MSRI - Workshop

Wednesday, March 25, 2020 12:16 PM

How to test your MTC's ?

Julia Plavnik - IU -

MTC are everywhere!

↳ connects diff. areas in math:

→ VOA's, CFT's, low-dim top. (TQFT), Q.G. & H.A (repr. th.), subfactors, planar alg., op. alg.

↳ connects with physics: top. phases of matter, TQC, Q.I.

Def.: ("rough") $\mathcal{B}$ is MTC over $\mathbb{C}$ if

- $\mathcal{B}$ is abelian & $\mathbb{C}$-linear,
- $(\mathcal{B}, \otimes, a, \mathbb{1}, l, r)$ monoidal
- $\mathcal{B}$ rigid
- $\mathcal{B}$ s.s.
- $\mathcal{B}$ finite
- $\mathbb{1}$ simple

$\mathcal{B}$ fusion / $\mathbb{C}$

- $\mathcal{B}$ braided $\rightsquigarrow$ c braid
- $\mathcal{B}$ pre-modular: $\theta_X : X \xrightarrow{\sim} X$ twist $\rightsquigarrow$ T

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ \underbrace{c_{Y,X} \circ c_{X,Y}}_{c^2_{X,Y} \rightsquigarrow S} \quad \text{(twist)}$$

$$\theta_{X^*} = \theta_X^* \quad \text{(ribbon)}$$

- $\mathcal{B}$ non-degenerate: $Z_2(\mathcal{B}) = \{X \in \mathcal{B} \,/\, c_{Y,X} c_{X,Y} = id_{X\otimes Y} \ \forall Y \in \mathcal{B}\}$

↳ $Z_2(\mathcal{B}) = \mathrm{Vec} = \langle \mathbb{1} \rangle$.

$\updownarrow$

$\det(S) \neq 0$.

Motivation for testing:

- classification
- construction
- examples
- properties / invariants

2014: classify MTC's integral non-group-theor. of dim 36 $\rightsquigarrow$ constructed from:

- $Z(\mathcal{T}_4)$ [GNN],
- $SU(3)_3$
- "new" rank 10 fusion ring

Q: categorifies to an MTC? **YES!**

Construct an MTC from $SU(3)_3 = \mathcal{C}$ by "twisting" the fusion rules by $G(\mathcal{C})$ $\longrightarrow$ **testing!**

2016: MME of super-modular cat.

By "twisting" ...
$G(\mathcal{C}) \rightsquigarrow$ ZESTING!

2016: MME of super-modular cat.

Conj. MME exist.

Fact: If there is one MME then $\exists$ 16 of them.

$PSU(2)_{4m+2} \subseteq SU(2)_{4m+2} \overset{\text{ZESTING}}{\rightsquigarrow}$ 8 MME

$\cap$

$SO(2m+1)_3 \overset{\text{ZESTING}}{\longrightarrow}$ 8 MME

2018/19: Grossman & Izumi $\rightsquigarrow$ modular data

Realizations?

Bonderson - Rowell - Wang: Realizations $\nearrow$ Zesting, $\searrow$ Gauging.

Goal: C. Delaney, C. Galindo, E. Rowell, Q. Zhang

Ad hoc constr. $\longrightarrow$ systematic construction

GENERAL FRAMEWORK FOR ZESTING.

Zesting construction:

$\mathcal{B}$ MTC $\longrightarrow$ Twist fusion rules by A

MTC

$\mathcal{B} = \overbrace{\bigoplus_{a \in A} \mathcal{B}_a}$, $A \doteq U(\mathcal{B}) \overset{\downarrow}{\simeq} G(\mathcal{B})$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$

$\lambda : G \times G \to (\mathcal{C}_e(\mathcal{C}_{ad}))_{pt}$

$\mathcal{B}_a \times \mathcal{B}_b \xrightarrow{\otimes} \mathcal{B}_{ab}$

$\mathcal{B}_e = \mathcal{B}_{ad}$ = Adjoint subc. $= \langle X \otimes X^* / X \text{ simple} \rangle$

$(\mathcal{B}_{ad})_{pt}$ is a symm. fusion cat.

Remark: This can be done more general BUT HARDER!

$\underset{\mathcal{B}_a}{\overset{\cap}{X_a}} \,\tilde{\otimes}\, \underset{\mathcal{B}_b}{\overset{\cap}{Y_b}} = \underbrace{X_a \otimes Y_b}_{\overset{\cap}{\mathcal{B}_{ad}}} \otimes \lambda(a,b) \in \mathcal{B}_{ab}$

$\lambda(a,b) \in G(\mathcal{B}) \cap \underbrace{\mathcal{B}_{ad}}_{= \mathcal{B}_e} \doteq (\mathcal{B}_{ad})_{pt}$

$\lambda : A \times A \longrightarrow (\mathcal{B}_{ad})_{pt}$

$\lambda(a,b) \rightsquigarrow$ 2-cocycle normalized

$\mathcal{B}_{ad}$ doesn't change (f.r.)

