# Unstable Stokes waves: A new periodic Evans function approach

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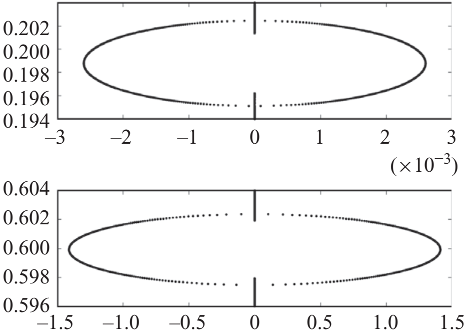
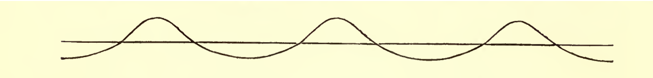



(Figures from [Deconinck and Oliveras; 2009])

Stokes in his 1847 paper made significant contributions to

- periodic traveling waves
- at the *free* surface
- two dimensional and irrotational flow
- acted on by gravity, no surface tension

e.g., the 'Stokes expansion'



(Figure from [Stokes; 1847])

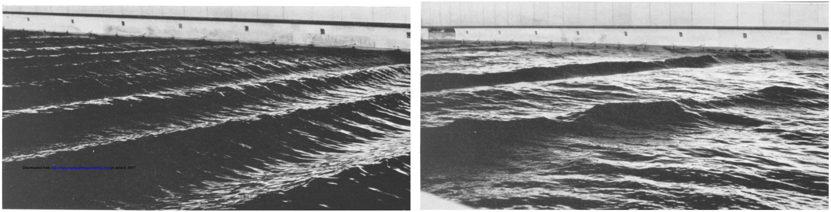
Existence theory (=rigorous proofs) of Stokes waves

- [Nekrasov; 1921], [Levi-Civita; 1925] in the infinite depth, and [Struik; 1926] in the finite depth, for small amplitude
- [Krasovskii; 1960, 1961],... for large amplitude

and many more.

"For a long time no doubt has remained, therefore, that [Stokes waves] are theoretically possible as states of perfect dynamic equilibrium." ([Benjamin and Feir; 1967])

# [Benjamin; 1967] experimentally found




The original caption reads: "Photographs of a progressive at two stations, illustrating disintegration due to instability: (left) view near to wavemaker; (right) view at 200ft. farther from wavemaker. Fundamental wavelength, 7.2ft."

[Benjamin; 1967] and [Whitham; 1967] predicted that
a Stokes wave of *small* amplitude is unstable in *deep* water, so that

$$\boxed{\text{(the wave number)}\times\text{(the fluid depth)} > 1.3627\ldots,}$$

namely, the Benjamin-Feir or modulational instability.

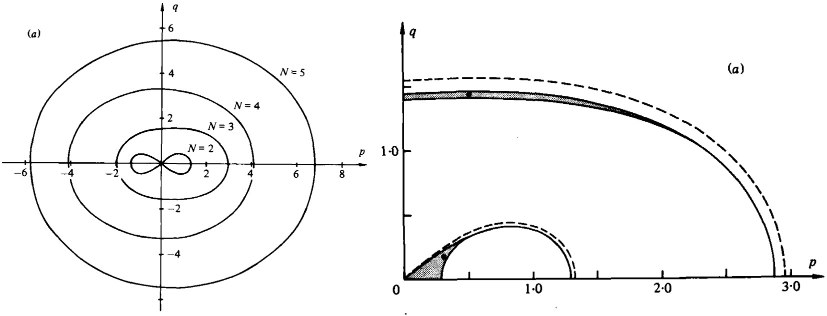
Corroborating results arrived the same time, but independently,
by Lighthill, Zakharov, Ostrovsky, Benney, Newell, . . . .
"The idea was emerging when the time was indeed ripe."
([Zakharov and Ostrovsky; 2008])

[Bridges and Mielke; 1995] proved spectral instability,
rigorously justifying the formal arguments in the 1960s.

But some fundamental issues remained open, e.g.,
the spectrum away from $0 \in \mathbb{C}$.

