MSRI
TC & TQFT Conference
March 2020 ①

Thank you for this opportunity.

"1-dimensional Tangle Hypothesis"

Joint with John Francis, using of joint work with Nick Rozenblyum.*

def/ **Tangle Hypothesis** (After Lurie, after Baez-Dolan) Let $n>1$.

Let $\mathcal{X}$ be an $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category.

1st There is an action $\Omega\mathbb{R}P^{n-1} \curvearrowright \mathrm{Obj}(\mathcal{X}^{\mathrm{duals}})$ on the moduli space of (R&L)-dualizable objects in $\mathcal{X}$.

"Tangle Category"

2nd Evaluation at $(\{0\}\subset\mathbb{R}^{n-1}) \in \mathrm{Bord}_1^{\mathrm{fr}}(\mathbb{R}^{n-1})$ factors as an $\Omega\mathbb{R}P^{n-1}$-equivariant equivalence between $(\infty,1)$-categories:

$$\begin{tikzcd}
 & \mathrm{Obj}(\mathcal{X}^{\mathrm{duals}}) \arrow[d] \\
\mathrm{Fun}^{\mathcal{E}_{n-1}}(\mathrm{Bord}_1^{\mathrm{fr}}(\mathbb{R}^{n-1}), \mathcal{X}) \arrow[ur, dashed, "\simeq", "\Omega\mathbb{R}P^{n-1}\text{-equiv.}"'] \arrow[r, "ev_{\{0\}}"'] & \mathcal{X}
\end{tikzcd}$$

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Cor def/ Bordism Hypothesis

Proof Take $(n-1)\longrightarrow\infty$.

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* • "The Cobordism Hypothesis" w/ Francis (ArXiv)
• "Factorization Homology I: Higher Categories" w/ Francis-Rozenblyum
• "A stratified homotopy hypothesis" w/ Francis-Rozenblyum

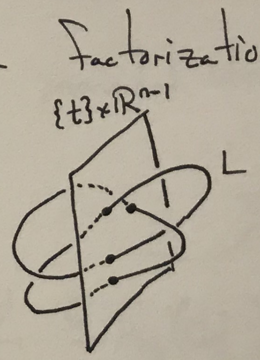
Practical Implications

- Each (1<n)-framed compact tangle $L' \subset \mathbb{R}^n$ determines a map

⛤ $$\mathrm{Obj}(\mathcal{X}^{\mathrm{duals}}) \xrightarrow{\quad Z^{\bullet}(L) \quad} \mathrm{End}_{\mathcal{X}}(\mathbb{1})$$ between spaces.

- Each transverse hyperplane $\{t\}\times\mathbb{R}^{n-1} \pitchfork L$ determines a factorization

$$Z^{\bullet}(L) : \mathbb{1} \xrightarrow{Z^{\bullet}(L_{\leq t})} Z^{\bullet}(L \cap \{t\}\times\mathbb{R}^{n-1}) \xrightarrow{Z^{\bullet}(L_{\geq t})} \mathbb{1} .$$



Eg| Let $U_q\mathfrak{g}$ be a quantum group $/ \mathbb{C}(q)$ (q generic).

Drinfeld ⤳ the 1-category $\mathrm{Rep}^{\mathrm{f.dim}}(U_q\mathfrak{g})$ has an $\mathcal{E}_2$-monoidal structure.

⛤ $$\mathrm{Obj}(\mathrm{Rep}^{\mathrm{f.dim}}(U_q\mathfrak{g})) \xrightarrow{\quad Z^{\bullet}(L) \quad} \mathrm{End}_{\mathrm{Rep}^{\mathrm{f.dim}}(U_q\mathfrak{g})}(\mathbb{1}) \xrightarrow{\simeq} \mathbb{C}(q)$$

$$\overset{\circlearrowright}{\Omega\mathbb{RP}^2} \qquad V \longmapsto \text{Reshetikhin-Turaev invariant } (\mathfrak{sl}_2)$$

$\pi_0 = O(1)$: duals

$\pi_1 = \mathbb{Z}$: ribbon, trivialized

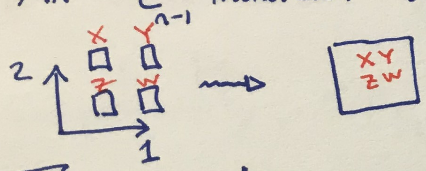
some Definitions

Def • An $\mathcal{E}_1$-monoidal $(\infty,1)$-category is $\underbrace{\text{an associative algebra in } \mathrm{Cat}_{(\infty,1)}}_{\text{"monoidal } (\infty,1)\text{-category"}}$.

