A Higher-order Water Wave Model: a New Approach

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French association Women and Mathematics

French association Women & Science

European Women in Mathematics

European Platform of Women Scientists

Corona Crisis : Impact on junior and women mathematicians

970 signatures by Jan. 15, 2021

An open letter from the EWM

16 institutions endorsed it

Let us be clear about one fact:

We did not experience the crisis equally.

Untenured faculty lost more. Women lost more. Caregivers lost more. The more vulnerable the population, the greater the disadvantage.

No one chooses a pandemic, but now we can choose how to respond. We are writing to advocate a **proactive policy to support current employees in temporary positions and future job applicants** in Mathematics in light of the Corona Crisis. We focus on:

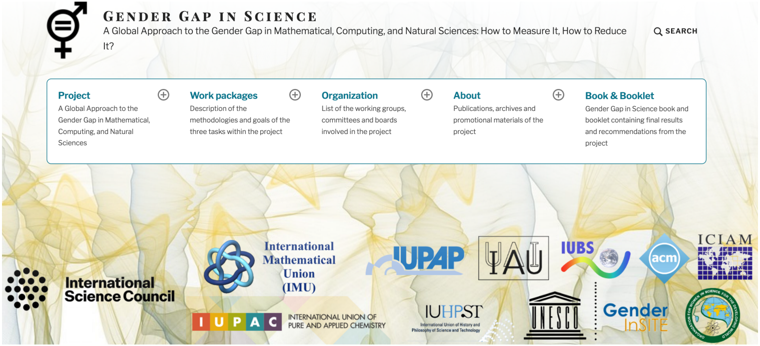
- Untenured mathematicians, because the loss of travel and training opportunities, the slow-down in research productivity, and the uncertainty of the job market is most likely to have a long-term impact on their careers.
- Women, because statistically, women shoulder more of the burden of caregiving (for children and the elderly) and domestic tasks (for which help and other supports recently disappeared).
- Parents, because the shuttering of daycares and schools left them stranded. Suddenly and unexpectedly, parents had to provide constant care for young children and home-schooling for older children.

A proactive policy should not be gender-blind*:*
While acknowledging the role that some men play in caregiving, we recognize that statistically, women play a significantly larger role. Hence we are concerned that **we may lose talented female mathematicians during and following this crisis**. Women may choose to leave their profession or reduce their hours. Women in temporary positions may choose security and "settle" for lesser positions. Young women may opt not to pursue careers in science. The COVID-19 pandemic has exacerbated existing gender inequities in mathematics and other sciences. **And gender-blind measures do not correct gender inequity.**

We advocate the following **proactive measures**:

- We encourage universities, governments, and funding agencies to invest in **extending the contracts** of researchers in temporary positions to offset the loss of productivity during the crisis. We advocate that these extensions give **particular consideration to women**. Perhaps savings due to cancelled travel and workshops can be redirected for this purpose.
- We encourage universities and funding agencies to award **release from teaching or teaching reductions** to untenured mathematicians who lost significant research time to digital teaching and caregiver responsibilities, again giving **particular consideration to women**. In case such measures are not possible, we advocate for allocating additional support via student assistants or other resources to reduce the teaching demands on junior colleagues.
- Evaluators on Hiring, Tenure, Prize, Grant, and other committees should be reminded that the crisis has impacted individuals very differently. It should be not the years past PhD but an **academic age, corrected for parental and other leaves**, that is the standard quantifier measured by committee members. Women with dependent children should be automatically eligible (although not required) **to subtract up to 12 months from their academic age** – for the purpose of hiring, grant eligibility, tenure deadlines, etc – due to disruptions from the COVID-19 pandemic. Men with minor children or researchers involved in eldercare during the crisis will be eligible if they can demonstrate that they were responsible for caregiving.
- We advocate **flexibility in deadlines and meeting times** especially for women with dependent children. The disruptions of the crisis may mean that it takes longer to review an article, finish a grant application, or return galley proofs. An early afternoon meeting might not be possible. Circumstances vary and allowing open conversations about needs and constraints is a necessary condition for a healthy workplace.

32,500 Answers / 3,083,185 Publications Analyzed
67 Good Practices / 21 Recommendations (parents, teachers, institutions, laboratories, international organizations)



Publication Patterns

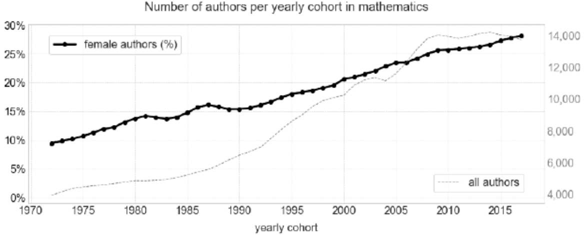


Figure 13: Number of active (publishing) mathematicians since 1970 and percentage of them that are women.