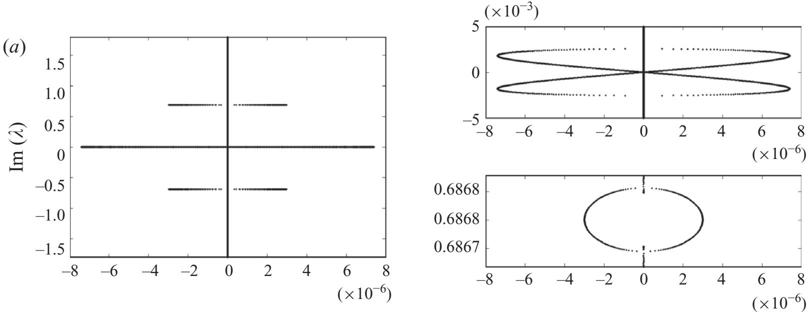
[McLean; 1982] numerically found instability when the unperturbed wave is ‘resonant’ with two infinitesimal perturbations:

$$k(\lambda) - k'(\lambda) = n\kappa, \quad n \neq 0, \in \mathbb{Z}.$$



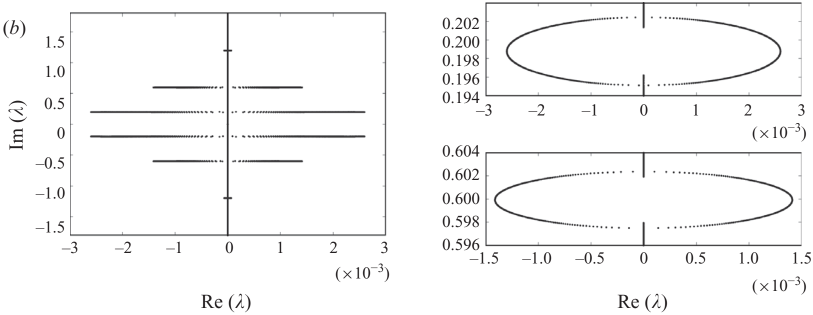

$h = 2.0$ and $a = 0.2$. resonances (a) and instability (b)

[Deconinck and Oliveras; 2009] numerically found




The original caption reads: Spectrum for $h = 1.5$ and $a = 0.01$ (left). Enlargements are shown (right) for the region near the origin (top) and on the imaginary axis near $0.68i$ (bottom).

Also,




The original caption reads: Spectrum for $h = 0.5$ and $a = 0.01$ (left). Enlargements are shown (right) for the region near $0.1489i$ (top) and $0.5212i$ (bottom).

This talk: the first proof of spectral instability away from $0 \in \mathbb{C}$.

What [Bridges and Mielke; 1995] did:

1. Locate the spectrum of $\mathcal{L}(\varepsilon, 0)$ at $0 \in \mathbb{C}$, explicitly for $|\varepsilon| \ll 1$.
   $\varepsilon =$ the amplitude parameter, $k =$ the Floquet exponent
2. Track the eigenvalues of $\mathcal{L}(\varepsilon, k)$ near $0 \in \mathbb{C}$ for $k \ll 1$.

What we do:

1. Locate the *full* spectrum of $\mathcal{L}(0, k)$ for all $k \in \mathbb{R}$.
2. Track the spectrum of $\mathcal{L}(\varepsilon, k)$ for $|\varepsilon| \ll 1$.

Also, do *not* resort to nonlocal operators, and use a periodic Evans function for cylindrical domains, and other ODE techniques.

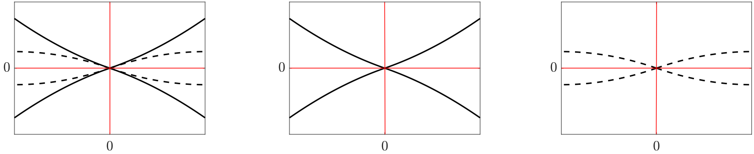
## Result 1. The Benjamin–Feir instability

A small amplitude and $2\pi/\kappa$ periodic Stokes wave in water of depth $= 1$ is spectrally unstable if

$$\begin{aligned}\mathrm{ind}(\kappa) =:& -\mu_0(\kappa)^{-2}(\cosh(2\kappa)+1)^2(10\cosh(2\kappa)^2+8\cosh(2\kappa)-9)\\ &+\mu_0(\kappa)^{-1}(8\cosh(2\kappa)^4+8\cosh(2\kappa)^3+4\cosh(2\kappa)^2+28\cosh(2\kappa)+24)\\ &-4\cosh(4\kappa)-32>0\end{aligned}$$

or, equivalently, $\kappa > \kappa_c \approx 1.362782756726421$.

