Boundary layer methods I

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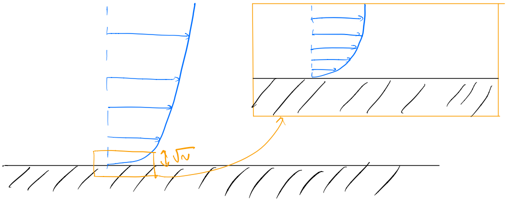
Outline

Plan

Motivation

Definition:

Boundary layer= zone of small width, located close to a boundary, in which a quantity has strong variations ($\Leftrightarrow$ large gradient). Ubiquitous in fluid mechanics (e.g. Prandtl boundary layer; Ekman boundary layer...)



Goals of these lectures:

- Identify situations in which BL are created;
- Present a general method to construct linear /semi-linear BL ;
- Explain the limits of this general method.

About singular perturbation problems

In many physical situations, presence of singular perturbation operators:

General definition: Consider a PDE

$$\mathcal{A}^\epsilon[u^\epsilon] = 0, \tag{SPP}$$

where $\mathcal{A}^\epsilon$ is a differential operator with the following properties:

- $\mathcal{A}^\epsilon$ depends on a small parameter ϵ;
- $\mathcal{A}^\epsilon$ is of order $d \geq 1$;
- If $u^\epsilon \to \bar{u}$ in some strong sense, then $\bar{u}$ is a solution of $\bar{\mathcal{A}}[\bar{u}] = 0$, where $\bar{\mathcal{A}}$ is an operator of order $d' < d$.

Then (SPP) is a singular perturbation problem.
Generically,

$$\text{SPP} + \text{boundary} = \text{BL}.$$

Example: $-\epsilon\partial_{xx} u_\epsilon + u_\epsilon = f$ in $(0, 1)$;

Example 1: the rotating fluids equation

$$\partial_t u + u \cdot \nabla u + \frac{1}{\mathrm{Ro}} e_3 \wedge u + \nabla p - \epsilon \Delta u = 0,$$
$$\mathrm{div}\, u = 0$$

in a domain $\Omega := \mathbb{T}^2 \times (0,1)$.
Boundary conditions:

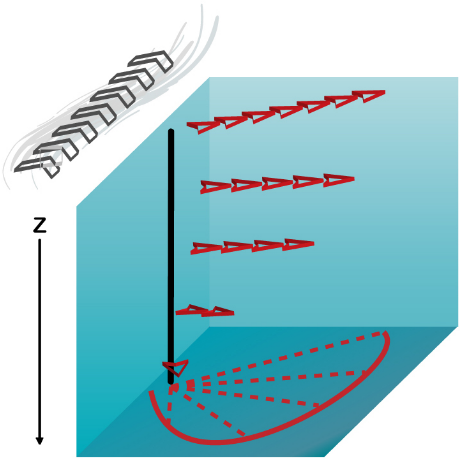
$$\partial_z u_h = \tau, \quad u_3 = 0 \quad \text{at } z = 1 \text{ (wind forcing)},$$
$$u_{|z=0} = 0 \quad \text{at } z = 0 \text{ (friction on the bottom.)}$$

Question: Limit as $\mathrm{Ro}, \epsilon \to 0$?
$\to$ Apparition of Ekman layers! Responsible for Ekman pumping:

- Transfer of momentum coming from the wind to the whole system;
- Dissipation of energy because of friction on the bottom.

[Chemin, Desjardins, Gallagher, Grenier; Grenier, Masmoudi; Gérard-Varet; D. , Saint-Raymond...]

Picture of Ekman layer



Source: Wikipedia

Example 2: the Sverdrup model

$$\partial_x \psi - \epsilon \Delta^2 \psi = F \text{ in } \Omega,$$
$$\psi_{|\partial\Omega} = \partial_n \psi_{|\partial\Omega} = 0.$$

ψ = stream function.
Remark: friction on the bottom, bottom topography and advection have been neglected.
Question: Limit as $\epsilon \to 0$?
$\to$ Apparition of Munk layers (western boundary currents) in the vicinity of western boundaries. Complicated phenomena close to northern/southern boundaries.
[Desjardins-Grenier; D. , Saint-Raymond]

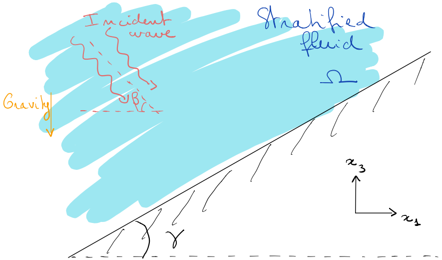
Example 3: the Boussinesq model for stratified fluids

$$\begin{aligned} \partial_t u + \delta u \cdot \nabla u + \nabla p - \nu \Delta u &= -b e_3, \\ \partial_t b + \delta u \cdot \nabla b - N^2 u_3 - \kappa \Delta b &= 0, \\ \operatorname{div} u &= 0, \\ u_{|\partial\Omega} = 0, \quad \partial_n b_{|\partial\Omega} &= 0 \end{aligned}$$

in a domain $\Omega := \{(x_1, x_3) \in \mathbb{R},\ -x_1 \sin\gamma + x_3 \cos\gamma > 0\}$.