• For $n>2$, $\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}) := \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{Alg}_{\mathcal{E}_{n-2}}(\mathrm{Cat}_{(\infty,1)}))$.

Thm Dunn, Fiedorowicz, Lurie

An $\mathcal{E}_{n-1}$-monoidal structure on $\mathcal{X}$ is $(n-1)$ compatible monoidal structures



Obs $\mathrm{CAlg}(\mathrm{Cat}_{(\infty,1)}) \to \cdots \xrightarrow{\mathrm{Fgt}} \mathrm{Alg}_{\mathcal{E}_n}(\mathrm{Cat}_{(\infty,1)}) \xrightarrow{\mathrm{Fgt}} \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}) \xrightarrow{\mathrm{Fgt}} \cdots \xrightarrow{\mathrm{Fgt}} \mathrm{Alg}_{\mathcal{E}_2}(\mathrm{Cat}_{(\infty,1)}) \xrightarrow{\mathrm{Fgt}} \mathrm{Alg}_{\mathcal{E}_1}(\mathrm{Cat}_{(\infty,1)})$

"Symmetric Monoidal" "Braided Monoidal" "Monoidal"

Rmk $\mathrm{Spaces}^{*/}_{\geq n} \xrightarrow{\Omega^n} \mathrm{Alg}_{\mathcal{E}_n}(\mathrm{Cat}_{(\infty,1)})$, it's fully faithful.

Rmk $\mathrm{CAlg}(\mathrm{Cat}_{(k,d)}) \underset{k<n-1}{\overset{\mathrm{Fgt}}{\xrightarrow{\ \sim\ }}} \mathrm{Alg}_{\mathcal{E}_n}(\mathrm{Cat}_{(k,d)})$.

Def $V \in \mathcal{X}$ is <u>(R&L) dualizable</u> if it is in $\mathrm{ho}(\mathcal{X})$

$\mathcal{X}$: $\mathcal{E}$-monoidal $(\infty,1)$-category

$\mathrm{ho}(\mathcal{X})$: underlying monoidal (homotopy) $(1,1)$-category.

Framings A $(1\subset n)$-framed tangle is

- $L \overset{\text{proper}}{\subset} M$ ← $\boxed{n\text{-manifold}}$, possibly w/ ∂ or codim=1 defect, or $(n-1)$-manifold
  codim = n-1
- an n-framing on M: $\tau_M \oplus \varepsilon_M^{n\text{-dim}} \underset{\varphi}{\cong} \varepsilon^n_M$
- a 1-framing on L: $\tau_L \oplus \varepsilon_L^{1\text{-dim}} \underset{\psi^\tau}{\cong} \varepsilon^1_L$
- a trivialization: $\nu_{L\subset M} \underset{\psi^\nu}{\cong} \varepsilon^{n-1}_L$
- Compatibility: $\varphi|_L \sim \psi^\tau \oplus \psi^\nu$.

$$\begin{tikzcd}
 & & \ast \arrow[d] & \\
 & & \mathbb{R}P^{n-1} \arrow[d] \arrow[dr] \arrow[dr, phantom, "\vee", near start] & \\
L \arrow[rr, "{(\tau_L\oplus\varepsilon_L^{1\text{-dim}},\ \nu_{L\subset M})}"'] \arrow[dr, "inc"'] \arrow[uurr, dashed, bend left, "\psi"] \arrow[urr, dashed, "\text{Gauss}\varphi"'] & & BO(1)\times BO(n-1) \arrow[dr, "\oplus"] & \ast \arrow[d] \\
 & M \arrow[rr, "\tau_M\oplus\varepsilon_M^{n\text{-dim}}"'] \arrow[urr, dashed, "\varphi"] & & BO(n)
\end{tikzcd}$$

Eg • $S^1 \subset \mathbb{R}^2$ has no $(1\subset 2)$-framings
- $S^1 \subset \mathbb{R}^3$ has $\pi_0 \mathrm{Map}(S^1, \Omega \mathbb{R}P^2) \cong$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$-many $(1\subset 3)$-framings
- $S^1 \subset \mathbb{R}^\infty$ has $O(1)$-many $(1\subset\infty)$-framings.