An update on the spectral curves:




spectral curves — figure-8 — figure-$\infty$

## Result 2. High-frequency instability, or the lack thereof

Spectral instability near $\lambda \in i\mathbb{R}$, for which $k(\lambda) - k'(\lambda) = 2\kappa$
if $0.86430\ldots < \kappa < 1.00804\ldots$

*No* spectral instability for $k(\lambda) - k'(\lambda) = n\kappa$, $n \geqslant 3$
at the order of $\varepsilon^2$.

Elucidates some numerical findings but not all.

Because infinitesimally small amplitude $\lll$ small amplitude

# Step 1. Reformulate the water wave problem

Let $\boxed{u = \phi_x \quad \text{and} \quad y \mapsto \frac{y}{1+\eta(x,t)}}$ ("flattening" coordinates).

The water wave problem becomes

$$\boxed{\begin{array}{ll} \phi_x - \frac{y\eta_x\phi_y}{1+\eta} - u = 0, & 0 < y < 1 \\ u_x - \frac{y\eta_x u_y}{1+\eta} + \frac{\phi_{yy}}{(1+\eta)^2} = 0, & 0 < y < 1 \\ \eta_t + (u-1)\eta_x - \frac{\phi_y}{1+\eta} = 0, & y = 1 \\ \phi_t - u + \frac{(u-1)\eta_x\phi_y}{1+\eta} + \frac{u^2}{2} - \frac{\phi_y^2}{2(1+\eta)^2} + \mu\eta = 0, & y = 1 \\ \phi_y = 0, & y = 0 \end{array}}$$

## Step 2. Linearize about a Stokes wave of small amplitude

$$
\begin{array}{ll}
\phi_x - u - \frac{y\eta_x(\varepsilon)}{1+\eta(\varepsilon)}\phi_y - (\cdots)\eta_x + (\cdots)\eta = 0, & 0 < y < 1 \\
u_x + \frac{\phi_{yy}}{(1+\eta(\varepsilon))^2} - (\cdots)u_y - (\cdots)\eta_x + (\cdots)\eta = 0, & 0 < y < 1 \\
\lambda\eta + (u(\varepsilon)-1)\eta_x - \frac{\phi_y}{1+\eta(\varepsilon)} + \eta_x(\varepsilon)u + (\cdots)\eta = 0, & y = 1 \\
\zeta - u = 0, & y = 1 \\
\phi_y = 0, & y = 0
\end{array}
$$

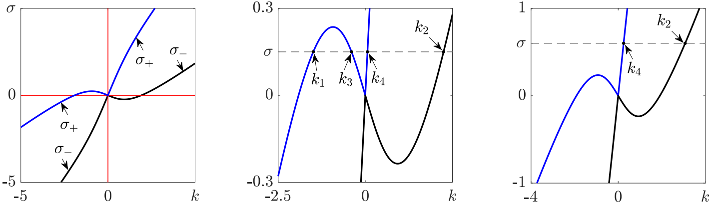
where $\eta = \eta(\phi_y(\cdot,1), \phi(\cdot,1), \zeta)$.

Abstractly, $\boxed{\mathbf{u}_x = \mathbf{L}(\lambda)\mathbf{u} + \mathbf{B}(x;\lambda,\varepsilon)\mathbf{u}}$

where $\mathbf{L}(\lambda)\mathbf{u} = \begin{pmatrix} u \\ -\phi_{yy} \\ [-\mu_0\phi_y - \lambda^2\phi + 2\lambda\zeta]_{y=1} \end{pmatrix}$.

## Step 3. The spectrum for $\varepsilon = 0$

$$\lambda = i\sigma,\ \sigma = \boxed{k \pm \sqrt{\mu_0 k \tanh(k)}},\ k \in \mathbb{R}.$$




For $\lambda = 0$, the eigenspece$=\mathrm{span}\{\phi_j(0)\}$, $j = 1, \ldots, 4$.
For $\lambda = i\sigma$, $\sigma > \sigma_{crit}$, the eigenspece$=\mathrm{span}\{\phi_j(\sigma)\}$, $j = 2, 4$.