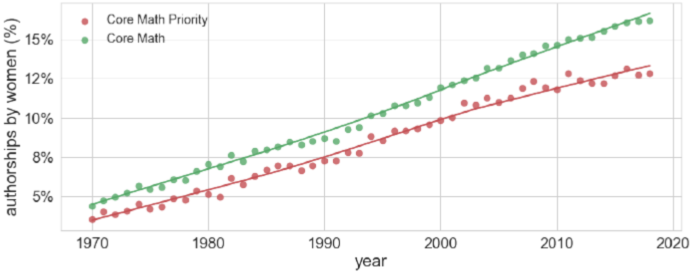


Figure 40: Percentage of fractional authorships from women in the Core Math (green) and

Outline of the talk

I. Introduction to long-crested waves in shallow water

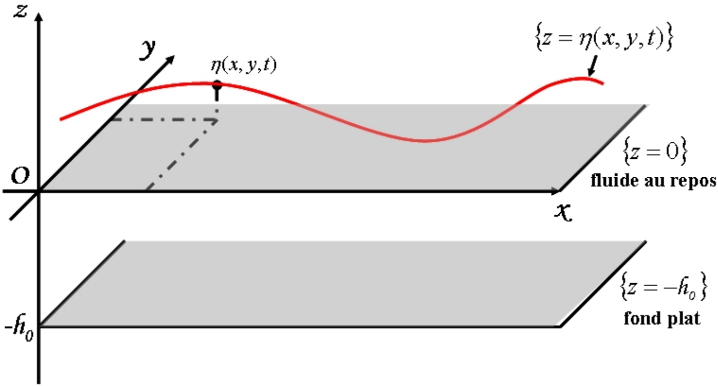
II. The problem under study

III. Local well-posedness

IV. Global well-posedness and temporal growth

V. Conclusion

I. Introduction to long-crested waves in shallow water



Liquid height : $h(x,t) = h_0 + \eta(x,t)$ (one-way propagation).

Hypotheses

- No surface tension. No viscosity.
- Irrotational flow.
  Incompressible perfect fluid.
- Principal direction of propagation : $x-$axis. 2D flow
- Long waves. Small amplitudes.
  $\alpha = \frac{a}{h_0} << 1,$ $\beta = (\frac{h_0}{\ell})^2 << 1,$
  (non linear effects) (dispersive effects)
  where $a = \max_{x,t} |\eta|$, and ℓ is the smallest wave length for which the flow has significative energy
  $S = \frac{\alpha}{\beta}$ close to 1 (Stokes number)
  Boussinesq approximation

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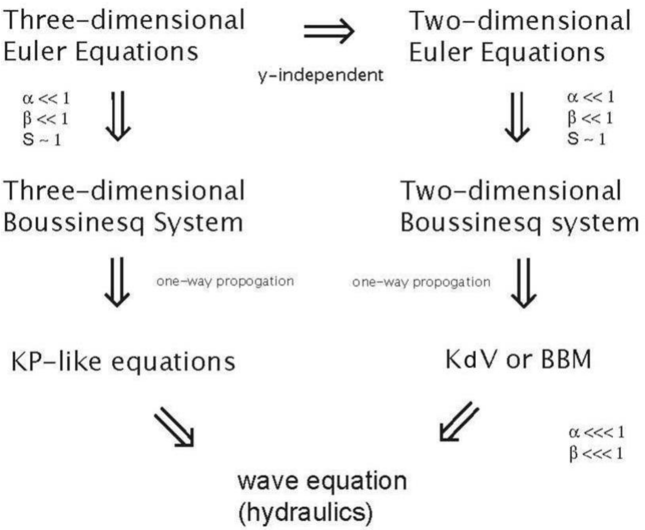
Scheme of different approximations



Higher-order models

Approximations on the Boussinesq time

Unidirectional models valid for a few waves or on the time interval $[0, T]$ with

$$T = O(1/\alpha) = O(1/\beta).$$

- Korteweg-de Vries equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0.$$

- BBM equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0.$$

II.The problem under study

The variables

- $\eta = \eta(x,t)$: perturbation of the free surface from the rest state.
- $x \in \mathbb{R}$, $t \in \mathbb{R}_+$.