Def (sketch) (see ArXiv's preprint) Let M be an $(n-1)$-framed $(n-1)$-manifold.

$\mathrm{Bord}_1^{fr}(M)$ is the $(\infty,1)$-category:

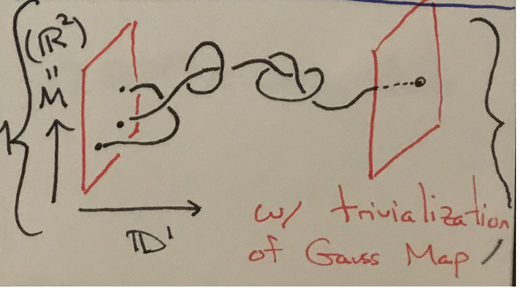
Obj = Moduli Space of $(1\subset n)$-framed tangles in M

= Moduli Space of compact 0-submanifolds w/ $(1\subset n)$-framing

$\simeq \coprod_{r\geq 0} \mathrm{Conf}_r(M) \underset{\Sigma_r}{\times} (\Omega \mathbb{R}P^{n-1})^{\times r}$

Terminology $\mathrm{Bord}_1^{fr}(\mathbb{R}^{n-1})$ is the $d=1$ tangle $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category.

Mor = Moduli Space of $(1\subset n)$-framed tangles in $\mathbb{D}^1 \times M$



w/ trivialization of Gauss Map ✓

Composition = Concatenate along common faces.

⑤

Prop "1st" $\Omega\mathbb{RP}^{n-1} \curvearrowright \mathrm{Obj}(\mathcal{X}^{\mathrm{duals}})$.

Proof $\mathrm{Homo}(\Omega\mathbb{RP}^{n-1}, \mathrm{Aut}(\mathrm{Obj})) \underset{B}{\simeq} \mathrm{Map}_*(\mathbb{RP}^{n-1}, B\mathrm{Aut})$

$\cong !$

$$\simeq \mathrm{Map}_*\left(\mathrm{Thom}\left(\mathbb{RP}^{n-2} \xrightarrow[\text{Canonical}]{} BO(1)\right), B\mathrm{Aut}\right) \simeq \lim\left(\mathbb{RP}^{n-2} \longrightarrow \overset{\substack{O(1)\\ \circlearrowright \\ \mathbb{R}^+}}{BO(1)} \longrightarrow \mathcal{S}paces_*^{op} \xrightarrow{\mathrm{Map}_*(-,B\mathrm{Aut})} \mathcal{S}paces\right)$$

$\mathrm{Thom}(X \to BO(1)) := \mathrm{Colim}(X \to BO(1) \longrightarrow \mathcal{S}paces_*)$

$$\underset{\substack{\text{Unstraightening}\\ \text{(aka Grothendieck)}\\ \text{Construction}}}{\simeq} \mathrm{Map}_{/BO(1)}\left(\mathbb{RP}^{n-2}, \overbrace{\underset{\substack{\circlearrowleft\\ O(1)}}{\mathrm{Map}_*(\mathbb{R}^+, B\mathrm{Aut})}}^{\mathrm{Aut} \circlearrowleft \text{inverses}}{}_{hO(1)}\right) \simeq \mathrm{Map}^{O(1)}(\mathbb{S}^{n-2}, \mathrm{Aut})$$

$$\cup$$

$$\left(\mathbb{S}^{n-2} \ni p \longmapsto (-)^\vee_p \in \mathrm{Aut}(\mathrm{Obj}(\mathcal{X}^{\mathrm{duals}}))\right)$$

▨

$\mathcal{E}_1$-monoidal	$\mathcal{E}_2$-monoidal	$\cdots$	Symmetric Monoidal
$\Omega\mathbb{RP}^1 \simeq \mathbb{Z}$, R/L-duals	$\pi_0\Omega\mathbb{RP}^2 = \mathbb{Z}/2\mathbb{Z}$, duals $\pi_1\Omega\mathbb{RP}^2 = \mathbb{Z}$, Serre $\pi_2\Omega\mathbb{RP}^2 = \mathbb{Z}$, ? $\vdots$		$\Omega\mathbb{RP}^\infty \simeq O(1)$, duals.

