Radial symmetry of stationary and uniformly-rotating solutions of 2D fluid equations

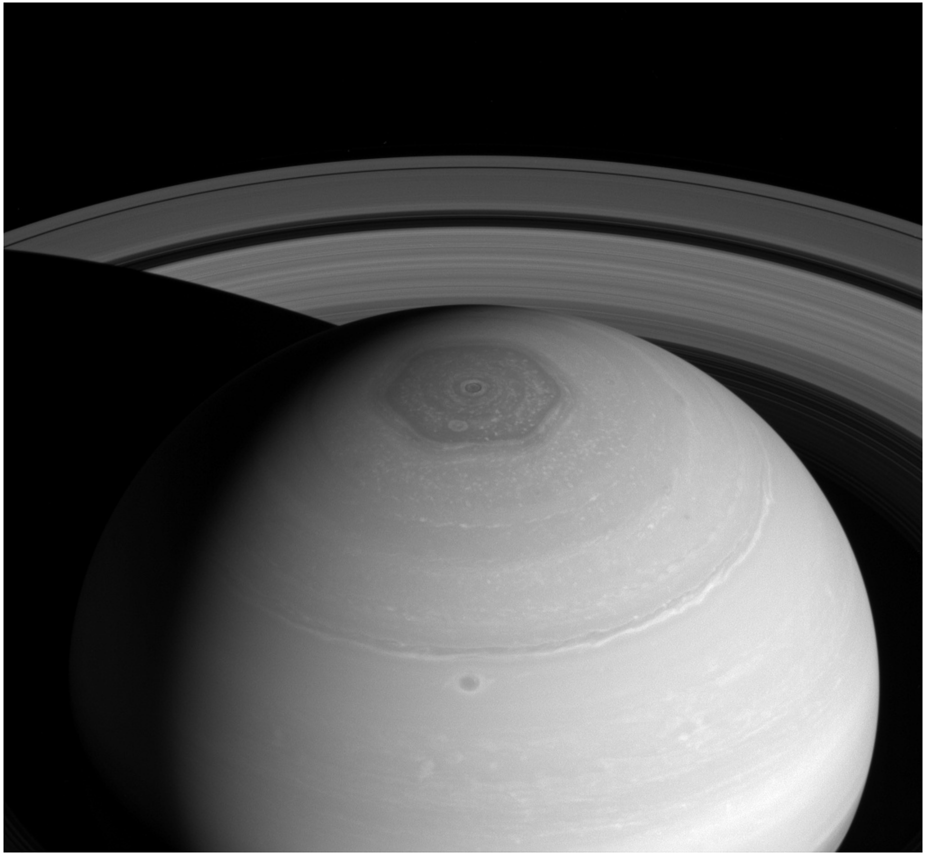
Yao Yao
Georgia Tech

joint work with
Javier Gómez-Serrano, Jaemin Park, and Jia Shi

Connections Workshop: Mathematical problems in fluid dynamics

Jan 22, 2021

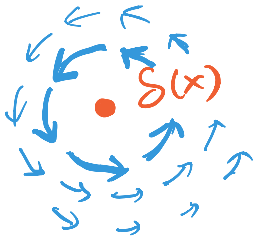
Saturn's hexagon



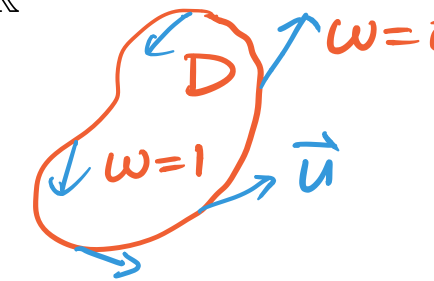
Rotating hexagon on the north pole of Saturn. Source: Wikipedia

Stationary/rotating patch for 2D Euler equation

- 2D Euler equation in vorticity form:



$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp (\mathcal{N} * \omega) \\ \omega(0, \cdot) = \omega_0, \end{cases}$$



where $\mathcal{N} = \frac{1}{2\pi} \log |x|$.

