Introduction to Water Waves
Lecture 1

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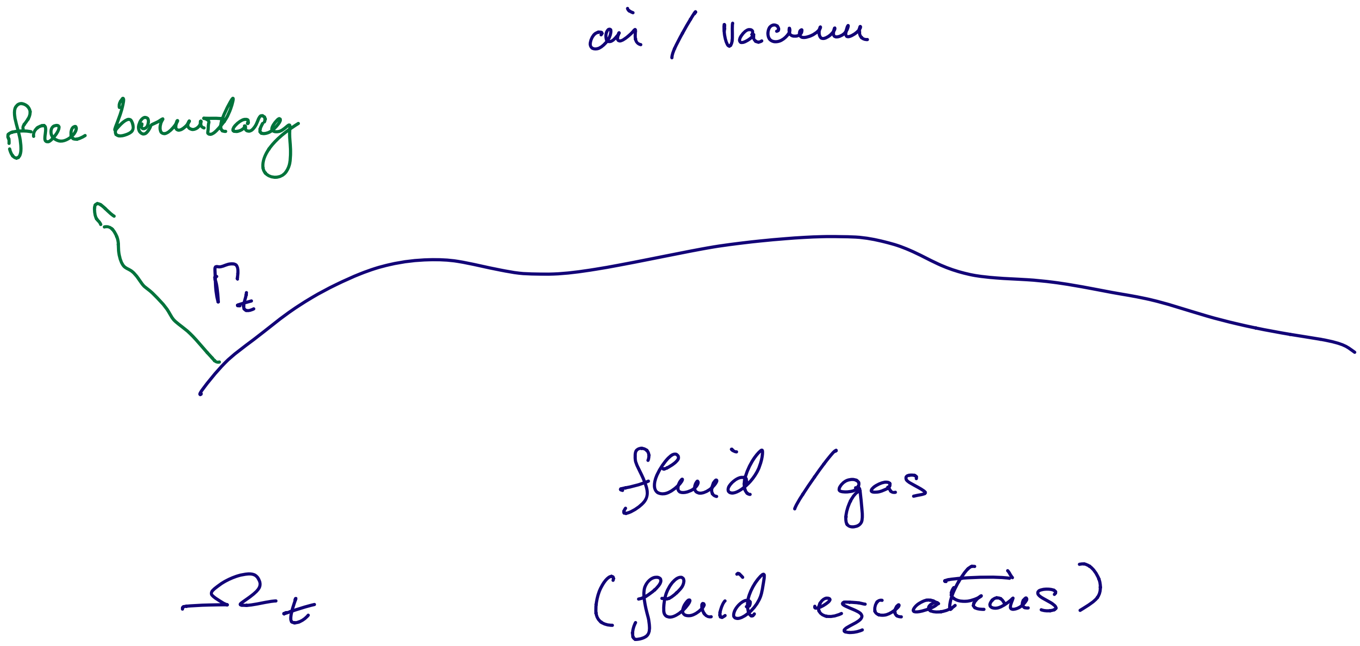
Free boundary problems in fluid dynamics



Free boundary problems in fluid dynamics

Fluid equations:

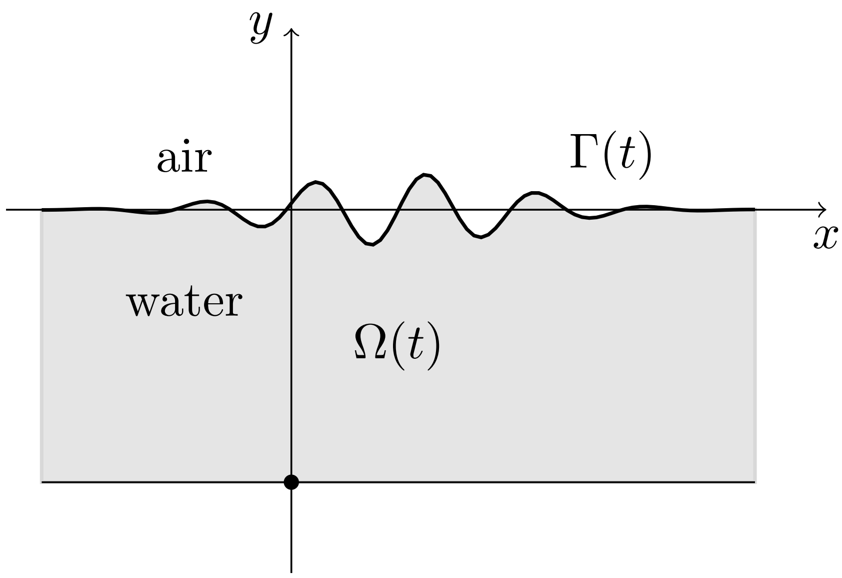
- Euler or Navier-Stokes equations
- Compressible or incompressible
- Gas vs fluid
- rotational or irotational

Boundary conditions:

- *kinematic*: free boundary moves with particle flow
- *dynamic*: balance of forces on free boundary (Newton's law)

$\rightarrow$ MSRI seminar Tuesdays 8:00 and 9:30

Free bdr problems for incompressible Euler



- Water flows inside the fluid domain
- Fixed bottom
- Free boundary motion (top)
- infinite or periodic domain

The incompressible Euler equation

Fluid motion in an open set:

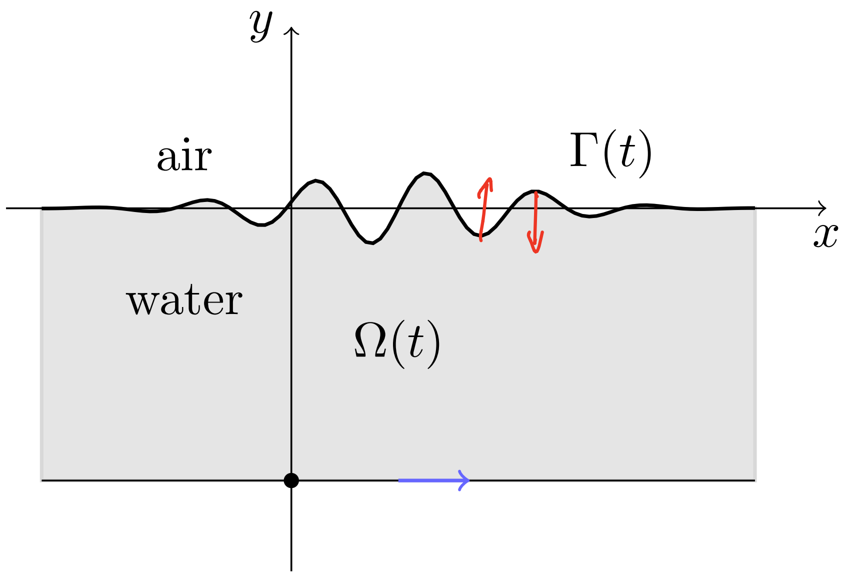
- $v = v(x,t)$ fluid velocity
- $p = p(x,t)$ fluid pressure
- incompressible flow, $\nabla \cdot v = 0$.
- Euler vs. Navier-Stokes

$\rho = 1$

$$\rho(\partial_t + v \cdot \nabla)v = \nabla p - g\mathbf{j} + \mu \Delta v \qquad \text{(Newton's law)}$$

- g = gravity
- μ = viscosity (resistance to shear stress)
- inviscid fluid: $\mu = 0$

Boundary conditions



Boundary conditions on Γ_t:

$$\begin{cases} \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup \Gamma_t & \text{(kinematic)} \\ p = -2\sigma \mathbf{H} \quad \text{on } \Gamma_t & \text{(dynamic)} \end{cases}$$

$\mathbf{H}$ = mean curvature of the boundary, σ = surface tension

Vorticity and irrotational flows

Vorticity = instantaneous rotation of a fluid

$$\omega = \nabla \times v \qquad \text{(curl of } v\text{)}$$

For solutions to Euler equations, ω satisfies a transport equation:

$$(\partial_t + v \cdot \nabla)\omega = (\omega \cdot \nabla)v$$

0 in 2d

Irrotational fluid: $\omega = 0$ (propagated along the flow)
Then there exists a *velocity potential* ϕ so that

$$v = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega_t,$$

which is uniquely determined by its values on the free boundary.