## Step 4. Reduce to finite dimensions

Let $\lambda = i\sigma + \delta$, $\delta \in \mathbb{C}$ and $|\delta| \ll 1$, and

$$\mathbf{u}_x = \mathbf{L}(i\sigma)\mathbf{u} + \mathbf{B}(x;\sigma,\delta,\varepsilon)\mathbf{u}.$$

Let $\mathbf{v} = \Pi(\sigma)\mathbf{u}$ and $\mathbf{w} = (\mathbf{1} - \Pi(\sigma))\mathbf{u}$
$\Pi(\sigma) =$ the projection onto the eigenspace, and

$$\boxed{\mathbf{v}_x = \mathbf{L}(i\sigma)\mathbf{v} + \Pi(\sigma)\mathbf{B}(x;\sigma,\delta,\varepsilon)(\mathbf{v} + \mathbf{w}(x,\mathbf{v};\sigma,\delta,\varepsilon))}$$

Let $\mathbf{v}(x;\sigma,\delta,\varepsilon) = \sum_j a_j(x;\sigma,\delta,\varepsilon)\boldsymbol{\phi}_j(\sigma)$, $\mathbf{a} = (a_j)$
The periodic Evans function is

$$\boxed{\Delta(\lambda,k;\varepsilon) = \det(\mathbf{a}(T;\sigma,\delta,\varepsilon) - e^{ikT}\mathbf{I}),}\ T = 2\pi/\kappa \text{ the period.}$$

$$\boxed{\mathrm{spec} = \{\lambda \in \mathbb{C} : \Delta(\lambda,k,\varepsilon) = 0 \quad \text{for some } k \in \mathbb{R}\}}$$

## Step 5. Exapand the Evans function. The Benjamin–Feir instability

For $\delta = i0 + \delta$, $|\delta| \ll 1$ for $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$,

$$\mathbf{a}(T) = \begin{pmatrix} e^{-i\kappa T} & 0 & 0 & 0 \\ 0 & e^{i\kappa T} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & T & 1 \end{pmatrix}$$

$$+ \delta \begin{pmatrix} a_{11}^{(1,0)} & 0 & 0 & 0 \\ 0 & a_{11}^{(1,0)} & 0 & 0 \\ 0 & 0 & a_{33}^{(1,0)} & 0 \\ a_{41}^{(1,0)} & a_{41}^{(1,0)} & \frac{T}{2} a_{33}^{(1,0)} & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & a_{13}^{(0,1)} & 0 \\ 0 & 0 & a_{13}^{(0,1)} & 0 \\ 0 & 0 & 0 & 0 \\ a_{41}^{(0,1)} & -a_{41}^{(0,1)} & a_{43}^{(0,1)} & 0 \end{pmatrix}$$

$$+ \delta^2 \begin{pmatrix} a_{11}^{(2,0)} & 0 & * & 0 \\ 0 & (a_{11}^{(2,0)})^* & * & 0 \\ a_{31}^{(2,0)} & a_{31}^{(2,0)} & * & a_{34}^{(2,0)} \\ * & * & * & a_{44}^{(2,0)} \end{pmatrix} + \delta\varepsilon \begin{pmatrix} a_{11}^{(1,1)} & a_{11}^{(1,1)} & * & a_{14}^{(1,1)} \\ a_{11}^{(1,1)} & a_{11}^{(1,1)} & * & a_{14}^{(1,1)} \\ a_{31}^{(1,1)} & * & * & 0 \\ * & * & * & 0 \end{pmatrix} + \cdots$$

$$\sum_{j=1}^{4}\left(\frac{d}{dx}a_{jk}^{(m,n)}\right)\boldsymbol{\phi}_j(0) = -\, i\kappa\, a_{1k}^{(m,n)}\boldsymbol{\phi}_1(0) + i\kappa\, a_{2k}^{(m,n)}\boldsymbol{\phi}_2(0)$$

$$+\, a_{3k}^{(m,n)}\boldsymbol{\phi}_4(0) + \boldsymbol{\Pi}(0)\mathbf{f}_k^{(m,n)}(x;0),$$