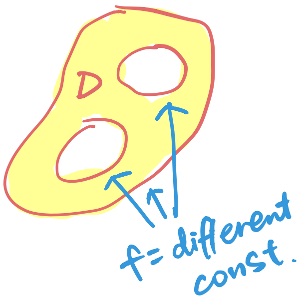
- Vortex patch: If $\omega_0(x) = 1_D(x)$ for bounded domain D,

$\omega(t, x) = 1_{D^t}(x)$, where $D^t = X_t(D)$ and X_t is the flow map of u.

- Global regularity of patches in $C^{1,\gamma}$ was proved by Chemin '93, a shorter proof by Bertozzi–Constantin '93.

- If $D^t = R_{\Omega t} D$ (rotation of D by angle Ωt), we say that D is a uniformly-rotating patch with angular velocity Ω. (If $\Omega = 0$, D becomes a stationary patch).

Symmetric or not?



D rotates with angular velocity Ω

$$\iff \underbrace{(u(x) - \Omega x^{\perp})}_{=\nabla^{\perp}(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2)} \cdot \vec{n}(x) = 0 \text{ on } \partial D$$

$$\iff \underbrace{1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2}_{=:f} = C_i \text{ on each component of } \partial D.$$

- Simple observation: Any radial D satisfies this for any $\Omega \in \mathbb{R}$.

Question

Under what condition must a stationary/rotating patch be radially symmetric?

Symmetric or not?

Positive answer in the following cases: if $\omega_0 = 1_D$, then

- Fraenkel '00: If D is simply-connected and $\Omega = 0$, it must be a disk. Proof based on the moving plane method.

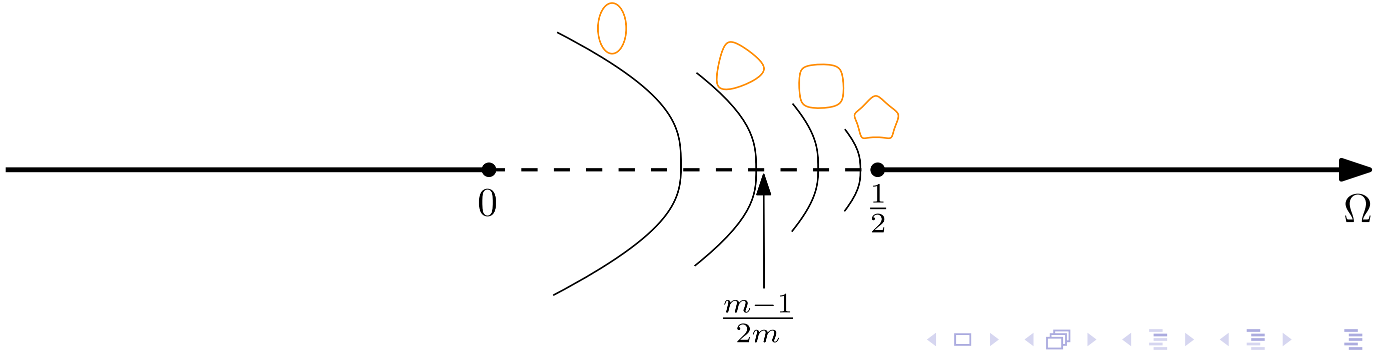
Consider stream function $\varphi(x) = 1_D * N$.

So φ satisfies $\Delta\varphi = 1_D = 1_{\{\varphi<c\}}$.
$\Rightarrow$ can apply moving planes.

- Hmidi '14: If D is convex and $\Omega < 0$, it must be a disk.
- Hmidi '14: If D is simply-connected and $\Omega = 1/2$, it must be a disk.

Non-radial uniformly rotating solutions

- Kirchhoff vortex (1876): any ellipse of semiaxis a, b is a rotating patch with $\Omega = \frac{ab}{(a+b)^2}$.
- Deem–Zabusky '78: numerical evidence of rotating patches with m-fold symmetry.
- Burbea '82 proved that there exists a family of m-fold rotating patches bifurcating from the disk at $\Omega = \frac{m-1}{2m}$. The case $m = 2$ corresponds to Kirchhoff ellipses.
- Boundary regularity: Hmidi–Mateu-Verdera '13, Castro–Córdoba–Gómez-Serrano '15
- Global bifurcation: Hassainia–Masmoudi–Wheeler '17

