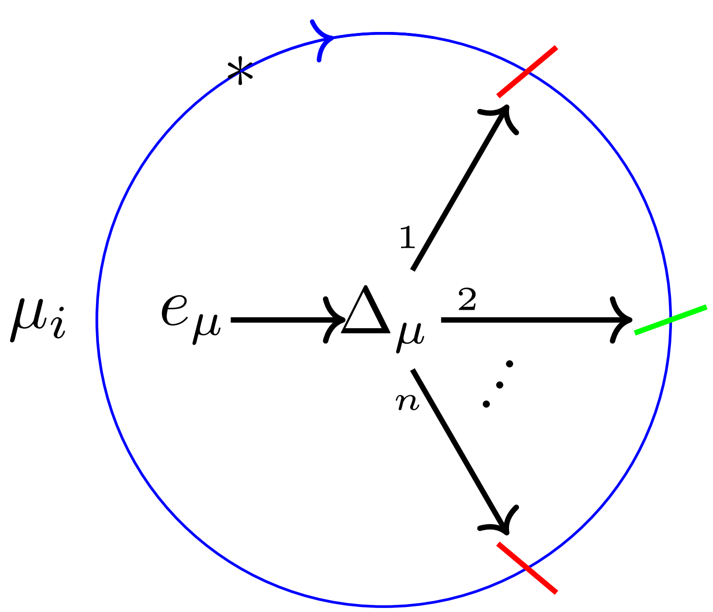
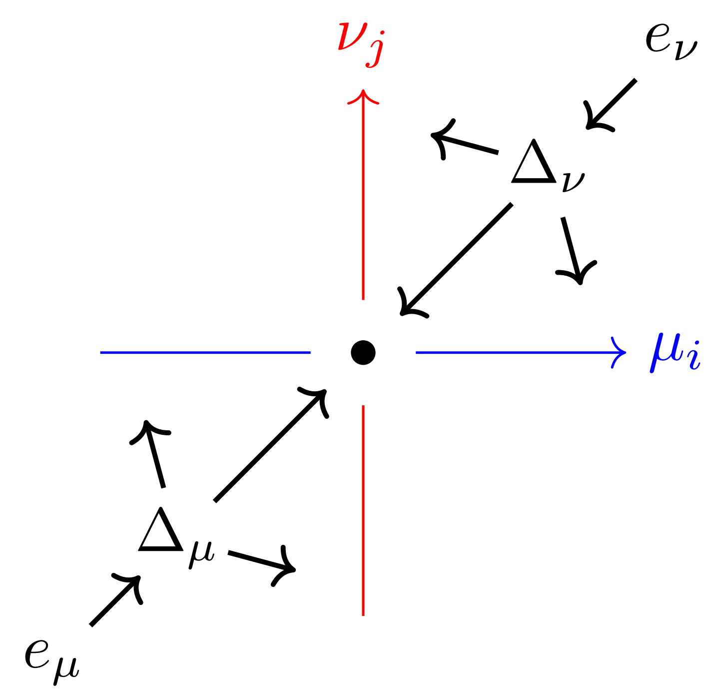
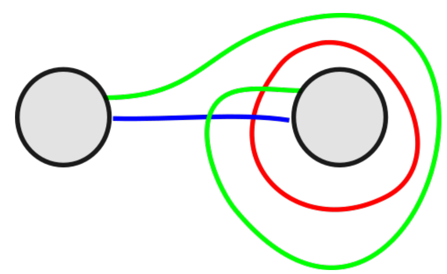
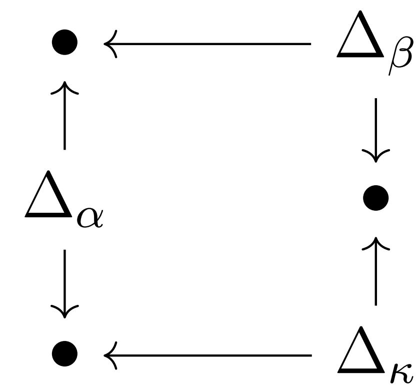




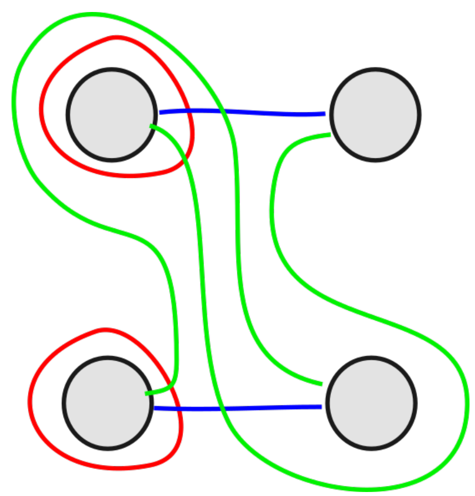
$Z(M; \mathcal{H})$ is defined to be the contraction of the tensors with some normalization factor.