$$\mathbf{f}_k^{(m,n)}(x;0) = \sum_{m',n'}\mathbf{B}^{(m',n')}(x;0)\Big(\mathbf{w}_k^{(m-m',n-n')}(x;0) + \sum a_{jk}^{(m-m',n-n')}\boldsymbol{\phi}_j(0)\Big),$$

and $\mathbf{w}_k^{(m,n)}$ solves

$$\begin{aligned}
&\phi_{xx} + \phi_{yy} = ((\mathbf{1} - \boldsymbol{\Pi}(0))\mathbf{f}_k^{(m,n)}(x;0))_{1\,x} + ((\mathbf{1} - \boldsymbol{\Pi}(0))\mathbf{f}_k^{(m,n)}(x;0))_2 && 0 < y < 1\\
&u = \phi_x - ((\mathbf{1} - \boldsymbol{\Pi}(0))\mathbf{f}_k^{(m,n)}(x;0))_1 && 0 < y < 1\\
&\zeta_x = -\mu_0\phi_y + ((\mathbf{1} - \boldsymbol{\Pi}(0))\mathbf{f}_k^{(m,n)}(x;0))_3 && y = 1\\
&\zeta = u && y = 1\\
&\phi_y = 0 && y = 0.
\end{aligned}$$

$$
\begin{aligned}
&\Delta(\lambda, n\kappa+\gamma;\varepsilon) = \det(\mathbf{a}(T;0,\lambda,\varepsilon) - e^{i\gamma T}\mathbf{I}) \\
=&d^{(4,0,0)}\lambda^4 + d^{(3,1,0)}\lambda^3\gamma + d^{(2,2,0)}\lambda^2\gamma^2 + d^{(1,3,0)}\lambda\gamma^3 + d^{(0,4,0)}\gamma^4 + \cdots \\
&+o((|\lambda|+|\gamma|)^4 + |\lambda|^3|\varepsilon|^2 + |\lambda|^2|\varepsilon|^3 + |\gamma|^3|\varepsilon|^2 + |\gamma|^2|\varepsilon|^3 \\
&\qquad + |\lambda\gamma||\varepsilon|^2(|\lambda|+|\gamma|+|\varepsilon|) + |\lambda||\varepsilon|^5 + |\gamma||\varepsilon|^5)
\end{aligned}
$$

Let
$\lambda_j(k_j(0)+\gamma,\varepsilon) = \alpha_j^{(1,0)}\gamma + \alpha_j^{(2,0)}\gamma^2 + \alpha_j^{(1,1)}\gamma\varepsilon + o(|\gamma|^2 + |\gamma||\varepsilon|)$,
and solve $\Delta(\lambda_j(k_j(0)+\gamma,\varepsilon), k_j(0)+\gamma;\varepsilon) = 0$.

# High frequency instability. Why resonance?

$$a_{jk}^{(m,n)}(x) = e^{ik_{2j}x}\Big\langle \sum \int_0^x e^{-ik_{2j}x'}\mathbf{B}^{(m',n')}(x') \Big(\mathbf{w}_k^{(m-m',n-n')}(x') + \sum a_{j'k}^{(m-m',n-n')}(x')\boldsymbol{\phi}_{2j'}\Big)\, dx', \boldsymbol{\psi}_{2j}\Big\rangle,$$

involving, e.g.,

$$\int_0^x e^{i(k_{2j}-k_{2j'})x'}\sin(p\kappa x')\, dx' = \begin{cases} \dfrac{p\kappa - p\kappa\cos(p\kappa x)e^{i(k_{2j}-k_{2j'})x}}{p^2\kappa^2-(k_{2j}-k_{2j'})^2} + \cdots & |k_{2j}-k_{2j'}| \neq p\kappa, \\ \pm\dfrac{i}{2}x + \dfrac{1-e^{\pm 2ip\kappa x}}{4p\kappa} & k_{2j}-k_{2j'} = \pm p\kappa, \end{cases}$$