Question: behavior as $\kappa, \nu, \delta \to 0$?

Partial answer: depends on relative sizes of parameters... + critical reflection problem. [Dauxois, Young; Bianchini, D., Saint-Raymond]



Plan

General framework

Simplification: $\mathcal{A}_\epsilon$=linear diff. operator with constant coeff.

$$\mathcal{A}_\epsilon[U^\epsilon] := \sum_{\alpha\in\mathbb{N}^N, |\alpha|\leq d} a_\alpha(\epsilon)\nabla^\alpha U^\epsilon = f$$

in the half-space $x_N > 0$.

Assumptions:

- a_α is polynomial in ϵ;
- $a_\alpha(0) = 0$ if $|\alpha| = d$;
- $\exists\alpha \in \mathbb{N}^d$ with $|\alpha| < d$ such that $a_\alpha(0) \neq 0$.

Goal: identify the BL sizes and profiles.

Modal solutions

[Eckhaus; Van Dyke; Gérard-Varet, Paul]
Look for solutions of

$$\sum_{\alpha\in\mathbb{N}^N, |\alpha|\le d} a_\alpha(\epsilon)\nabla^\alpha U = 0$$

in the form $U = \exp(i\xi' \cdot x' - \lambda x_N)\mathfrak{U}$, $\mathfrak{U} \in \mathbb{C}^K$, $\xi' \in \mathbb{R}^{N-1}$, $\lambda \in \mathbb{C}$.
After plugging into PDE, obtain linear system

$$\mathbb{A}(\epsilon; \xi', \lambda)\mathfrak{U} = 0, \quad (1)$$

where $\mathbb{A}(\epsilon; \xi', \lambda) \in \mathcal{M}_K(\mathbb{C})$, with polynomial coefficients.
Non-zero solution of (1) iff

$$P_{\epsilon,\xi'}(\lambda) := \det \mathbb{A}(\epsilon; \xi', \lambda) = 0.$$

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Boundary layer sizes and profiles

At this stage: look for solutions of

$$P_{\epsilon,\xi'}(\lambda) = \det \mathbb{A}(\epsilon; \xi', \lambda) = 0,$$

where $P_{\epsilon,\xi'}$ is a polynomial in λ (say of degree m), with complex coefficients that are polynomials in ϵ and ξ'.

Fact # 1: $P_{\epsilon,\xi'}$ has exactly m complex (possibly multiple) roots $\lambda_1, \cdots, \lambda_m$.

But we are only interested in the roots s.t. $\Re(\lambda) > 0$.

Fact # 2: as $\epsilon \to 0$ (with ξ' fixed), all roots λ_i behave as $\epsilon^{q_i}\mu_i$, for some $\mu_i \in \mathbb{C}$, $q_i \in \mathbb{Q}$.

Fact # 3: the number of BC that can be lifted by the BLs is

$$p := \dim \mathrm{Vect}\ \left\{ V \in \ker \mathbb{A}(\epsilon; \xi', \lambda_i),\ \Re(\lambda_i) \gg 1 \right\}$$

Fact # 4: the BL size is $(\Re\lambda)^{-1}$. Possible superposition of BLs.

General result and open problems

Summary: each root λ_i is s.t. $\lambda_i \sim \epsilon^{q_i}\mu_i$, $q_i \in \mathbb{Q}$, $\mu_i \in \mathbb{C}$.
$q_i = q_i(\xi'), \mu_i = \mu_i(\xi')$.

Theorem: [Gérard-Varet, Paul] As long as q_i is independent of ξ' and μ_i does not vanish (=non-degeneracy) [...], an approximate solution can be constructed up to any order.

Limitations/open problems:

- Flat boundary/constant coefficients (but can be generalized);
- Linear equation (at least for the BL);
- Theory breaks down when degeneracy occurs.

Remark: Equivalent to (but slightly different from) framework of matched asymptotic expansions.