Special case: 2-d with constant vorticity

Water waves

WW= Free boundary problems for irrotational incompressible Euler.

Two main unknowns:

- the free surface Γ_t.
- The velocity potential ϕ [determined by its trace the free boundary]

$\Delta \phi = 0$

Key idea: The fluid equation reduces to an equation of motion for the free boundary ! [Zakharov '76]

Two equations on the top:

(i) Kinematic boundary condition

(ii) Bernoulli law = integrated Euler

$$\phi_t + \frac{1}{2}|\nabla \phi|^2 + gy + p = 0 \qquad \text{in } \Omega_t$$

restricted to the top, where $p = -2\sigma \mathbf{H}$.

Dictionary:

- *gravity waves*: $g > 0$, $\sigma = 0$.
- *capillary waves*: $g = 0$, $\sigma > 0$.

Gravity waves

Capillary waves

Gravity/capillary waves

Question 1: Local behavior of water waves

Question 2: Long time behavior of water waves

Scattering: waves propagating from a source

Periodic traveling waves

Solitary wave (soliton)

Choices of coordinates

Choice of coordinates = gauge freedom

Eulerian coordinates (x, t): Particles are moving in a fixed frame. Flat geometry.

Lagrangian coordinates (X, t): Frame moves along particle trajectories. Curved geometry.

$$(\partial_t + \nabla \cdot v)X = 0$$

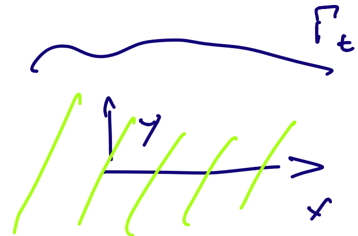
Holomorphic coordinates (α, t): (2-d only) Both particles and frame move. Conformally flat geometry.

Arclength coordinates (s, t): (2-d only) Both particles and frame move, flat top geometry

Water waves in Eulerian coordinates

Velocity potential

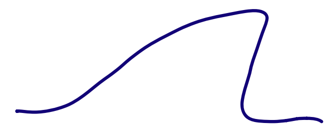
$$v = \nabla \phi, \qquad \Delta \phi = 0 \quad \text{in } \Omega_t$$



Bernoulli law = integrated Euler equations

$$\phi_t + \frac{1}{2}|\nabla \phi|^2 + g y + p = 0 \qquad \text{in } \Omega_t$$

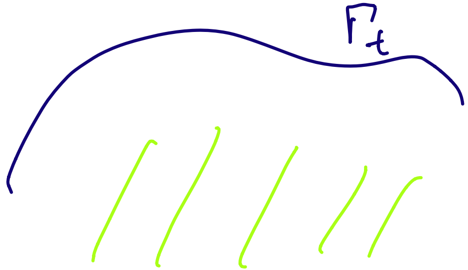
Equations reduced to the boundary in Eulerian formulation.
Variables: η = elevation, $\Gamma_t = \{y = \eta(x)\}$, $\psi = \phi_{|\Gamma_t}$.