Choices

- Dimensionless variables : $x = \ell\tilde{x}, \eta = a\tilde{\eta}, t = \sqrt{h_0/g}\,\tilde{t}$.
- Boussinesq approximation : S close to 1, $\alpha << 1$, $\beta << 1$.
- Rescaling of the variables.

II. The problem under study

Fifth order wave model (Bona, Carjaval, Panthee, Scialom, 2018)

$$\begin{aligned} &\eta_t + \eta_x - \gamma_1 \beta \eta_{xxt} + \gamma_2 \beta \eta_{xxx} + \delta_1 \beta^2 \eta_{xxxxt} + \delta_2 \beta^2 \eta_{xxxxx} \\ &+ \frac{3}{4}\alpha(\eta^2)_x + \alpha\beta\Big(\gamma(\eta^2)_{xx} - \frac{7}{48}\eta_x^2\Big)_x - \frac{1}{8}\alpha^2(\eta^3)_x = 0 \end{aligned} \tag{1}$$

Initial condition

$$\eta(\cdot, 0) = \eta_0.$$

Parameters $\gamma_1, \gamma_2, \delta_1, \delta_2, \gamma$

$$\delta_1 > 0, \gamma_1 > 0,$$

$$\gamma_1 + \gamma_2 = \tfrac{1}{6}, \quad \gamma = \tfrac{1}{24}(5 - 18\gamma_1), \quad \delta_2 - \delta_1 = \tfrac{19}{360} - \tfrac{1}{6}\gamma_1.$$

See also when dealing with global well-posedness and $\gamma = \frac{7}{48}$.

II. The problem under study (Bona, Carjaval, Panthee, Scialom, 2018)

Theorem A. Local existence.

For any $s \geq 1$ and for given $\eta_0 \in H^s(\mathbb{R})$, there exist a time $T_\eta = \frac{C_s}{||\eta_0||_s(1+||\eta_0||_s)}$ and a unique function $\eta \in \mathcal{C}([0, T_\eta]; H^s)$ which is a solution of equation (1), posed with initial condition η_0.
The solution η varies continuously in $\mathcal{C}([0, T_\eta]; H^s)$ as η_0 varies in H^s.

Theorem B. Global existence.

Let $\gamma = \frac{7}{48}$. Then the solution to problem (1) given by Theorem A is global in $H^s(\mathbb{R})$ for $s \geq 1$, and if $s \geq 2$, the solution is bounded in $H^2(\mathbb{R})$, independently of t.

Derivation of an integral equation

Change of variables : $\tilde{x} = \beta^{-1/2}(x - \frac{\delta_2}{\delta_1}t)$, $\tilde{t} = \beta^{-1/2}t$.

Change of unknown function : $u(\tilde{x}, \tilde{t}) = \alpha\eta(x, t)$.

$$\begin{cases} u_t + \left(1 - \frac{\delta_2}{\delta_1}\right)u_x - \gamma_1 u_{xxt} + \left(\gamma_2 + \gamma_1\frac{\delta_2}{\delta_1}\right)u_{xxx} + \delta_1 u_{xxxxt} \\ \qquad + \frac{3}{4}\left(u^2\right)_x + \gamma\left(u^2\right)_{xxx} - \frac{7}{48}\left(u_x^2\right)_x - \frac{1}{8}\left(u^3\right)_x = 0, \\ \qquad u(x, 0) = u_0(x) = \alpha\eta_0\left(\beta^{\frac{1}{2}}x\right). \end{cases}$$

No more term with a fifth-order x-derivative!

Derivation of an integral equation

Take the Fourier transform, solve the resulting equation, take the inverse Fourier transform, and obtain

$$\begin{aligned} u &= u_0 + \int_0^t K * \left[\left(1 - \frac{\delta_2}{\delta_1}\right) u + \frac{3}{4} u^2 - \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \right] (x,s)\, ds \\ &\qquad + \int_0^t L * \left(\left(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1 \right) u + \gamma u^2 \right) (x,s)\, ds \\ &= u_0 + \int_0^t \left[\left(\left(1 - \frac{\delta_2}{\delta_1}\right) K + \left(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1 \right) L \right) * u \right. \\ &\qquad \left. + \left(\tfrac{3}{4}K + \gamma L\right) * u^2 - \tfrac{7}{48} K * u_x^2 - \tfrac{1}{8} K * u^3 \right] (x,s)\, ds \\ &=: \mathcal{A} u, \end{aligned}$$

where K and L are integral kernels.
Moreover $L = K''$ outside of 0.