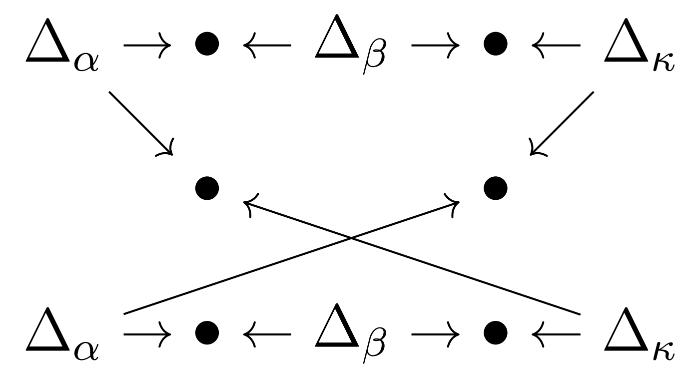
$\mathbb{CP}^2$





$S^2 \times S^2$





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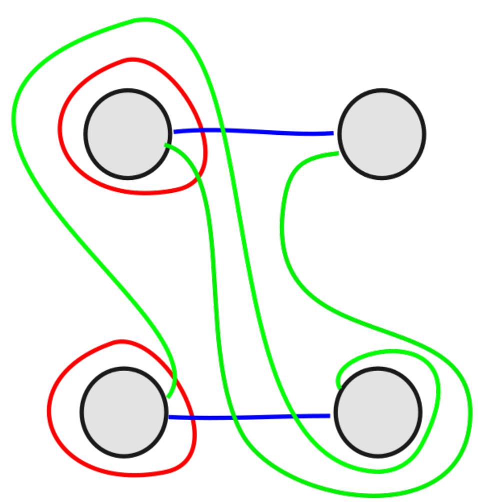

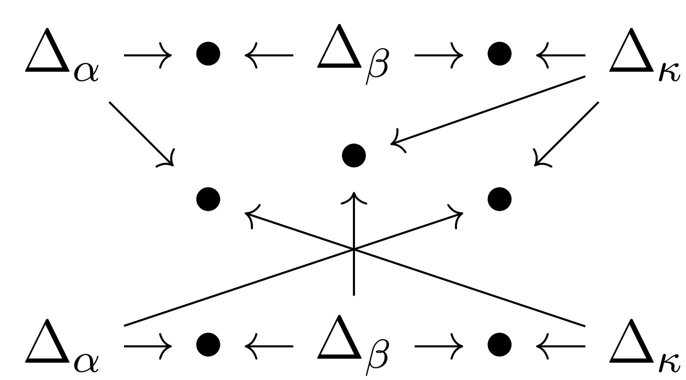

Theorem [Chaidez, Cotler, C-]

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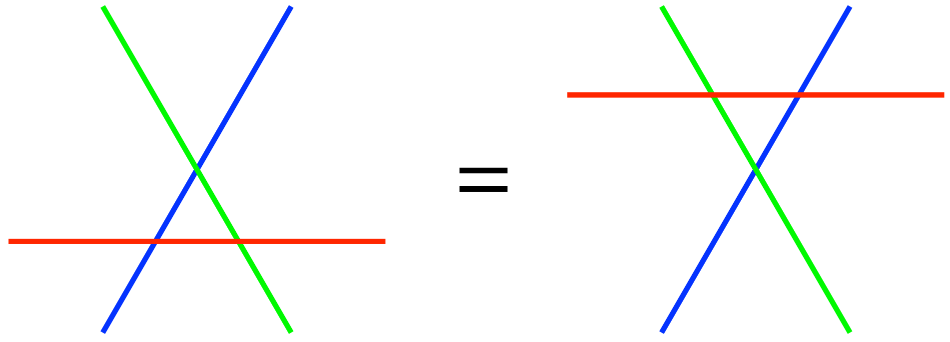