$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta - \sigma \mathbf{H}(\eta) + \dfrac{1}{2}|\nabla \psi|^2 - \dfrac{1}{2}\dfrac{(\nabla \eta \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases}$$

$$\mathbf{H}(\eta) = \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right), \qquad G(\eta) = \text{ Dirichlet to Neuman operator}$$

The Dirichlet to Neuman operator



Dirichlet problem:

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_t \\ \phi = \psi & \text{in } \Gamma_t \end{cases}$$

D-N map:

$$\psi = \phi_{|\Gamma_t} \qquad \longrightarrow \qquad G(\eta)\psi = \frac{1}{\sqrt{1+|\nabla\eta|^2}} \frac{\partial\phi}{\partial\nu}_{|\Gamma_t}$$

(Dirichlet) (Neuman)

- Elliptic pseudodifferential operator of order 1 in ψ.
- Also depends on the free surface, i.e. on η !

Hamiltonian structure (Zakharov)

$\frac{1}{2}\int |\nabla \phi|^2 \, dx\, dy$

Conserved energy (Hamiltonian):

$$H(\eta, \psi) = \int_{\mathbb{R}^d} \frac{1}{2} g \eta^2 + \sigma\left(\sqrt{1 + |\nabla \eta|^2} - 1\right) + \frac{1}{2} \psi \cdot G(\eta)\psi \; dx$$

Kinetic energy

$$\begin{cases} \eta_t = \dfrac{\delta H}{\delta \psi} \\ \psi_t = -\dfrac{\delta H}{\delta \eta} \end{cases}$$

$\omega = \int d\eta \wedge d\psi$

Horizontal momentum (Noether law - invariance to translations):

$$M_j = \int_{\mathbb{R}^d} \eta \, \partial_j \psi \; dx$$

Symmetries

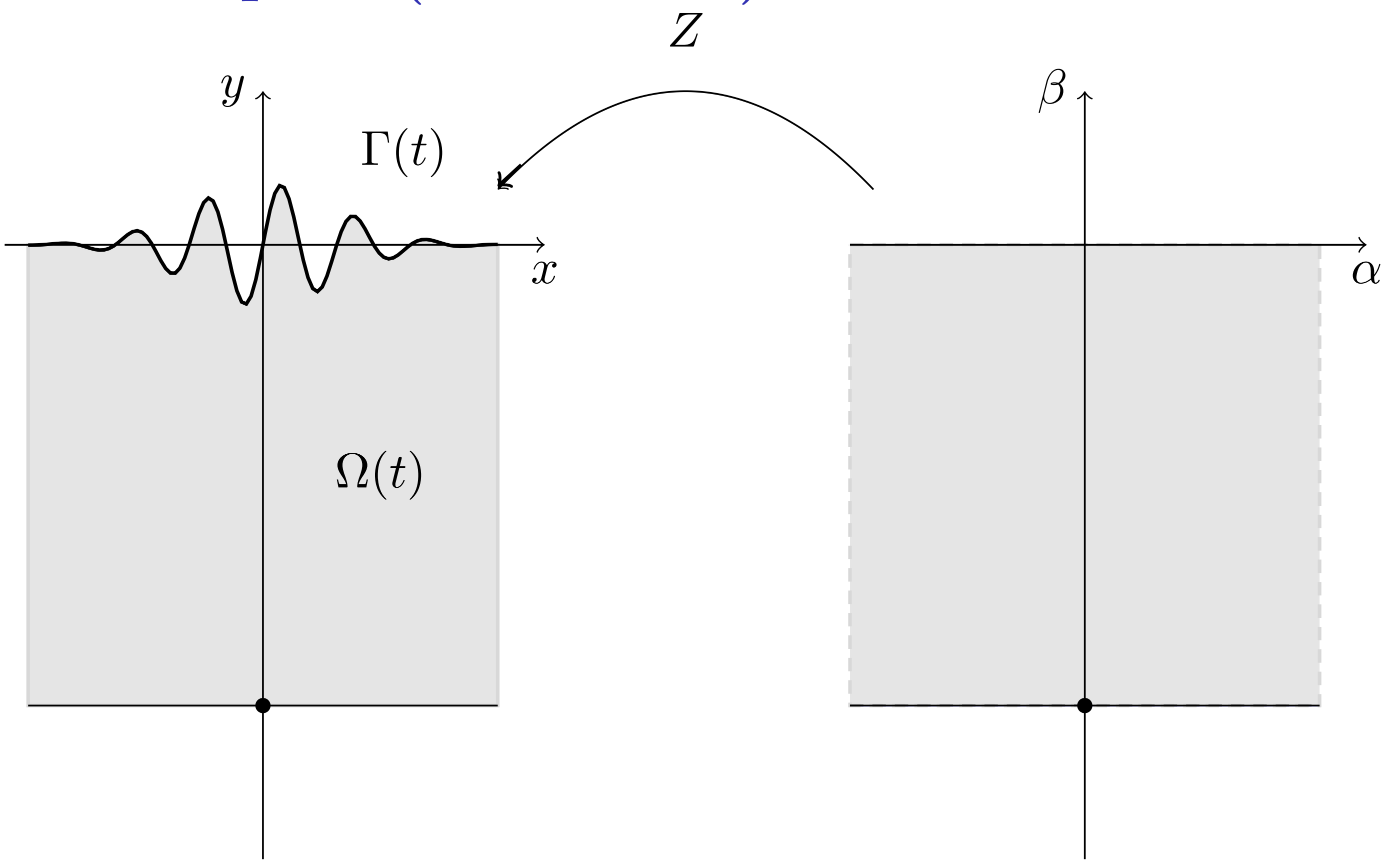
- Translations in α and t.
- Galilean invariance
- Scaling
  - gravity waves in deep water:

$$(\eta(t,x),\psi(t,x)) \to (\lambda^{-2}\eta(\lambda t,\lambda^2 x),\lambda^{-3}\psi(\lambda t,\lambda^2 x))$$

  - capillary waves in deep water:

$$(\eta(t,x),\psi(t,x)) \to (\lambda^{-2}\eta(\lambda^3 t,\lambda^2 x),\lambda^{-3}\psi(\lambda^3 t,\lambda^2 x))$$

Holomorphic (conformal) coordinates



The conformal map

Holomorphic (conformal) coordinates

Holomorphic coordinates:

$$Z : \{\Im z \leq 0\} \to \Omega_t, \qquad \alpha + i\beta \to Z(\alpha + i\beta)$$

Boundary condition at infinity:

$$Z(\alpha) - \alpha \to 0 \ \ (nonperiodic) \qquad Z(\alpha) - \alpha \ periodic \ \ (periodic)$$

Free boundary parametrization:

$$Z : \mathbb{R} \to \Omega_t, \qquad \alpha \to Z(\alpha)$$

(W, Q)

Perturbation of steady state:

$$W = Z - \alpha$$

Holomorphic velocity potential ($v = \nabla \phi$, $q =$ stream function):

$v = \nabla^{\perp} q$ $\quad$ $q = ct.$

$$Q = \phi + iq$$

Holomorphic variables: (W, Q). → functions with negative freq. in deep water.

Water waves in holomorphic coordinates

[Zakharov & al '96, Wu '96, Hunter-Ifrim-T '14]

- P - Projection onto negative wavenumbers

Fully nonlinear equations for *holomorphic* variables ($W = Z - \alpha, Q$):

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha + P\big[|R|^2\big] - igW + i\sigma P\left[\dfrac{W_{\alpha\alpha}}{J^{1/2}(1 + W_\alpha)} - \dfrac{\bar{W}_{\alpha\alpha}}{J^{1/2}(1 + \bar{W}_\alpha)}\right] = 0. \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2, \qquad R = \frac{Q_\alpha}{1 + W_\alpha}.$$

$= v_x - i v_y$

Conserved energy (Hamiltonian):

$$E(W, Q) = \int \Im(Q\bar{Q}_\alpha) + \frac{1}{2}g\left(|W|^2 - \Re(\bar{W}^2 W_\alpha)\right) + \frac{1}{4}\sigma(J^{\frac{1}{2}} - 1 - \Re W_\alpha)\, d\alpha$$