Use the Residue Theorem, for example, to calculate :

$$K(x) = \begin{cases} \dfrac{\operatorname{sgn}(x)}{2\sqrt{\gamma_1^2 - 4\delta_1}} \left\{ e^{-\rho_1|x|} - e^{-\rho_2|x|} \right\} & \text{for} \quad \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{1}{4\delta_1^{\frac{3}{4}}} x e^{-\rho_0\,|x|} & \text{for} \quad \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)}{\sqrt{4\delta_1 - \gamma_1^2}} e^{-\rho|x|\cos\omega} \sin\big(\rho|x|\sin\omega\big) & \text{for} \quad 0 < \gamma_1 < 2\sqrt{\delta_1}, \end{cases}$$

and

$$L(x) = \begin{cases} \dfrac{\operatorname{sgn}(x)}{2\sqrt{\gamma_1^2 - 4\delta_1}} \left\{ \rho_1^2 e^{-\rho_1 |x|} - \rho_2^2 e^{-\rho_2 |x|} \right\} & \text{for } \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{1}{4\delta_1^{\frac{3}{4}}} (-2\rho_0 \operatorname{sgn}(x) + \rho_0^2 x) e^{-\rho_0 |x|} & \text{for } \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)\rho^2}{\sqrt{4\delta_1 - \gamma_1^2}} e^{-\rho |x| \cos\omega} \sin\big(2\omega - \rho |x| \sin\omega\big) & \text{for } 0 < \gamma_1 < 2\sqrt{\delta_1}. \end{cases}$$

Agreeable regularity properties, vanishing at $\pm\infty$.

III. Local well-posedness

Lemma. Existence of local solutions in continuous function spaces.

The integral equation is locally well posed in the space $C_b^1(\mathbb{R})$. Precisely, for any value $r > 0$, there is $T = T_r$ such that for all initial data $u_0 \in C_b^1(\mathbb{R})$ with $\|u_0\|_{C_b^1(\mathbb{R})} \leq r$, there is a unique solution $u \in C(0, T_r; C_b^1(\mathbb{R}))$.

Moreover, the correspondence between the initial data u_0 and the associated solution u in $C(0, T_r; C_b^1(\mathbb{R}))$ is a Lipschitz continuous mapping on any bounded subset of $C_b^1(\mathbb{R})$.

Proof. Use a contraction mapping argument to solve $\mathcal{A}u = u$, where

$$\mathcal{A}u = u_0 + \int_0^t \Bigg[\left(\left(1 - \frac{\delta_2}{\delta_1}\right) K + \left(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1\right) K'' \right) * u(x,s) + \left(\tfrac{3}{4} K + \gamma K'' \right) * u^2(x,s) - \tfrac{7}{48} K * u_x^2(x,s) - \tfrac{1}{8} K * u^3(x,s) \Bigg] ds.$$

Theorem. Existence of local solutions in Sobolev spaces.

The integral equation is locally well posed in the space $H^1(\mathbb{R})$. Precisely, for any value $M > 0$, there is $T = T_M$ such that for all initial data $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1(\mathbb{R})} \leq M$, there is a unique solution $u \in C(0, T_M; H^1(\mathbb{R}))$.

Moreover, the correspondence between the initial data u_0 and the associated solution u in $C(0, T_M; H^1(\mathbb{R}))$ is a Lipschitz continuous mapping on the ball of radius M of $H^1(\mathbb{R})$.

IV. Global well-posedness and temporal growth

Here $\gamma = \frac{7}{48}$.The equation has a Hamiltonian structure.

Invariants

Both quantities are independent of time for solutions $u \in C(0, T_M; H^1(\mathbb{R}))$:

$$\int_{-\infty}^{\infty} \left(u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2 \right) dx,$$

$$I(u) = \int_{-\infty}^{\infty} \left[\left(\frac{\delta_2}{\delta_1} - 1 - \frac{1}{2} u \right) u^2 + \left(\gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} + \frac{7}{24} u \right) u_x^2 + \frac{1}{16} u^4 \right] dx.$$

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IV. Global well-posedness and temporal growth

Theorem. Temporal growth.

The integral equation is globally well posed in the space $H^k(\mathbb{R})$ for any $k \geq 2$. If the initial data u_0 lies in $H^m(\mathbb{R})$ for a certain $m \geq 2$ and the solution u emanating from u_0 is bounded in $H^2(\mathbb{R})$ independently of t, then the temporal growth bounds

$$\|u(\cdot,t)\|_k \leq c(1+t)^{\frac{k-2}{2}}, \quad \text{for} \;\; t \geq 0 \;\; \text{and} \;\; k = 2, 3, \cdots m \qquad (2)$$

hold. Here, the constant c depends only on $\|u_0\|_m$ and the assumed bound on the $H^2(\mathbb{R})$-norm of u.