# God is in the detail!

and

$$\begin{aligned}
b_{1,2}^{(0,1)} = p_{2,1}\Big(&\frac{\kappa s(s_2(2)-2k_2c_2(2))}{4k_2} + \frac{k_2\kappa s(2k_2-s_2(2))}{4} - \frac{k_2^2\kappa s_2(k_2c_2s-\kappa cs_2)}{k_2^2-\kappa^2} \\
&+ \frac{2\kappa^2c_2(k_2cs_2-\kappa c_2s)}{\mu_0(k_2^2-\kappa^2)} - \frac{2k_2^2\kappa^2c_2(k_2cs_2-\kappa c_2s)}{\mu_0(k_2^2-\kappa^2)} + \frac{2k_2\kappa^2\sigma c_2(k_2cs_2-\kappa c_2s)}{\mu_0(k_2^2-\kappa^2)} \\
&- \frac{2\kappa^2\sigma c_2(k_2cs_2-\kappa c_2s)}{k_2\mu_0(k_2^2-\kappa^2)} - \frac{k_2^2\kappa\sigma c_2(k_2c_2s-\kappa cs_2)}{\mu_0(k_2^2-\kappa^2)} + \frac{k_2\kappa\sigma^2c_2(k_2c_2s-\kappa cs_2)}{\mu_0(k_2^2-\kappa^2)} \\
&- \frac{c_2(k_2^2-1)}{k_2\mu_0^2}(k_2\kappa^2\mu_0c_2c - k_2^2\kappa\mu_0s_2s + 2k_2\kappa\mu_0\sigma s_2s \\
&\qquad - \kappa\sigma^3c_2s - k_2\kappa\mu_0c_2s + \kappa\mu_0\sigma c_2s + k_2\kappa\mu_0^2c_2s + k_2\kappa\sigma^2c_2s) \\
&+ k_2\kappa s_2\Big(\frac{\kappa c_2c}{k_2^2-\kappa^2} - \frac{k_2s_2s}{k_2^2-\kappa^2} + \frac{c_2s(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} - \frac{2k_2\kappa cs_2}{(k_2^2-\kappa^2)^2}\Big) \\
&- k_2^2\kappa^2s_2\Big(\frac{k_2c_2c}{k_2^2-\kappa^2} - \frac{\kappa s_2s}{k_2^2-\kappa^2} - \frac{cs_2(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} + \frac{2k_2\kappa c_2s}{(k_2^2-\kappa^2)^2}\Big) \\
&+ \frac{k_2\kappa\sigma c_2}{\mu_0}\Big(\frac{\kappa c_2c}{k_2^2-\kappa^2} - \frac{k_2s_2s}{k_2^2-\kappa^2} + \frac{c_2s(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} - \frac{2k_2\kappa cs_2}{(k_2^2-\kappa^2)^2}\Big) \\
&- \frac{k_2^2\kappa^2\sigma c_2}{\mu_0}\Big(\frac{k_2c_2c}{k_2^2-\kappa^2} - \frac{\kappa s_2s}{k_2^2-\kappa^2} - \frac{cs_2(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} + \frac{2k_2\kappa c_2s}{(k_2^2-\kappa^2)^2}\Big) \\
&- \frac{\kappa\sigma^2c_2}{\mu_0}\Big(\frac{\kappa c_2c}{k_2^2-\kappa^2} - \frac{k_2s_2s}{k_2^2-\kappa^2} + \frac{c_2s(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} - \frac{2k_2\kappa cs_2}{(k_2^2-\kappa^2)^2}\Big) \\
&+ \frac{k_2\kappa^2\sigma^2c_2}{\mu_0}\Big(\frac{k_2c_2c}{k_2^2-\kappa^2} - \frac{\kappa s_2s}{k_2^2-\kappa^2} - \frac{cs_2(k_2^2+\kappa^2)}{(k_2^2-\kappa^2)^2} + \frac{2k_2\kappa c_2s}{(k_2^2-\kappa^2)^2}\Big)\Big) \\
&- \frac{\kappa\ p_{1,1}s(1)s}{\mu_0}(-c_2\sigma^2 + k_2c_2\sigma + 2k_2\mu_0s_2),
\end{aligned}$$