Set-up for finite depth

h = depth

[Harrop-Griffith -Ifrim -T.'16]

$$H \to \mathcal{T}_h, \qquad \mathcal{T}_h = -i\tanh(hD), \quad \text{Tilbert transform}$$

Holomorphic functions:

$$\Im W = \mathcal{T}_h \Re W,$$

Anti-holomorphic:

$$\Im W = -\mathcal{T}_h \Re W,$$

Orthogonal w.r. to

$$\langle W_1, W_2 \rangle = \langle \mathcal{T}_h \Re W_1, \mathcal{T}_h \Re W_2 \rangle_{L^2} + \langle \Im W_1, \Im W_2 \rangle_{L^2}$$

P = orthogonal projection onto holomorphic functions

The differentiated equation

→ Self-contained equation for differentiated variables (W_α, Q_α).

→ Self-contained equation for *good variables* $(\mathbf{W} = W_\alpha, R = \dfrac{Q_\alpha}{1+W_\alpha})$:

$$\begin{cases} \mathbf{W}_t + b\mathbf{W}_\alpha + \dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}} = (1+\mathbf{W})M, \\ R_t + bR_\alpha - i(g+a)\left(\dfrac{\mathbf{W}}{1+\mathbf{W}}\right) = ia, \end{cases}$$

where

$$b = \Re F = 2\Re(R - P(R\bar{Y})), \qquad a = 2\Re P(R\bar{R}_\alpha).$$

$1 - \frac{1}{z_\alpha} =$

$$Y := \frac{\mathbf{W}}{1+\mathbf{W}}, \qquad M = 2\Re P[R\bar{Y}_\alpha - \bar{R}_\alpha Y]$$

Purely cubic equation in (Y, R):

$$\begin{cases} Y_t + bY_\alpha + |1-Y|^2 R_\alpha = (1-Y)M, \\ R_t + bR_\alpha - i(g+a)Y = -ia, \end{cases}$$

Linearization around 0

In deep water, as a system:

$$\begin{cases} W_t + Q_\alpha = 0 \\ Q_t - igW + i\sigma\partial_\alpha^2 W = 0, \end{cases}$$

or as a second order equation:

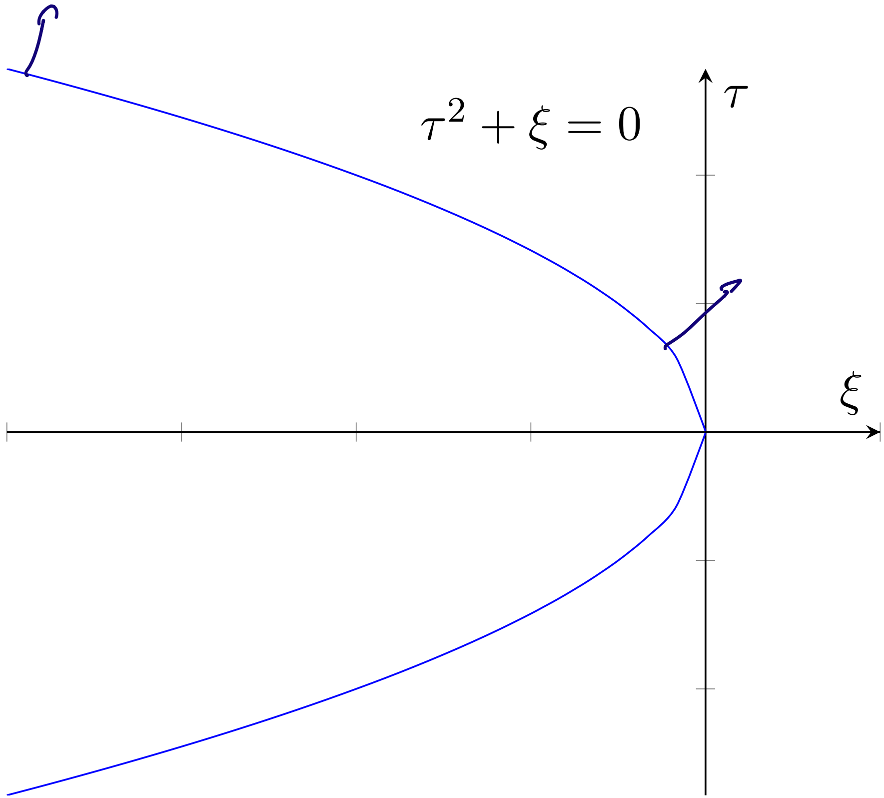
$$W_{tt} = -ig\partial_\alpha W + i\sigma\partial_\alpha^3 W$$

