# Vanishing viscosity and conserved quantities for 2D incompressible flow

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Connections: Mathematical problems in fluid dynamics
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# Euler equations

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  Involves studying optimal conditions for energy flux to vanish.

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- wild solutions: no control on integrability of vorticity

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Existence of such weak solutions is known (DiPerna, Majda 87), but uniqueness is open, except for the case $p = \infty$.

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$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

2. For almost every $t \in (0, T)$, $\text{div}\, u(t, \cdot) = 0$, in the sense of distributions.

Existence of such weak solutions is known (DiPerna, Majda 87), but uniqueness is open, except for the case $p = \infty$. We call a weak solution *conservative* if the $L^2$-norm of velocity is constant in time.

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*Fix* $T > 0$ *and let* $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$.

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$$\partial_t \left( \frac{|u^\varepsilon|^2}{2} \right) + \operatorname{div} \left[ u^\varepsilon \left( \frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] = u^\varepsilon \cdot \mathcal{R}^\varepsilon. \qquad (3)$$

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because $u^\varepsilon \to u$ in $L^\infty(L^6(\mathbb{T}^2))$, $\nabla u^\varepsilon = \zeta^\varepsilon * \nabla u \to \nabla u$ in $L^\infty(L^{3/2}(\mathbb{T}^2))$ and $u\cdot\nabla u \in L^\infty(L^{6/5}(\mathbb{T}^2))$.

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$S_q$ is a convolution with a mollifier, hence smooth. Can argue easily that energy flux for $S_q[f]$ vanishes if $f \in W^{1,3/2}$ – easy adaptation of argument for $\omega \in L^{3/2}$ with $S_q$ in place of the convolution with a mollifier.

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**Theorem (Cheskidov, Lopes Filho, N-L, Shvydkoy; 2016)**

*There exists a divergence free vector field* $u \in B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, *for any* $1 \leq p < 3/2$, *such that* $\limsup_{q\to\infty} \Pi_q[u] \neq 0$.

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Suggests exists dynamical mechanism preventing anomalous dissipation in 2D

Note. The div-free vector field $u$ in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

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Suggests exists dynamical mechanism preventing anomalous dissipation in 2D even for supercritical (less than $1/3$ regular) flows

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Obs. $1 < p < 3/2$ ‘Onsager supercritical’.

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$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

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Using strong convergence of initial data, together with the known fact that there are no energy concentrations for the vanishing viscosity limit with vorticity in $L^p$, $p > 1$, we complete the proof.

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Natural question: regularity conditions for conservation of $\|\omega(t, \cdot)\|_{L^p}$?

More generally, regularity conditions for $\omega$ to be *renormalized solution* of the transport equation?

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Question: convergence of vorticity only weak or can it be improved?

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First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in $L^2$, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in $L^p$, **forcing** $g^\nu \in L_t^1 L^p$. Then,

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*Let* $T > 0$*,* $\omega_0 \in L^p(\mathbb{T}^2)$*,* $1 < p < \infty$*,*

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$$\omega^\nu \to \omega^E \text{ strongly in } C(0, T; L^p(\mathbb{T}^2)),$$

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- Vorticity weak solutions obtained as limits of smooth approximations or the vortex blob method are also renormalized.

Thank you!