$$\begin{aligned}
b_{1,3}^{(0,1)} = -p_{2,2}\Big(&\frac{2ik_2^2s(k_2c_4s_2-k_4c_2s_4)}{k_4(k_2^2-k_4^2)} - \frac{2ik_2^2k_4s(k_2c_4s_2-k_4c_2s_4)}{k_2^2-k_4^2} \\
&+ \frac{i(k_4^2-1)c_4}{k_4\mu_0^2}(k_2\mu_0^2s_2s + k_2\kappa^2\mu_0s_2s + 2\kappa\mu_0\sigma^2c_2c - 2k_2\kappa\mu_0\sigma c_2c \\
&\qquad - \kappa^2\sigma^2c_2s - 2k_2\kappa\mu_0^2cs_2 + k_2\kappa^2\sigma c_2s) \\
&+ \frac{ik_2\kappa^2s_2}{k_4}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&- ik_2k_4\kappa^2s_2\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&+ \frac{ik_2\kappa^2\sigma c_2}{k_4\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&- \frac{ik_2k_4\kappa^2\sigma c_2}{\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&+ \frac{ik_4\kappa^2\sigma^2c_2}{\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&- \frac{i\kappa^2\sigma^2c_2}{k_4\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big)\Big),
\end{aligned}$$



and

$$\begin{aligned}
b_{1,4}^{(0,1)} = -p_{2,2}\Big(&\frac{k_2k_4\kappa s(k_2c_2s_4-k_4c_4s_2)}{k_2^2-k_4^2} + \frac{k_2k_4\kappa s_2(k_4c_4s-\kappa cs_4)}{k_4^2-\kappa^2} \\
&- \frac{2k_2\kappa^2c_2(k_4cs_4-\kappa c_4s)}{k_4\mu_0(k_4^2-\kappa^2)} + \frac{2k_2k_4\kappa^2c_2(k_4cs_4-\kappa c_4s)}{\mu_0(k_4^2-\kappa^2)} + \frac{k_2k_4\kappa\sigma c_2(k_4c_4s-\kappa cs_4)}{\mu_0(k_4^2-\kappa^2)} \\
&+ \frac{2\kappa^2\sigma c_2(k_4cs_4-\kappa c_4s)}{k_4\mu_0(k_4^2-\kappa^2)} - \frac{2k_4\kappa^2\sigma c_2(k_4cs_4-\kappa c_4s)}{\mu_0(k_4^2-\kappa^2)} - \frac{k_4\kappa\sigma^2c_2(k_4c_4s-\kappa cs_4)}{\mu_0(k_4^2-\kappa^2)} \\
&+ \frac{\kappa c_4(k_4^2-1)}{k_4\mu_0^2}(-k_2^2\mu_0s_2s + 2k_2\mu_0\sigma s_2s + k_2\kappa\mu_0c_2c \\
&\qquad - \sigma^3c_2s + k_2\mu_0^2c_2s + k_2\sigma^2c_2s - k_2\mu_0c_2s + \mu_0\sigma c_2s) \\
&+ \frac{k_2^2\kappa s}{k_4}\Big(\frac{k_2c_2c_4}{k_2^2-k_4^2} - \frac{k_4s_2s_4}{k_2^2-k_4^2} - \frac{c_4s_2(k_2^2+k_4^2)}{(k_2^2-k_4^2)^2} + \frac{2k_2k_4c_2s_4}{(k_2^2-k_4^2)^2}\Big) \\
&+ k_2\kappa s\Big(\frac{k_2c_2c_4}{k_2^2-k_4^2} - \frac{k_4s_2s_4}{k_2^2-k_4^2} - \frac{c_4s_2(k_2^2+k_4^2)}{(k_2^2-k_4^2)^2} + \frac{2k_2k_4c_2s_4}{(k_2^2-k_4^2)^2}\Big) \\
&- k_2^2k_4\kappa s\Big(\frac{k_4c_2c_4}{k_2^2-k_4^2} - \frac{k_2s_2s_4}{k_2^2-k_4^2} + \frac{c_2s_4(k_2^2+k_4^2)}{(k_2^2-k_4^2)^2} - \frac{2k_2k_4c_4s_2}{(k_2^2-k_4^2)^2}\Big) \\
&- k_2^2k_4\kappa s\Big(\frac{k_2c_2c_4}{k_2^2-k_4^2} - \frac{k_4s_2s_4}{k_2^2-k_4^2} - \frac{c_4s_2(k_2^2+k_4^2)}{(k_2^2-k_4^2)^2} + \frac{2k_2k_4c_2s_4}{(k_2^2-k_4^2)^2}\Big) \\
&- k_2\kappa s_2\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&+ k_2k_4\kappa^2s_2\Big(\frac{k_4c_4c}{k_4^2-\kappa^2} - \frac{\kappa s_4s}{k_4^2-\kappa^2} - \frac{cs_4(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} + \frac{2k_4\kappa c_4s}{(k_4^2-\kappa^2)^2}\Big) \\
&- \frac{k_2\kappa\sigma c_2}{\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&+ \frac{k_2k_4\kappa^2\sigma c_2}{\mu_0}\Big(\frac{k_4c_4c}{k_4^2-\kappa^2} - \frac{\kappa s_4s}{k_4^2-\kappa^2} - \frac{cs_4(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} + \frac{2k_4\kappa c_4s}{(k_4^2-\kappa^2)^2}\Big) \\
&+ \frac{\kappa\sigma^2c_2}{\mu_0}\Big(\frac{\kappa c_4c}{k_4^2-\kappa^2} - \frac{k_4s_4s}{k_4^2-\kappa^2} + \frac{c_4s(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} - \frac{2k_4\kappa cs_4}{(k_4^2-\kappa^2)^2}\Big) \\
&- \frac{k_4\kappa^2\sigma^2c_2}{\mu_0}\Big(\frac{k_4c_4c}{k_4^2-\kappa^2} - \frac{\kappa s_4s}{k_4^2-\kappa^2} - \frac{cs_4(k_4^2+\kappa^2)}{(k_4^2-\kappa^2)^2} + \frac{2k_4\kappa c_4s}{(k_4^2-\kappa^2)^2}\Big)\Big) \\
&- \frac{\kappa\ p_{1,2}}{\mu_0}s(1)s(-c_2\sigma^2 + k_2c_2\sigma + 2k_2\mu_0s_2).
\end{aligned}$$