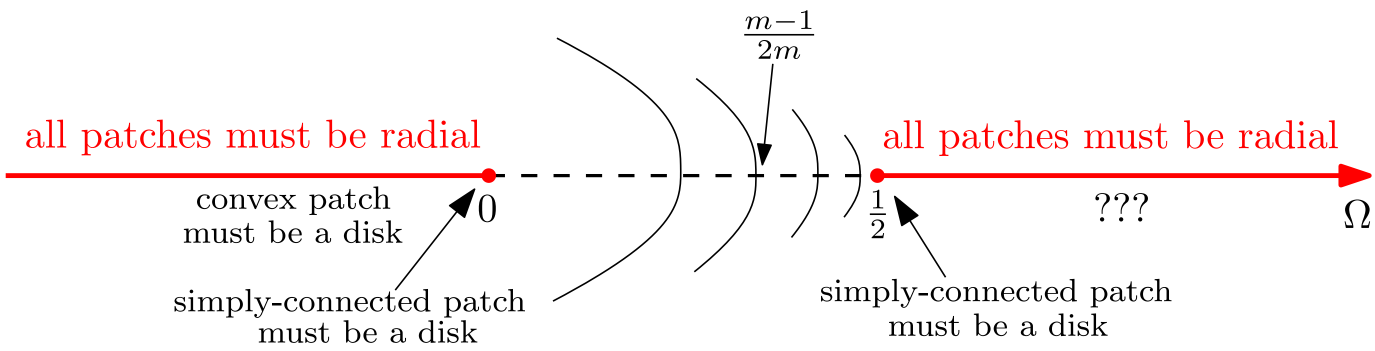


No non-trivial patch for $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$

Theorem (Gómez-Serrano, Park, Shi, and Y., '19)

Let D be a stationary/rotating patch (not necessarily connected or simply-connected) with angular velocity Ω.

- *If $\Omega < 0$ or $\Omega \geq 1/2$, then D must be radially symmetric.*
- *And if $\Omega = 0$, then D is radial up to a translation.*



- Instead of moving plane method, our proof has a calculus-of-variation flavor.

Simply-connected patch are radial for $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$

- The proof is very short for simply-connected patch D. Towards a contradiction, assume D is not a disk, and it is stationary/rotating with $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$.
- Idea: Consider the first variation of the "energy functional"

$$E[D] = -\int_{\mathbb{R}^2} \frac{1}{2} 1_D (1_D * \mathcal{N}) - \frac{\Omega}{2} |x|^2 1_D \, dx$$

along a carefully chosen deformation of D.

- For the transport equation $\rho_t + \nabla \cdot (\rho \vec{v}) = 0$ with initial data $\rho(x, 0) = 1_D$, we have

$$\frac{d}{dt} E[\rho]\Big|_{t=0} = -\int_D \vec{v}(x) \cdot \nabla \Big(\underbrace{(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2}_{=: f(x)} \Big) dx =: \mathcal{I}$$

- On the one hand, using $f = C$ on ∂D, divergence theorem gives $\mathcal{I} = 0$ for any smooth $\vec{v}$ with $\nabla \cdot \vec{v} = 0$ in D.

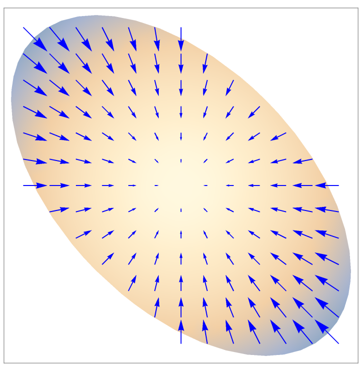
Perturbing D by a divergence-free vector field

- On the other hand, if D is simply-connected and not a disk, we construct an explicit smooth $\vec{v}$ with $\nabla \cdot \vec{v} = 0$ in D, and show that $\mathcal{I} \neq 0$ if $\Omega \in (-\infty, 0] \cup [1/2, \infty)$.
- We define $\vec{v} : \overline{D} \to \mathbb{R}^2$ as