In particular, if $\gamma = \frac{7}{48}$, then these inequalities are valid and the constant c depend only on $\|u_0\|_m$.

Remark. In the case $\gamma = \frac{7}{48}$, because of the Hamiltonian structure, $||u(t)||_2$ is bounded independently of $t \geq 0$.

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IV. Global well-posedness and temporal growth

Theorem. Global well-posedness in $H^1(\mathbb{R})$.

Assume that

$$\gamma = \frac{7}{48}, \quad \delta_2 > \delta_1 > 0 \ \text{ and } \ \gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} > 0.$$

Under these hypotheses, there are positive numbers τ and τ_1, depending only on γ_1, γ_2, δ_1 and δ_2, such that if $u_0 \in H^1(\mathbb{R})$ and $\|u_0\|_1 \leq \tau$, then

$$|\, u(\cdot, t)|_\infty \leq \|u(\cdot, t)\|_1 \leq \tau_1$$

for all $t \geq 0$.

Physically relevant values of the parameters

$$\gamma_1 = \gamma_2 = \frac{1}{12}, \ \chi_1 := \frac{\delta_2}{\delta_1} - 1 = \frac{7}{180\delta_1} > 0,$$

$$\chi_2 := \gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} = \frac{1}{12}(2 + \frac{7}{190\delta_1}) > \frac{1}{6}.$$

IV. Global well-posedness and temporal growth

Proof. Define $\chi = \min(\chi_1, \chi_2)$. Suppose $||u_0||_1 \leq \tau$, for some $\tau > 0$, and derive estimates for the second invariant $I(u) = I(u_0)$, say

$$\chi||u(t)||_1^2 - \frac{1}{2}||u(t)||_1^3 \leq I(u) = I(u_0) \leq \max(\chi_1, \chi_2)\tau^2 + \frac{1}{2}\tau^3 + \frac{1}{16}\tau^4$$

$$\leq \frac{\chi^3}{2} \quad \text{for } \tau \text{ small enough}$$

First, note that if $||u(t)||_1 \leq \chi$, then $\frac{\chi}{2}||u(t)||_1^2 \leq I(u) = I(u_0)$.

Second, choose $\overline{\tau}$ such that for $||u_0||_1 \leq \overline{\tau}$, we have $I(u_0) \leq \frac{\chi^3}{8}$.

Finally, define $\tau = \min(\frac{\chi}{2}, \overline{\tau})$.

Proof (cont'd). By continuity, if $||u_0||_1 \leq \chi$, the solution satisfies $||u(t)||_1 \leq \chi$ on some time interval $[0, T]$, for a $T > 0$.

Suppose there is a maximal time of existence T^*. Then, $||u(T^*)||_1 = \chi$ and for $t > T^*$ in a neighbourhood of T^*, $||u(t)||_1 > \chi$.

Then the above estimates yield, for t in a neighbourhood of T^*, $t > T^*$,

$$||u(T^*)||_1 \leq \frac{\chi}{2},$$

which is a contradiction with $||u(T^*)||_1 = \chi$.

Thus $||u(t)||_1 \leq \chi$ for all $t \geq 0$.

V. Conclusion : Results in order-one variables

Theorem

Suppose the initial-value problem (1) has γ_1 and δ_1 positive. Then this problem is locally well posed in $H^s(\mathbb{R})$ for any $s \geq 1$, in $C_b^k(\mathbb{R})$ for any $k = 1, 2, \cdots$ and in $C_b^{k,\sigma}(\mathbb{R})$ for any $\sigma \in (0, 1]$ and $k = 1, 2, \cdots$.

If in addition $\gamma = \frac{7}{48}$, then it is globally well posed in $H^s(\mathbb{R})$ for $s \geq 1$. If $s = k + \sigma \geq 2$ then the solution is globally bounded in $H^2(\mathbb{R})$ and satisfies the growth bounds

$$||u(t)||_s \leq c(1 + t)^{\frac{s-2+\sigma}{2}}$$

for $t \geq 0$. Moreover, still in the case $\gamma = \frac{7}{48}$, for α and β sufficiently small and $s \geq 1$, the solution is globally bounded in $H^1(\mathbb{R})$.

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Thank you for your attention!

And stay safe...

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