Nearly Theorem, but still

<u>Conj</u> (A-Francis) (ArXiv) Factorization Homology (of $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-categories) defines a (easy)!

<u>Fully faithful</u> (hard) functor

$$\int : \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})^{\mathrm{duals}} \xrightarrow{\text{°°\{partial "beta"\}}} \mathrm{coPShv}(\mathrm{Mfd}^{\mathrm{sfr}}_{n-1,n})$$

with values $(\ast_{n-1})\ \int_{\mathbb{R}^{n-1}} \mathcal{X} \simeq \mathrm{Obj}(\mathcal{X})$

$(\ast_n)\ \int_{\mathbb{R}^n} \mathcal{X} \simeq \mathrm{End}_{\mathcal{X}}(\mathbb{1})$.

<u>Eg</u> A $(1<n)$-framed tangle $L \subset \mathbb{R}^n$ determines a composite morphism in $\mathrm{Mfd}^{\mathrm{sfr}}_{n-1,n}$:

$$\mathbb{R}^{n-1} \xrightarrow{\text{c.cr}} L \times \mathbb{R}^{n-1} \xrightarrow{\text{o.emb}} \mathbb{R}^n .$$

<u>Practical Implication</u> $\mathcal{X} \in \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})^{\mathrm{duals}}$

<u>Eg</u> $\rightsquigarrow$

$$\begin{tikzcd}
\int_{\mathbb{R}^{n-1}} \mathcal{X} \arrow[r] \arrow[d, "\wr", "(\ast_{n-1})"'] & \int_{L \times \mathbb{R}^{n-1}} \mathcal{X} \arrow[r] \arrow[d, "\wr", "\underline{\text{Calc}}"'] & \int_{\mathbb{R}^n} \mathcal{X} \arrow[d, "\wr", "(\ast_n)"'] \\
\mathrm{Obj}(\mathcal{X}) \arrow[r] & \mathrm{HH}(\mathcal{X}) \arrow[r, "\times \pi_0 L"] & \mathrm{End}_{\mathcal{X}}(\mathbb{1})
\end{tikzcd}$$

An object in $\mathrm{Mfd}^{\mathrm{sfr}}_{n-1,n}$ is

- an "n-manifold w/ codim-1 defects", M,
- an injection of its tangent constructible bdl:

$$\tau_M \hookrightarrow \varepsilon^n_M .$$

"solid-n framing"

Generating morphisms are:

- "<u>closed-creation</u>": opposite of proper constructible bundles, w/ framing data
- "<u>refining away a defect</u>"
- "<u>open embeddings</u>", compatible w/ framing data

(See AFR1 and AFR2)

Heuristic

$\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category with duals

Associated directed graph.

$$\int_M \mathcal{X} := \left\{ \underset{(1<n)\text{-framed}}{L^1 \subset M} \, , \; \underset{\pi_0\text{-surj}}{\overset{\text{finite}}{S \subset L}} \, , \; \mathcal{C}(S\subset L) \xrightarrow{\ell} \mathcal{X} \right\}$$

framed n-manifold

Map between directed graphs.

For $n=3$, $\int_M \mathcal{X}$ abstracts Skein-modules.

Q) Literally? We're Not yet sure.

Improved Heuristic (after Beilinson-Drinfeld)

1st $\mathrm{Mod}^{fr}_{graph}(M)$ = a moduli space of $(1<n)$-framed ~~tangles~~ graphs in M.

Two graphs in Same stratum if isotopic.