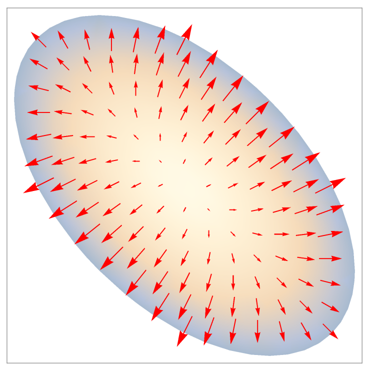
$$\vec{v}(x) := -\vec{x} - \nabla p,$$

where p solves the Poisson equation $\begin{cases} \Delta p = -2 & \text{in } D, \\ p = 0 & \text{on } \partial D. \end{cases}$

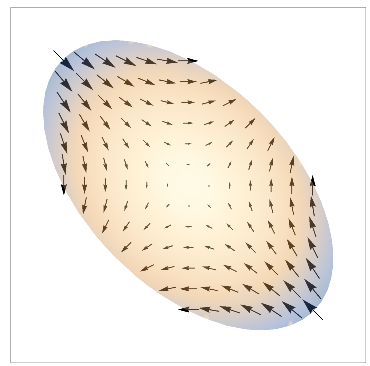
- Note that $\nabla \cdot \vec{v} = 0$ in D.



$-\vec{x}$



$-\nabla p$



$\vec{v} = -\vec{x} - \nabla p$

Obtaining a contradiction for $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$

- For such v, an explicit computation gives

$$\mathcal{I} = \int_D x \cdot \nabla(1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2)dx + \int_D \nabla p \cdot \nabla f dx$$
$$= \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx + (2\Omega - 1)\int_D p dx$$

- For $|D|$ fixed, $\int_D |x|^2 dx$ is minimized if and only if D is a disk.
- Talenti '76: If p solves $\Delta p = -2$ in D with $p = 0$ on ∂D, we have

$$\int_D p\, dx \leq \frac{1}{4\pi}|D|^2,$$

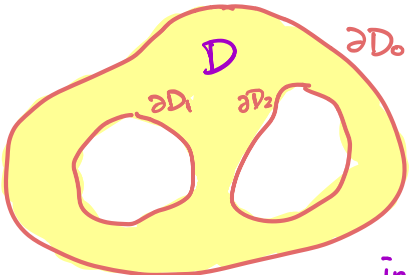
with "=" achieved if and only if D is a disk.

- Combining them, we have $\mathcal{I} \geq 0$ if $\Omega \leq 0$, $\mathcal{I} \leq 0$ if $\Omega \geq \frac{1}{2}$, with "=" achieved if and only if D is a disk.

Dealing with non-simply-connected patches

- If D is not simply-connected,

$$f = \mathcal{N} * \omega - \frac{\Omega}{2}|x|^2 = C_i \text{ on } \partial D_i \quad \Longrightarrow \quad \mathcal{I} = \int_D \vec{v} \cdot \nabla f \, dx \neq 0!$$



- If $\vec{v}$ is divergence free and satisfies $\int_{\partial D_i} \vec{v} \cdot n \, d\sigma = 0$, we still have $\mathcal{I} = 0$.
- Idea: still let $\vec{v} = -\vec{x} - \nabla p$, but modify p into $\Delta p = -2$ in D, $p = c_i$ on ∂D_i for suitable c_i. Also need to modify the proof of Talenti's theorem for such p.
- Such modification gives us that any connected patch (not necessarily simply-connected) must be radial for $\Omega \leq 0$ or $\Omega \geq 1/2$.

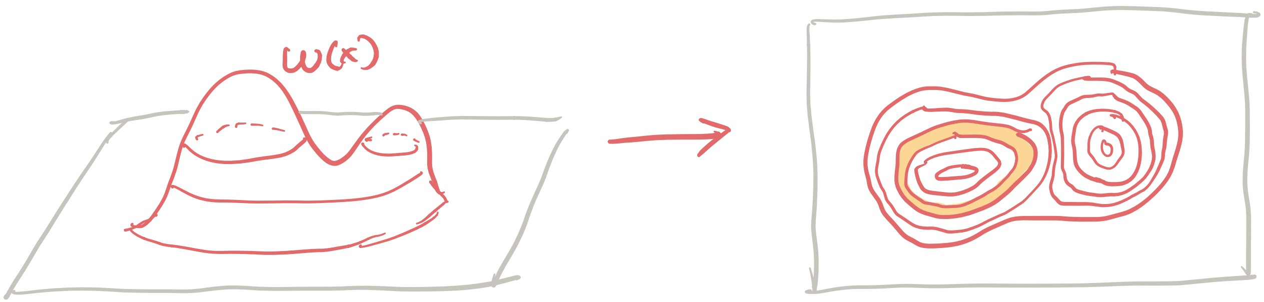
Stationary patch/smooth solution

For smooth stationary solutions we can also say the following:

Theorem (Gómez-Serrano, Park, Shi, and Y., '19)

Assume ω is a smooth stationary solution with compact support (or fast decay at infinity). If ω does not change sign, it must be radial up to a translation.

- Idea of proof: approximate a smooth ω by step functions, then apply the previous perturbation for each layer.



- Note: If vorticity is allowed to change sign, one can construct nonradial compactly-supported stationary solutions. (Gómez-Serrano–Park–Shi, forthcoming).

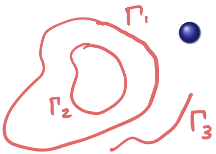
Stability results for radially symmetric steady states:

- Bedrossian–Coti Zelati–Vicol '17: Inviscid damping results for linearized equation around a smooth radial steady state
- Jia–Ionescu '19 Axi-symmetrization for nonlinear equation near point vortex solutions

Other rigidity results for steady 2D Euler equation:

- Hamel–Nadirashvili '17: any steady state in $\mathbb{R}^2$ without a stagnation point is a shear flow (moving plane methods).
- Hamel–Nadirashvili '19: generalization to annulus, exterior of disk.
- Constantin–Drivas–Ginsberg '20: Rigidity and flexibility result in a periodic channel.

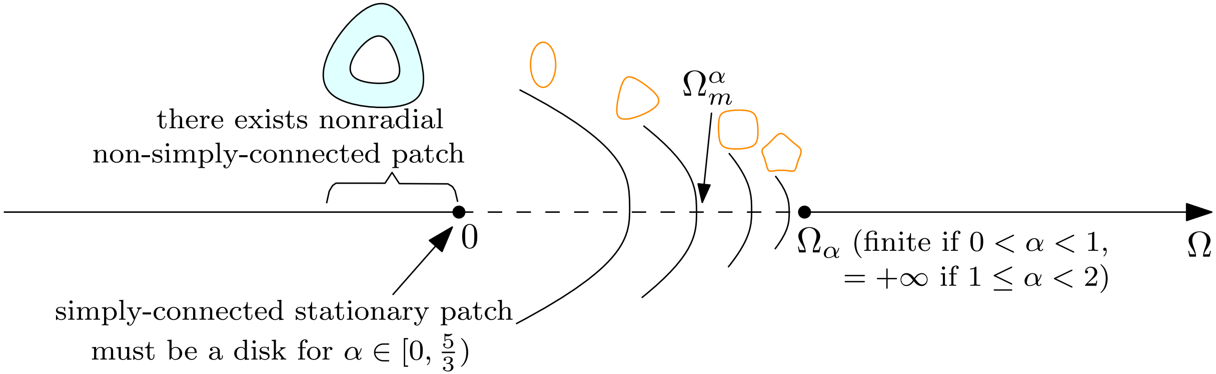
Rigidity results for more singular solutions:



- Gómez-Serrano–Park–Shi–Y. '20: Any stationary vortex sheet with positive strength concentrated on smooth curves with finite length must be radially symmetric.

SQG and generalized SQG

- Consider the Biot-Savart law $u = \nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \omega = \nabla^\perp (\mathcal{K}_\alpha * \omega)$, for $\alpha \in (0,2)$. ($\alpha = 0 \Rightarrow$ 2D Euler; $\alpha = 1 \Rightarrow$ SQG) $C_\alpha |x|^{-\alpha}$ for $\alpha>0$
- Existence of patch/smooth rotating solution (for some $\Omega > 0$) given by Castro–Córdoba–Gómez-Serrano '16.
- For $0 < \alpha < 5/3$, all *simply connected* stationary patches are disks. (Reichel '09, Lu–Zhu '12, Choksi–Neumayer–Topaloglu '18, moving plane method).
- Non-simply-connected stationary patches are not necessarily radial: For $\alpha \in (0,2)$, Gómez-Serrano '18 showed there exists non-radial stationary patches bifurcating from an annulus.