Also,

$$b_{1,5}^{(0,1)} = \frac{i\kappa(-\sigma^2c_2 + k_2\sigma c_2 + k_2\mu_0s_2)}{k_2\mu_0}, \qquad b_{1,6}^{(0,1)} = \frac{k_2\mu_0s_2 - \sigma^2c_2 + k_2\sigma c_2}{\mu_0},$$

$$b_{1,7}^{(0,1)} = \frac{\kappa\ b_{1,4}^{(0,1)} + k_4b_{1,3}^{(0,1)}i - k_2b_{1,3}^{(0,1)}i}{-k_2^2 + 2k_2k_4 - k_4^2 + \kappa^2}, \qquad b_{1,8}^{(0,1)} = -\frac{\kappa b_{1,3}^{(0,1)} + k_2b_{1,4}^{(0,1)}i - k_4b_{1,4}^{(0,1)}i}{-k_2^2 + 2k_2k_4 - k_4^2 + \kappa^2},$$

and

$$b_{1,9}^{(0,1)} = \frac{4i\kappa^2(\mu_0b_{1,5}^{(0,1)} - k_2\kappa c_2i + \kappa\sigma c_2i)}{k_2(k_2^2-4\kappa^2)\mu_0}, \qquad b_{1,10}^{(0,1)} = \frac{2\kappa(\mu_0b_{1,5}^{(0,1)} - k_2\kappa c_2i + \kappa\sigma c_2i)}{(k_2^2-4\kappa^2)\mu_0}.$$

Pros: can accommodate surface tension, vorticity, ...
infinite depth?

| | |
|---|---|
| $\phi_{xx} + \phi_{yy} = 0$ (vorticity) | $0 < y < 1 + \eta(x,t)$ |
| $\phi_y = 0$ (infinite depth) | $y = 0$ |
| $\eta_t - \eta_x + \eta_x \phi_x = \phi_y$ | $y = 1 + \eta(x,t)$ |
| $\phi_t - \phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2) + \mu\eta = 0$ (surface tension) | $y = 1 + \eta(x,t)$ |

Cons: stability.

$\Delta(\lambda, p\kappa + \gamma; \varepsilon) = W(\lambda, \gamma, \varepsilon) h(\lambda, \gamma, \varepsilon)$,

and roots of $W(\lambda, \gamma, \varepsilon) = \lambda^4 + g_3(\gamma, \varepsilon)\lambda^3 + g_2(\gamma, \varepsilon)\lambda^2 + g_1(\gamma, \varepsilon)\lambda + g_0(\gamma, \varepsilon)$?