2nd $\mathcal{F}_{\mathcal{X}}$ a Constructible cosheaf on $\mathrm{Mod}^{fr}_{graph}(M)$, with coStalk at $L\subset M$ the space $\mathrm{Map}(\mathcal{C}(L), \mathcal{X})$.

3rd $\int_M \mathcal{X} :=$ cosheaf homology of $\mathcal{F}_{\mathcal{X}}$ over $\mathrm{Mod}^{fr}_{graph}(M)$.

Key Lemma ($\overset{\text{see}}{\text{ArXiv}}$)

$$\begin{tikzcd}[column sep=huge]
\int_{M} \mathrm{Bord}_1^{fr}(\mathbb{R}^{n-1}) \arrow[r, dashed, shift left=2, "\underline{\text{Eg}|}\ (\mathbb{R}^n \hookleftarrow\!\!\rightarrow M)"] & \hom_{\mathrm{Mfd}^{sfr}_{n-1,n}}(\mathbb{R}^{n-1}, M) \arrow[l, shift left=2, "\simeq" description, "\{0\}\subset\mathbb{R}^{n-1}"]
\end{tikzcd}$$

$$\left[\text{Natural in } M \in \mathrm{Mfd}^{sfr}_{n-1,n}\right]$$

In other words:

$$\int_{-} \mathrm{Bord}_1^{fr}(\mathbb{R}^{n-1}) \simeq \mathbb{R}^{n-1}$$

↖ representable copresheaf

Obs| Conj + Key Lemma $\Longrightarrow$ Tangle Hypothesis:

$$\hom_{\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})}\left(\mathrm{Bord}_1^{fr}(\mathbb{R}^{n-1}), \mathcal{X}\right) \xrightarrow[\underset{\text{Conj}}{\simeq}]{\int} \hom_{\mathrm{coPShv}(\mathrm{Mfd}^{sfr}_{n-1,n})}\left(\int \mathrm{Bord}_1^{fr}(\mathbb{R}^{n-1}), \int \mathcal{X}\right)$$

$$\xrightarrow[\simeq]{\text{Key Lem}} \hom_{\mathrm{coPShv}(\mathrm{Mfd}^{sfr}_{n-1,n})}\left(\mathbb{R}^{n-1}, \int \mathcal{X}\right) \xrightarrow[\simeq]{\text{Yoneda}} \int_{\mathbb{R}^{n-1}} \mathcal{X} \underset{(*)_{n-1}}{\simeq} \mathrm{Obj}(\mathcal{X}).$$

Sketch Proof of Conj

- $\underline{\mathbb{R}P}^{n-1} := \underset{\mathbb{R}P^{n-1}}{\text{Schubert-Stratified}}$, $\mathrm{Exit}(\underline{\mathbb{R}P}^{n-1}) = \left\{1 \overset{R}{\underset{L}{\rightrightarrows}} 2 \overset{R}{\underset{L}{\rightrightarrows}} \cdots \overset{R}{\underset{L}{\rightrightarrows}} n\right\} \xrightarrow{\text{localization}} \mathbb{R}P^{n-1}$.

- $$\begin{tikzcd}
S^{n-2} \arrow[r] & O(n-1)\times O(1) \\
p \arrow[r, mapsto] & (\mathrm{Ref}_p, -1)
\end{tikzcd}$$

Respects Involutions

$O(n-1)\times O(1) \curvearrowright \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})$ Change-o-Framing & op's $\rightsquigarrow$

$$\begin{tikzcd}
\mathrm{Exit}(\underline{\mathbb{R}P}^{n-1}) \arrow[r, dashed, "②"] \arrow[d] & \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})_{O(n-1)\times O(1)} \arrow[d] \\
\mathbb{R}P^{n-1} \arrow[r, "(\text{Prop } 1^{\text{st}})"'] \arrow[ur, dashed, "①"] & BO(n-1)\times BO(1)
\end{tikzcd}$$

$$\begin{tikzcd}
\rightsquigarrow \mathrm{Loc}_{\mathbb{R}P^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[d, hook, "\text{fully faithful}"] & \arrow[l, "(\infty,1)\text{-Cat of ①'s}"'] \\
\mathrm{Cbl}_{\underline{\mathbb{R}P}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) & \arrow[l, "(\infty,1)\text{-Cat of ②'s}"']
\end{tikzcd}$$