Plan

Setting of the problem

$$\partial_t u + \frac{1}{\epsilon} e_3 \wedge u + \nabla p - \epsilon \Delta u = 0,$$
$$\mathrm{div}\, u = 0, \quad t > 0,\ x_h \in \mathbb{T}^2,\ z \in (0,1)$$

Look for a BL solution $(u, p) = \exp(i(-\omega t + k_h \cdot x_h) - \lambda z) U$.

Linear system:

Eq. for BL size λ:

$$\epsilon^2(-i\omega + \epsilon(|k_h|^2 - \lambda^2))^2(|k_h|^2 - \lambda^2) - \lambda^2 = 0 :$$

Boundary layer sizes

$$\epsilon^2(-i\omega + \epsilon(|k_h|^2 - \lambda^2))^2(|k_h|^2 - \lambda^2) - \lambda^2 = 0 :$$

$\rightarrow$ 6 complex roots $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3,$ $\Re(\lambda_i) \geq 0$.
Regime #1: $k_h \neq 0$, $\omega = \Omega\epsilon^{-1}$, $|\Omega| < 1$:
$\lambda_i \sim C_i/\epsilon$ with $\Re(C_i) > 0$ $i = 1, 2$, $\lambda_3 = O(1)$ (not a BL !)
$\rightarrow$ Classical Ekman layers.

Regime #2: $k_h \neq 0$, $|\omega| = \epsilon^{-1}$: $\lambda_1 \sim C_1/\epsilon$, $\lambda_2, \lambda_3 \sim C_i' \epsilon^{-1/2}$
$\rightarrow$ Quasi-resonant boundary layers.
Regime #3: $k_h = 0$, $|\omega| < \epsilon^{-1}$: 2 classical Ekman layers, 1 root is exactly zero;
Regime #4: $k_h = 0$, $|\omega| = \epsilon^{-1}$: 1 classical Ekman layers, 2 roots are exactly zero.

Boundary layer sizes

$$\epsilon^2(-i\omega + \epsilon(|k_h|^2 - \lambda^2))^2(|k_h|^2 - \lambda^2) - \lambda^2 = 0 :$$

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Plan

The critical reflection problem for internal waves - 1

Consider, in $\Omega := \{(x_1, x_3) \in \mathbb{R},\ -x_1 \sin\gamma + x_3 \cos\gamma > 0\}$

$$\begin{aligned} \partial_t u + \nabla p - \nu \Delta u &= -b e_3, \\ \partial_t b - N^2 u_3 - \nu \Delta b &= 0, \\ \operatorname{div} u &= 0, \\ u_{|\partial\Omega} = 0, \quad \partial_n b_{|\partial\Omega} &= 0 \end{aligned}$$

Look for solutions of the form $(u, b, p) = \exp(i(kx - \omega t) - \lambda z)U$.

Linear system:

Determinant=polynomial of degree 6 in λ.

Behavior of the BL sizes

Important quantity: criticality parameter $\zeta := \sin^2\gamma - \omega^2$.

Theorem: Always 3 roots with $\Re(\lambda) > 0$, with the following behaviour:

$\lvert\zeta\rvert \gtrsim 1$	$\nu^{1/3} \ll \lvert\zeta\rvert \ll 1$	$\lvert\zeta\rvert \lesssim \nu^{1/3}$
2 roots $\propto \nu^{-1/2}$ 1 root $O(1)$	Roots of sizes ν/ζ^4, $\lvert\nu/\zeta\rvert^{1/2}$, $\nu^{-1/2}$	2 roots $\propto \nu^{-1/3}$ 1 root $\propto \nu^{-1/2}$

Experimental observations

[Gostiaux et al., Phys. Fluids, 2006]