Remark: $\tilde{\mathcal{B}}$ is still graded by A with trivial comp. $\mathcal{B}_{ad}$.

In part., this is an example of [ENO]'s ext. theory. $\rightsquigarrow$ obstructions!

$\mathcal{B}$ strict

$\tilde{\alpha}_{X_a, Y_b, Z_c} : (X_a \,\tilde{\otimes}\, Y_b) \,\tilde{\otimes}\, Z_c \to X_a \,\tilde{\otimes}\, (Y_b \,\tilde{\otimes}\, Z_c)$

$X_a \; Y_b \; \lambda(a,b) \; Z_c \; \lambda(ab,c)$

$\tilde{a}_{X_a, Y_b, Z_c}$: $(X_a \otimes Y_b) \otimes Z_c \ \ldots$

$X_a \; Y_b \; \lambda(a,b) \; Z_c \; \lambda(ab,c)$

$\boxed{\lambda(a,b,c)}$

BRAID

$X_a \; Y_b Z_c \; \lambda(b,c) \; \lambda(a,bc)$

$\underbrace{\lambda(a,b,c)}_{\text{3-cochain}} : \; \lambda(a,b) \otimes \lambda(ab,c) \xrightarrow{\sim} \lambda(b,c) \otimes \lambda(a,bc)$

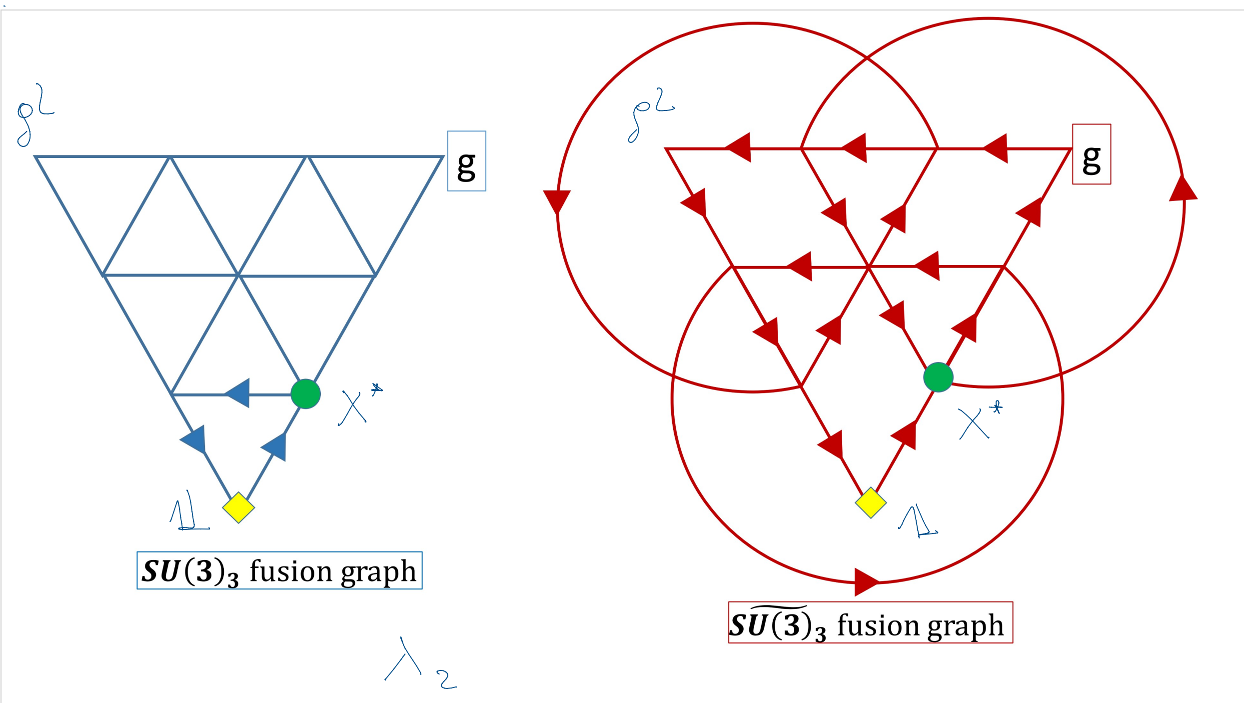
ASS. on $\tilde{\mathcal{B}}$ ⬠ $\longmapsto$ $H^4(A, \mathbb{C}^\times)$ OBSTR.
THAT NEEDS TO VANISH

Solutions form a Torsor over $\underbrace{H^3(A, \mathbb{C}^\times)}_{\text{Diff. ASS. CONSTR.}}$

Rk:
- Rank, dim, dim of simples is not changed
- Grading is preserved
- $\lambda \equiv 1 \rightsquigarrow$ Get [KW] results.