$\mathrm{Cbl}_{\underline{\mathbb{R}P}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \ni \left(\mathcal{X}_1 \overset{F_1^R}{\underset{F_1^L}{\rightrightarrows}} \mathcal{X}_2 \overset{F_2^R}{\underset{F_2^L}{\rightrightarrows}} \cdots \overset{F_{n-1}^R}{\underset{F_{n-1}^L}{\rightrightarrows}} \mathcal{X}_n \right)$ } Contravariant

$$\begin{tikzcd}
\rightsquigarrow \mathrm{Loc}^{\text{duals}}_{\mathbb{R}P^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[d, hook, "ff"] \\
\mathrm{Cbl}^{\text{duals}}_{\underline{\mathbb{R}P}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}))
\end{tikzcd}$$

$\cup$

$(\mathcal{X}_1 \rightrightarrows \mathcal{X}_2 \rightrightarrows \cdots \rightrightarrows \mathcal{X}_n)$ – Such that $\forall\, v \in \mathcal{X}_i$, $F_i^R(v)$ is R-dualizbl

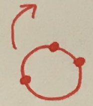
TOGETHER WITH: $F_i^R(-)^\vee \simeq F_i^L(-)$

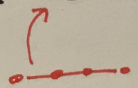
Obs

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})^{\text{duals}} \arrow[r, dashed, "\simeq"'] \arrow[d, hook] & \mathrm{Loc}^{\text{duals}}_{\mathbb{R}P^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[d, hook] \\
\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}) \arrow[r, dashed, "ff"] \arrow[dr] & \mathrm{Cbl}^{\text{duals}}_{\underline{\mathbb{R}P}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[d] \\
\mathcal{X} \arrow[dr, mapsto] & \mathrm{Cbl}_{\underline{\mathbb{R}P}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \\
 & \left(\mathcal{X} \overset{1.\text{duals}_1}{\underset{(-)^\vee}{\rightrightarrows}} \mathcal{X} \overset{1.\text{duals}_2}{\underset{(-)^\vee}{\rightrightarrows}} \cdots \underset{(-)^\vee}{\rightrightarrows} \mathcal{X}\right)
\end{tikzcd}$$

⑨

- $\mathcal{D}^{\mathrm{sfr}}_{n-1,n} \overset{\iota}{\underset{\mathrm{ff}}{\hookrightarrow}} \mathrm{Mfd}^{\mathrm{sfr}}_{n-1,n}$ generated under $\sqcup$ by $\tilde{\mathbb{S}}^1 \times \mathbb{R}^{n-1}$ and $\tilde{\mathbb{D}}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{R}^{n-1}$.





- $\mathcal{D}^{\$\mathrm{fr}}_{n-1,n}$: Objects

$$\begin{tikzcd}
 & & \mathrm{Exit}(\underline{\mathbb{RP}}^{n-1})^{\triangleright} \arrow[d] \\
 & (\mathrm{Vect}^{\mathrm{inj}}_{\geq n-1})_{/\mathbb{R}^n} \arrow[d] \arrow[r, phantom, "\simeq"] & (\mathbb{RP}^{n-1})^{\triangleright} \\
M \arrow[ur, "\mathrm{sfr}"] \arrow[r, "\tau_M"'] \arrow[uurr, dashed, bend left, "\$\mathrm{fr}"] & \mathrm{Vect}^{\mathrm{inj}}_{\geq n-1} &
\end{tikzcd}$$

in $\mathcal{D}^{\mathrm{sfr}}_{n-1,n}$ with a further lift.

<u>Eg</u>| $\mathbb{R}^{n-1}$ has n $\$\mathrm{fr}$ - structures: for each $1 \leq i \leq n$, $\mathbb{R}^{\underline{n}\setminus i}$.