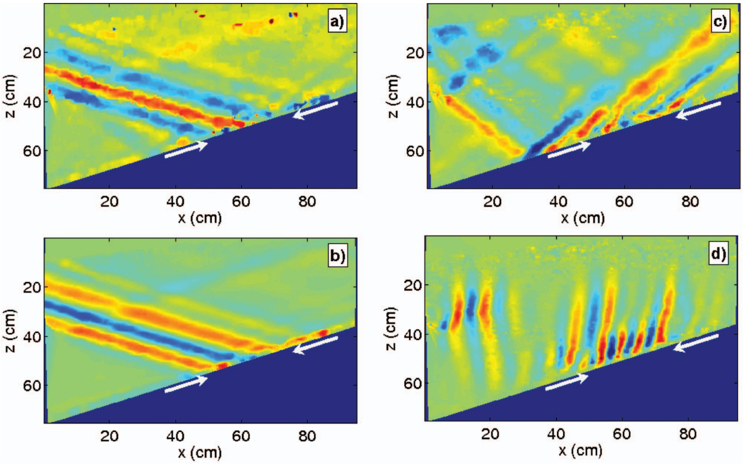


FIG. 5. (Color) False-color velocity pattern in the vertical (x,z) plane for the critical run 4 (see Table I). Panel (a) presents the instantaneous horizontal velocity field $u(x,z,t)$ while panel (b) shows the phase-averaged velocity $\langle u\rangle_1$. Panels (c) and (d) show, respectively, the second $\langle w\rangle_2$ and third $\langle w\rangle_3$ harmonics of the vertical velocity, $\langle w\rangle_n$, in the case of run 4 (see Table I). The two white arrows define the impinging region of the incident beam. In panels (c) and (d), the rays at the left of this region should thus not be taken into account: they have been generated by the screen. The maximum velocity in panels (a) and (b) is 2 mm s^{-1}, and in panels (c) and (d), 0.4 mm s^{-1}.

What about the next step?

- Once the construction of a generic BL is understood, build an approximate solution $U^\epsilon_{\text{app}} = U_{\text{int}} + U_{BL}$ so that:
  - $\mathcal{A}^\epsilon[U^\epsilon_{\text{app}}] = f + r^\epsilon$, where r^ϵ is a remainder that is sufficiently small in some energy norm;
  - U^ϵ_{app} satisfies **exactly** the BC (this is what the BL is for!).
- Write an equation for $U^\epsilon - U^\epsilon_{\text{app}}$, perform energy estimates... Typically, one ends up with

$$\|U^\epsilon - U^\epsilon_{\text{app}}\|_X \leq C_\epsilon \|r^\epsilon\|_Y$$

where X, Y are functional spaces, say $X = L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x)$, $Y = L^2_t(H^{-1}_x))$, C_ϵ is usually a (possibly negative) power of ϵ.
- **Goal:** have a sufficiently good approximation, i.e. such that

$$C_\epsilon \|r^\epsilon\|_Y \ll \|U^\epsilon_{\text{app}}\|_X.$$

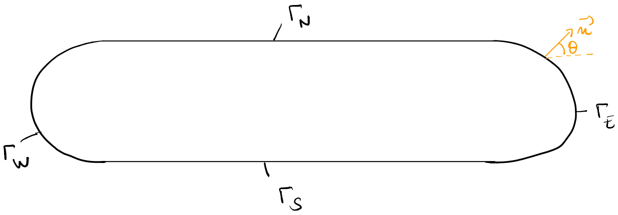
Remark: because of the singular perturbation, this may require several additional correctors!

Plan

Setting of the problem

(Simplified) Sverdrup equation:

$$\partial_x \psi^\epsilon - \epsilon \Delta^2 \psi^\epsilon = f \text{ in } \Omega,$$
$$\psi^\epsilon_{|\partial\Omega} = 0, \quad \partial_n \psi^\epsilon_{|\partial\Omega} = 0.$$



Previous formal analysis predicts: $(\partial_x = \cos\theta \partial_n + \frac{\sin\theta}{1+d\theta'}\partial_\tau)$

- BL on Γ_E, Γ_W of size $(\epsilon/|\cos\theta|)^{1/3}$: degenerates as $\theta \to \pm\pi/2$;
- BL on Γ_S, Γ_N of size $\epsilon^{1/4}$;
- BL on $\Gamma_W, \Gamma_N, \Gamma_S$ lifts 2 BC; BL on Γ_E lifts 1 BC.

Difficulties and results

Consequences of formal analysis:

- Interior part satisfies $\partial_x \psi_{\text{int}} = f$, $\psi_{\text{int}|\Gamma_E} = 0$;
- Western intensification of currents: $u^\epsilon = \nabla^\perp \psi^\epsilon \sim \epsilon^{-1/3}$ in a BL of size $\epsilon^{1/3}$ close to Γ_W.

But because of the degeneracy, no immediate conclusion...

Theorem:[D., Saint-Raymond]

- The sizes of BL are the ones predicted by the formal analysis;
- BUT the BL profiles are not! The profiles on Γ_N, Γ_S are non-intrinsic (satisfy a diffusion-like eq.);
- Complicated superposition in transition zone where $|\cos\theta| \ll 1$.

Remark: one should NOT try to connect the BL sizes!

Conclusion: when degeneracy occurs, analysis may still be possible, but the general previous analysis might give a wrong answer.

Summary

- Presentation of a general, systematic method to compute the boundary layer sizes and profiles when:
  - The equation is linear;
  - The boundary is flat (or smooth);
  - The situation is non-degenerate.
- BL sizes are given as the roots of a polynomial. Quite often, different regimes must be investigated.
- Even in linear cases, singularities may occur in degenetate cases (leading to mathematical difficulties).

Next lecture

- Extension of the methodology to rough boundaries;
- Analysis of semilinear equations.