# Vanishing viscosity and conserved quantities for 2D incompressible flow

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### Euler equations

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  - Extension to p = 3/2 follows from Cheskidov, Constantin, Friedlander, Shvydkov.
  - Involves studying optimal conditions for energy flux to vanish.

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Existence of such weak solutions is known (DiPerna, Majda 87), but uniqueness is open, except for the case  $p = \infty$ . We call a weak solution *conservative* if the  $L^2$ -norm of velocity is constant in time.

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Fix T > 0 and let  $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$  be a weak solution with  $\omega \equiv \text{ curl } u \in L^{\infty}(0, T; L^{3/2}(\mathbb{T}^2)).$ 

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January 21<sup>st</sup>, 2021

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because  $u^{\varepsilon} \to u$  in  $L^{\infty}(L^{6}(\mathbb{T}^{2}))$ ,  $\nabla u^{\varepsilon} = \zeta^{\varepsilon} * \nabla u \to \nabla u$  in  $L^{\infty}(L^{3/2}(\mathbb{T}^{2}))$ and  $u \cdot \nabla u \in L^{\infty}(L^{6/5}(\mathbb{T}^2))$ .

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### Theorem (Cheskidov, Lopes Filho, N-L, Shvydkoy; 2016)

There exists a divergence free vector field  $u \in B_{3,\infty}^{1/3} \cap W^{1,p}(\mathbb{T}^2)$ , for any  $1 \le p < 3/2$ , such that  $\limsup_{a \to \infty} \Pi_a[u] \ne 0$ .

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Obs. 1 'Onsager supercritical'.

Proof:

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Proof: Assume  $\omega_0 \in L^p(\mathbb{T}^2)$  for some p < 2, and  $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable  $\Longrightarrow \exists$  family  $\{u^{\nu}\}\$  of solutions of Navier-Stokes satisfying the corresponding conditions.

$$\partial_t \omega^{\nu} + \mathbf{u}^{\nu} \cdot \nabla \omega^{\nu} = \nu \Delta \omega^{\nu}.$$

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Gagliardo-Nirenberg ⇒

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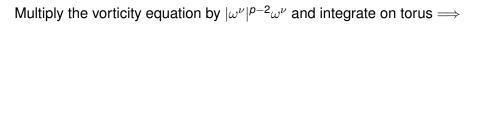
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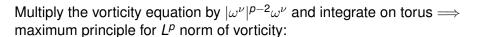
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Write 
$$y = y(t) = \|\omega^{\nu}\|_{L^{2}}^{2}$$
 and  $C_{0} = \|\omega_{0}^{\nu}\|_{L^{p}}^{-\frac{2p}{2-p}}$ .

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Rewriting in terms of vorticity yields

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Using strong convergence of initial data, together with the known fact that there are no energy concentrations for the vanishing viscosity limit with vorticity in  $L^p$ , p > 1, we complete the proof.

# $L^p$ -norms of vorticity

We consider conserved quantities for *vorticity* 

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We consider conserved quantities for *vorticity*  $\omega$  transported by div-free vector field:

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Natural question: regularity conditions for conservation of  $\|\omega(t,\cdot)\|_{L^p}$ ?

More generally, regularity conditions for  $\omega$  to be *renormalized solution* of the transport equation?

#### Consider the transport equation

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## Definition (DiPerna-Lions)

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Also: uniqueness for linear transport equation, Lagrangian formulation of transport, (notion of Lagrangian solution).

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Summary:

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## Question:

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Question: convergence of vorticity only weak or can it be improved?

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Nearly simultaneously Ciampa, Crippa, Spirito 2020, virtually same result but  $1 and <math>g^{\nu} = 0$ .

Discuss simpler case,

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where  $\omega^{\nu} = \text{curl } u^{\nu}$  and  $u^{\nu} \rightarrow u^{E}$  weak-\*  $L^{\infty}(0, T; L^{2})$ .

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Step 1  $\omega^{\nu} \rightharpoonup \omega^{E}$  weak-\*  $L^{\infty}(0, T; L^{p})$ ,  $\omega^{\nu}$  equicontinuous [0, T] to  $\mathcal{D}'$ 

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Indeed.

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$$\|\omega^{\nu}(t,\cdot)\|_{L^p} o \|\omega^{\mathcal{E}}(t,\cdot)\|_{L^p}$$
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Indeed.

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^{\nu}(t,\cdot)\|_{L^p}$$

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Step 4  $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$  strong  $L^p$ ,

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- If  $p = \infty$  get rate for  $C_t^0 L_x^q$  convergence (rate depends on  $L^1$ -modulus of continuity of  $\omega_0 \in L^\infty$ ),

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- Also extend energy conservation to full plane fluid domain.

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- Energy conservation in the case p = 1? No tools.
- Vorticity weak solutions obtained as limits of smooth approximations or the vortex blob method are also renormalized.

# Thank you!