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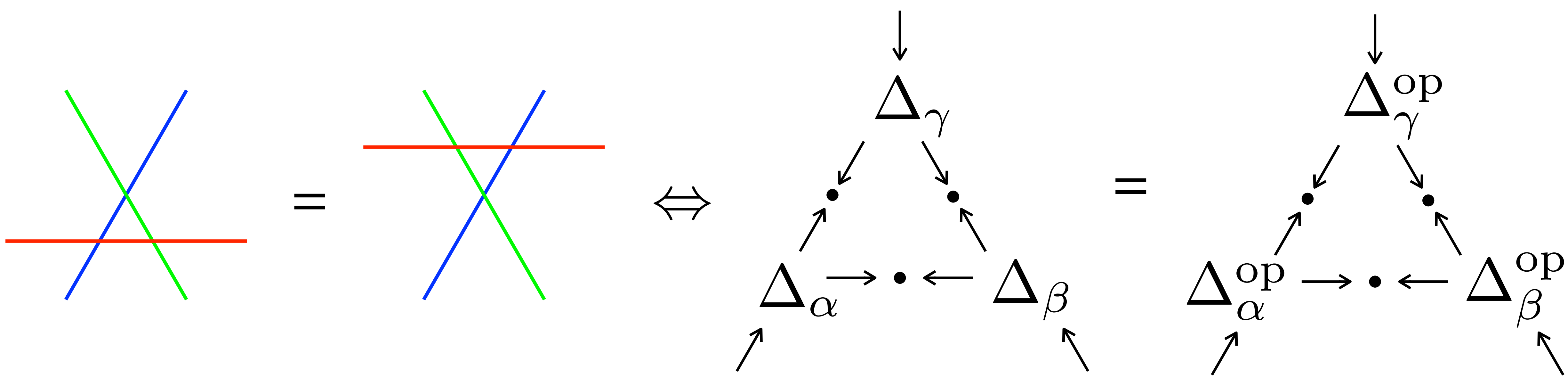

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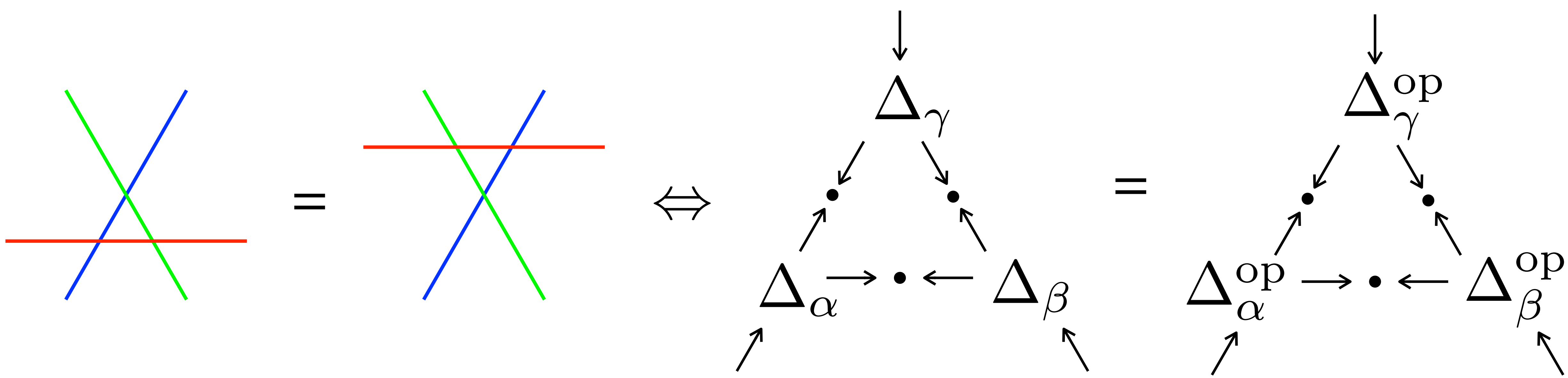

Proposition [Chaidez, Cotler, C-]

$$Z(M_1 \# M_2;\mathcal{H}) = Z(M_1;\mathcal{H})\, Z(M_2;\mathcal{H})$$

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Theorem [Chaidez, Cotler, C-]

If H is quasi-triangular semisimple, then

$$Z(M;\mathcal{H}_H) = \dim(H)^{1-\chi(M)}\ \mathrm{CY}(M;\mathrm{Rep}(H))$$

Crane-Yetter

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Conjecture [Chaidez, Cotler, C-]

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Kashaev invariant

If the above conjecture is right, then

Corollary: Crane-Yetter contains Kashaev invariant.

(conjectured by Williamson-Wang)

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Thank You

arXiv:1910.14662