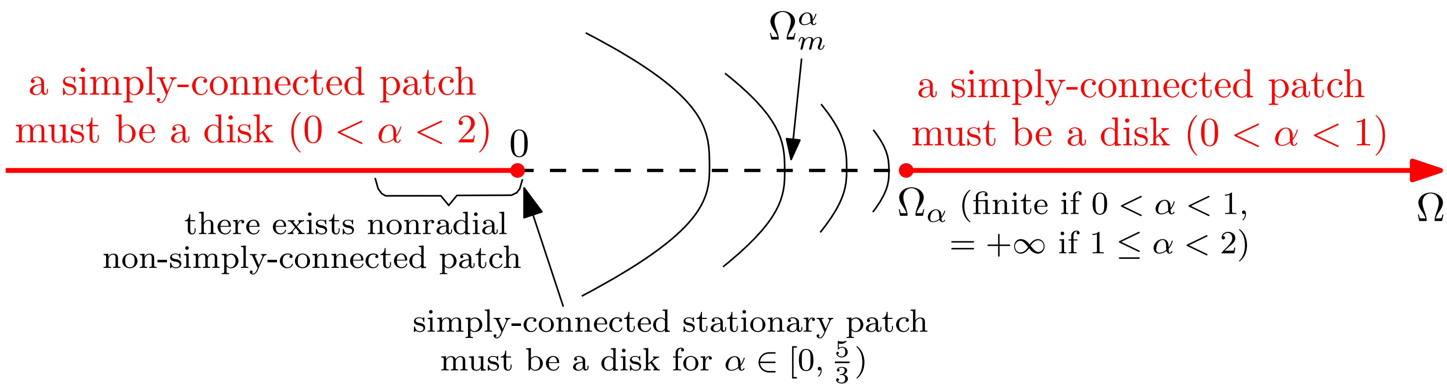


Symmetry of stationary/rotating patches

Theorem (Gómez-Serrano, Park, Shi, and Y., '19)

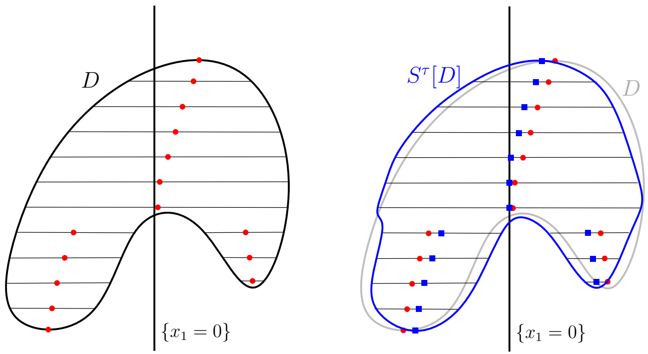
Let D be a simply-connected rotating patch with angular velocity Ω. Then:

- *For $\alpha \in (0,2)$, if $\Omega \le 0$, the patch must be a disk.*
- *For $\alpha \in (0,1)$, there exists a constant Ω_α (sharp and explicit) such that if $\Omega \ge \Omega_\alpha$ the patch must be a disk.*



Symmetry for $\Omega \leq 0$ case

- Known: $1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 =$ const on ∂D.
- Let $E[D] := \frac{1}{2}\int 1_D(1_D * \mathcal{K}_\alpha) - \frac{\Omega}{2}|x|^2 dx$.
- Let us perturb D by continuous Steiner symmetrization, in a similar spirit as Carrillo–Hittmeir–Volzone–Y. '19.



- Under this perturbation, $E[D]$ decreases to the first order of τ, i.e. $E[S^\tau[D]] - E[D] < -c\tau$.
- But using that $1_D * \mathcal{K}_\alpha - \frac{1}{2}\Omega|x|^2 = C$ on ∂D, we also have $E[S^\tau[D]] - E[D] = o(\tau)$, a contradiction.

Thank you for your attention!