Ex. (2014) $\mathcal{B} = SU(3)_3$ Bad

$Irr(\mathcal{B}) = \{ \underbrace{\mathbb{1}, g, g^2}_{\mathcal{B}_{pt}}, \underset{3}{Y} \} \oplus \{ \underset{2}{X}, gX, g^2X \} \oplus \{ X^*, g^2X^*, gX^* \}$



$SU(3)_3$ fusion graph

$\widetilde{SU(3)}_3$ fusion graph

$$\begin{cases} X \otimes X^* = \mathbb{1} \oplus Y \\ Y \otimes Y = \mathbb{1} \oplus g \oplus g^2 \oplus Y^2 \\ X \otimes X = X^* \oplus g X^* \end{cases}$$

$$\begin{cases} X^* \tilde{\otimes} X^* = X^* \otimes X^* \otimes g^2 \\ Y \otimes Y \end{cases}$$

$$\lambda_a(i,j) = \begin{cases} 1 & i+j < 3 \\ q^{a} & i+j \geq 3 \end{cases}$$

$a \in \{0,1,2\} \rightsquigarrow 3$ choices

$$\lambda_{a,b}(i,j,k) = \begin{cases} 1 & i+j < 3 \\ q^{kb} & i+j \geq 3 \end{cases} \qquad b \in \{0,1,2\}, \quad q = e^{\frac{2\pi i}{3}}$$

$\downarrow$ 3 choices

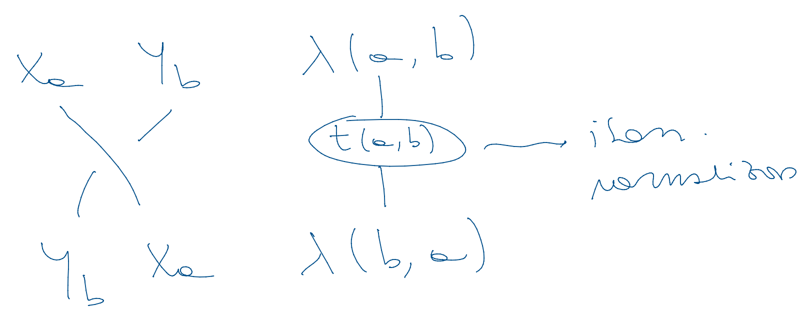
Total: 9 assoc. gradings.

General Q.:
- is $\tilde{\mathcal{B}}$ braided?
- is $\tilde{\mathcal{B}}$ pivotal/spherical?

Braided grading:

$$\tilde{c} : X_a \tilde{\otimes} Y_b \xrightarrow{\sim} Y_b \tilde{\otimes} X_a$$

$\downarrow$



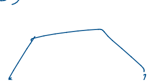
Remark: might lose examples

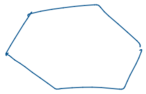
$\lambda \equiv 1$, c^{-1} will not appear

since $\tilde{c}\big|_{\mathcal{B}_{ad}} = c$

Ex. $SU(3)_3$ $\quad a \equiv 2b \ (3)$

Obstructions to braided grading:

coming  $\rightsquigarrow H^2(A, \mathbb{C}^\times)$, obstruction to ...

Coming  $\rightsquigarrow$ $H^2(A, \mathbb{C}^\times)$,
$H^2(A, \hat{A})$.

Pivotal & Ribbon:

$$\tilde{\theta}_{X_a} = f(a)\, \theta_{X_a} \qquad f : A \longrightarrow \mathbb{C}^\times$$

(i) $\tilde{\theta}_{X_a}$ twist $\longleftrightarrow$ $\dfrac{f(a+b)}{f(a)f(b)} \otimes \theta_{\lambda(a,b)}$ $\underbrace{\chi_{\lambda(a,b)}(a+b) = t^2(a,b)}_{\text{squares of } c_{\lambda(a,b)}}$,

(ii) $\tilde{\theta}_{X_a}$ ribbon $\longleftrightarrow$ $f(-a) \; \dots =$

Remark: we might loose some twists...
we can do a counting to
check this!

$SU(3)_3 \rightsquigarrow$ cyclic Tannakian grading

$\parallel$

$B \qquad \mathbb{Z}_3 \simeq \underbrace{B_{pt}}_{\text{symm. cyclic Tann.}} \subseteq B_{ad}$

$SU(4)_4 \rightsquigarrow$. 4×4 ass. grt.
$\downarrow \quad \downarrow$
$\lambda_a \quad \lambda_{a,b}$

64 ribbon cat. $\left\{ \begin{array}{l} \cdot\ 8 \text{ braid. } \& \\ \cdot\ 8\ f\text{'s ribbon grading} \end{array} \right.$

Testing:
- control of fusion rules
- modular data
- images of braid group repr.
- control charge changes. ... / non-pivotal ...

- IMAGES OF ...
- CONTROL CHARGE CHANGES.
- WE CAN GET NON-BRAIDED/NON-PIVOTAL ...

CHALLENGING:
- NOT ALWAYS EASY TO COMPUTE FUSION RULES
- WITH CLASS & CONTROL CHARGES ARE PROJECTIVE

REL. IX:
- → SOMETIMES IS FOR FREE
- →
- IT IS NICE THEORETICALLY BUT SOMETIMES TO GET EXPLICIT DATA.