Relative Geometric Langlands Handout

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Topological Field Theory

Dimension	Output
4	Number $\in \mathbb{C}$ Rarely well defined algebraically, requires analysis
3	(dg) vector space
2	(dg) category
1	$(\infty,2)$-category
0	$(\infty,3)$-category ? Rarely understood

"Arithmetic Field Theory"

Dimension	Settings
3	Global Arithmetic number fields, curves $C / \mathbb{F}_q$
2	Local Arithmetic local fields, e.g $\mathbb{Q}_p$, $\mathbb{F}_q((t))$ Global Geometric curves $\bar{C} / \bar{\mathbb{F}}_q$ Riemann surfaces
1	Local Geometric Punctured discs $\bar{\mathbb{F}}_q((t))$, $\mathbb{C}((t))$

Langlands Dual Groups

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
SO_{2n}	SO_{2n}
SO_{2n+1}	Sp_{2n}

Spherical Varieties

- Toric Varieties: $G = T$
- Flag varieties G/B
- Symmetric Spaces G/K
- "Group case" $G = H \times H \circlearrowright H$
- Whittaker G/N twisted by character ψ
- $SL_n \circlearrowright \mathbb{A}^n$, $GL_n \times GL_n \circlearrowright Mat_{n \times n}$

- $GL_{n+1} \times GL_n \circlearrowright GL_{n+1}$
- $SO_{n+1} \times SO_n \circlearrowright SO_{n+1}$
- $U_{n+1} \times U_n \circlearrowright U_{n+1}$

$\Big\}$ branching problems (Gan-Gross-Prasad)

- $\cdot \quad \cdot \quad \cdot$

Dual Spherical Varieties

G	$G^\vee$
Group	Group

usual, non-relative Langlands

$\mathbb{G}_m \circlearrowright \mathbb{A}^1$ $\qquad$ $\mathbb{G}_m \circlearrowright \mathbb{A}^1$

Tate's thesis

$PGL_2/\mathbb{G}_m$ $\qquad$ $SL_2 \circlearrowright \mathbb{A}^2$

Hecke $\qquad$ standard L-function

Point $\xleftrightarrow[\#]{\text{Tamagawa}}$ Whittaker

Neumann $\qquad$ Nahm pole

Whittaker $\xleftrightarrow{\text{Whittaker normalization}}$ point

G/B $\qquad$ $G^\vee/B^\vee$

Eisenstein $\qquad$ Eisenstein

$SO_{2n} \times SO_{2n+1}/SO_{2n+1}$ $\qquad$ $SO_{2n} \times Sp_{2n}$, std $\otimes$ std

Gan-Gross-Prasad $\qquad$ Θ - correspondence