In shallow water, as a system:

$$\begin{cases} W_t + Q_\alpha = 0 \\ Q_t - \mathcal{T}_h(gW - \sigma\partial_\alpha^2 W) = 0, \end{cases}$$

or as a second order equation:

$$W_{tt} = -\mathcal{T}_h(g\partial_\alpha W - \sigma\partial_\alpha^3 W)$$



Dispersion relation for gravity waves in deep water

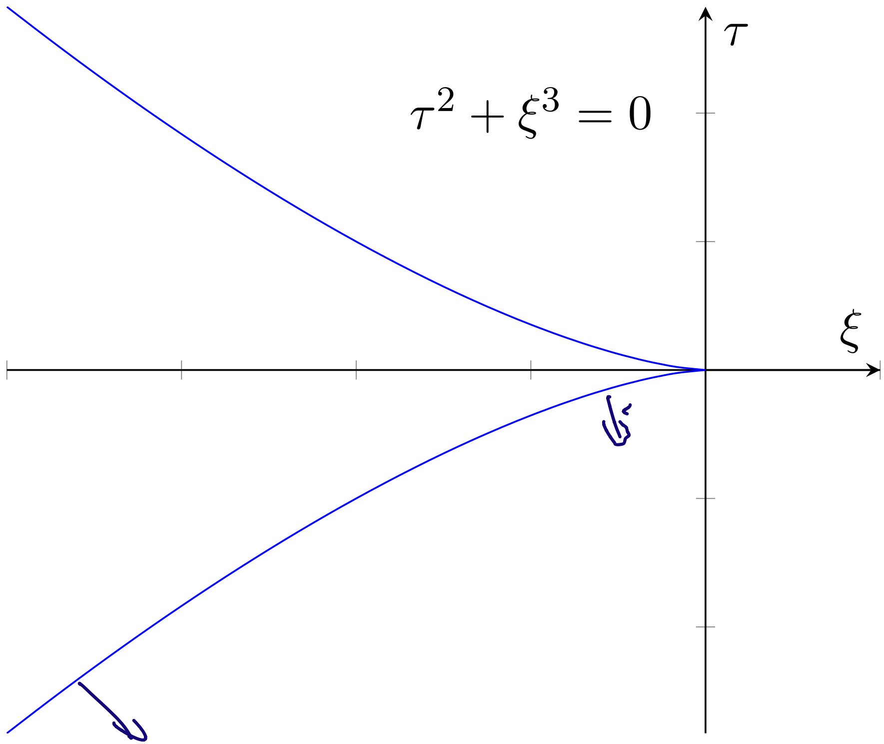


Figure: Dispersion relation for capillary waves in deep water

Standard questions:

1. Obtain local well-posedness in Sobolev spaces
   - high $\to$ low regularity (via energy estimates)
   - even lower regularity (using also dispersion)
2. Understand asymptotic equations in various regimes
   - low frequency asymptotics
   - wave packet asymptotics

3. Study long time solutions (i.e. the stability of the trivial steady state) in two settings:

   - lifespan bounds for small data
   - global solutions for small localized data if no solitons exist
   - Soliton resolution for small localized data if solitons exist

4. Understand solitons and near soliton dynamics
   - (non) existence of solitons
   - stability and asymptotic stability

Thank you !