- $\mathcal{D}^{\mathrm{vfr}}_{n-1,n} := \underbrace{\mathcal{D}\mathrm{isk}^{\mathrm{fr}}_{n-1,*} \wr \Delta^{\mathrm{op}}}_{\text{Familiar}} : \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}) \underset{(\text{Definitions})}{\overset{\mathrm{ff}}{\hookrightarrow}} \mathrm{coPShv}^{\mathrm{Seg}}(\mathcal{D}^{\mathrm{vfr}}_{n-1,n})$

$$\rightsquigarrow \mathrm{Exit}(\underline{\mathbb{RP}}^{n-1}) \times \mathcal{D}^{\mathrm{vfr}}_{n-1,n} \xrightarrow{\rho} \mathcal{D}^{\$\mathrm{fr}}_{n-1,n} \xrightarrow{\ell} \mathcal{D}^{\mathrm{sfr}}_{n-1,n}$$

<u>Claim A</u>| ℓ is (essentially) a localization.

<u>Claim B</u>| Let $1 \leq i < j \leq n$. The copresheaf

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{vfr}}_{n-1,n} \arrow[r, "\mathcal{E}^j(\mathbb{R}^{\underline{n}\setminus i})"] & \mathcal{S}\mathrm{paces} \\
D \arrow[r, mapsto] & \hom_{\mathcal{D}^{\$\mathrm{fr}}_{n-1,n}}(\mathbb{R}^{\underline{n}\setminus i}, \{j\} \times D)
\end{tikzcd}$$

satisfies a segal condition. Furthermore, its univalent-completion $\mathcal{E}^j(\mathbb{R}^{\underline{n}\setminus i})^{\wedge}_{\mathrm{univ}} \simeq \mathrm{Dual}_{\mathcal{E}_{n-1}}$ is the $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category corepresenting (right) dualizable objects.

Factorization Homology (of $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-categories) is the horizontal composite functor

$$\begin{tikzcd}
\int : \mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})^{\mathrm{duals}} \arrow[r, dashed, "\mathrm{obs}", "\simeq"'] \arrow[d, "\mathrm{ff}\checkmark"] & \mathrm{Loc}^{\mathrm{duals}}_{\mathbb{RP}^{n-1}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[r, dashed, "\mathrm{obs}"] \arrow[d, "\mathrm{ff}\checkmark"] & {}_{\mathrm{co}}\mathrm{PShv}^{\mathrm{seg}}(\mathcal{D}^{\mathrm{sfr}}_{n-1,n}) \arrow[r, "\overset{\checkmark}{\mathrm{ff}}", "\iota_! = \mathrm{LKE}"'] \arrow[d, "\mathrm{res} = \ell^*"', "\text{Ⓐ}\ \mathrm{ff}"] & {}_{\mathrm{co}}\mathrm{PShv}(\mathrm{Mfd}^{\mathrm{sfr}}_{n-1,n}) \\
\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)}) \arrow[r, dashed, "\mathrm{obs}", "\mathrm{ff}"'] \arrow[dr] & \mathrm{Cbl}^{\mathrm{duals}}_{\underline{\mathbb{RP}^{n-1}}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[r, dashed, "\text{Ⓑ}", "\mathrm{ff}"'] \arrow[d] & {}_{\mathrm{co}}\mathrm{PShv}^{\mathrm{seg}}(\mathcal{D}^{\$\mathrm{fr}}_{n-1,n}) & \\
& \mathrm{Cbl}_{\underline{\mathbb{RP}^{n-1}}}(\mathrm{Alg}_{\mathcal{E}_{n-1}}(\mathrm{Cat}_{(\infty,1)})) \arrow[r, leftrightarrow, "\mathrm{defs}", "\mathrm{ff}"'] & \mathrm{Cbl}_{\underline{\mathbb{RP}^{n-1}}}({}_{\mathrm{co}}\mathrm{PShv}^{\mathrm{seg}}(\mathcal{D}^{\mathrm{vfr}}_{n-1,n})) \arrow[u, "\mathrm{RKE} = \rho_*"] &
\end{tikzcd}$$

← Make Duals / Rotations "Directed"

↑ Pure Category Theory

↑ Intertwine Natural Symmetries: $\Omega\mathbb{RP}^{n-1}$

↑ Basic Geometry

↑ Global Geometry