Introduction to Water Waves
Lecture 2

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Low regularity well-posedness for water waves

Main issues/ features:

- fully nonlinear system $\rightarrow$ differentiate/ linearize/ paralinearize
- non-diagonal system $\rightarrow$ use Alinhac style *good variables*
- dispersive flow $\rightarrow$ use dispersive decay/ Strichartz estimates
- gauge independence $\rightarrow$ carefully choose coordinates
- complex (non)-resonant structure $\rightarrow$ use normal form methods

Well-posedness for nonlinear equations

Equation: $u_t = F(u)$ Linearization: $v_t = DF(u)v$

Para-diff: $u_t = T_{DF(u)}u + N(u)$ Linearized: $v_t = T_{DF(u)}v + N_{lin}(u)v$

main eq. ← $u_{kt} = DF(u_{<k})\, u_k + N_k(u)$ → perturbative

Well-posedness à la Hadamard+:

- Existence of regular solutions
  - Regularization/iteration scheme
- Uniqueness of regular solutions
  - Estimates for differences in a weaker topology
- Rough solutions as unique limits of smooth solutions
  - Lipschitz bounds for linearized equation in a weaker topology
  - Uniform propagation of higher regularity
- Continuous dependence on initial data
  - Lipschitz bounds for linearized equation in a weaker topology
  - Frequency envelopes

Low regularity well-posedness: a quick guide

Following [T., Bahouri-Chemin '98-00, nonlinear wave eqn.]

Step 1. Energy estimates:

$$\frac{d}{dt}E^s(u) \lesssim \|D^\sigma u\|_{L^\infty} E^s(u), \qquad E^s(u) \approx \|u\|_{H^s}^2$$

- Similar bounds for the linearized equation in H^{s_0} for a fixed s_0.
- Gives well-posedness in H^s if $H^s \subset C^\sigma$.

Step 2. Strichartz estimates:

$$\|D^\sigma u\|_{L^p L^\infty} \lesssim \|u\|_{H^s}$$

- Frequency localized, paradifferential
- Also for the linearized equation
- parametrices, dispersion on semiclassical time scales

Water waves: Alinhac's "good variable"

Idea: diagonalize the principal (transport) part of the equation.
Good variables for differentiated equation (Hunter-Ifrim-T. '14):

$$\left(\mathbf{W} = W_\alpha, R = \frac{Q_\alpha}{1 + W_\alpha} \right).$$

Differentiated equation [with omitted projections]:

$$\begin{cases} (\partial_t + b\partial_\alpha)\mathbf{W} + \dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}} R_\alpha = G(\mathbf{W}, R) \\ (\partial_t + b\partial_\alpha)R - i\dfrac{(g+a)\mathbf{W}}{1+\mathbf{W}} = K(\mathbf{W}, R) \end{cases}$$

$g+a \approx \frac{\partial p}{\partial n} > 0$

Taylor sign

where

$$b = 2\Re P\left[\frac{R}{1+\mathbf{W}}\right], \qquad a = 2\Im P[R\bar{R}_\alpha]$$

Taylor coefficient: $a \geq 0$, necessary for well-posedness.
[Wu,H-I-T] (deep water) + [Lannes, HG-I-T](shallow water)

Note: Good variable in Eulerian setting: Alazard-Burq-Zuily '11

Water waves: paradifferential equation

Slightly oversimplified:

$$\begin{cases} (\partial_t + T_b\partial_\alpha)w + r_\alpha = 0 \\ (\partial_t + T_b\partial_\alpha)r - iT_{g+a}w + i\sigma T_{J^{-\frac{1}{2}}}\partial^2 w = 0 \end{cases}$$

Scalar version:

$$(\partial_t + T_b\partial_\alpha)u + i((g+a)|D| + \sigma|D|^3)^{\frac{1}{2}}u = 0$$

more real coeff.

Energy functional

$$E(w,r) = \int (g+a)|w|^2 + \sigma J^{-1}|w_\alpha|^2 + \Im(r\bar{r}_\alpha)\,d\alpha$$
$$\approx g\|w\|_{L^2}^2 + \sigma\|w_\alpha\|_{L^2}^2 + \|r\|_{\dot{H}^{\frac{1}{2}}}^2$$

Low regularity local well-posedness: 2-d

Theorem

a) Gravity waves are locally well-posed for $(\mathbf{W}_0, R_0) \in H^{1+\sigma} \times H^{\frac{3}{2}+\sigma}$.

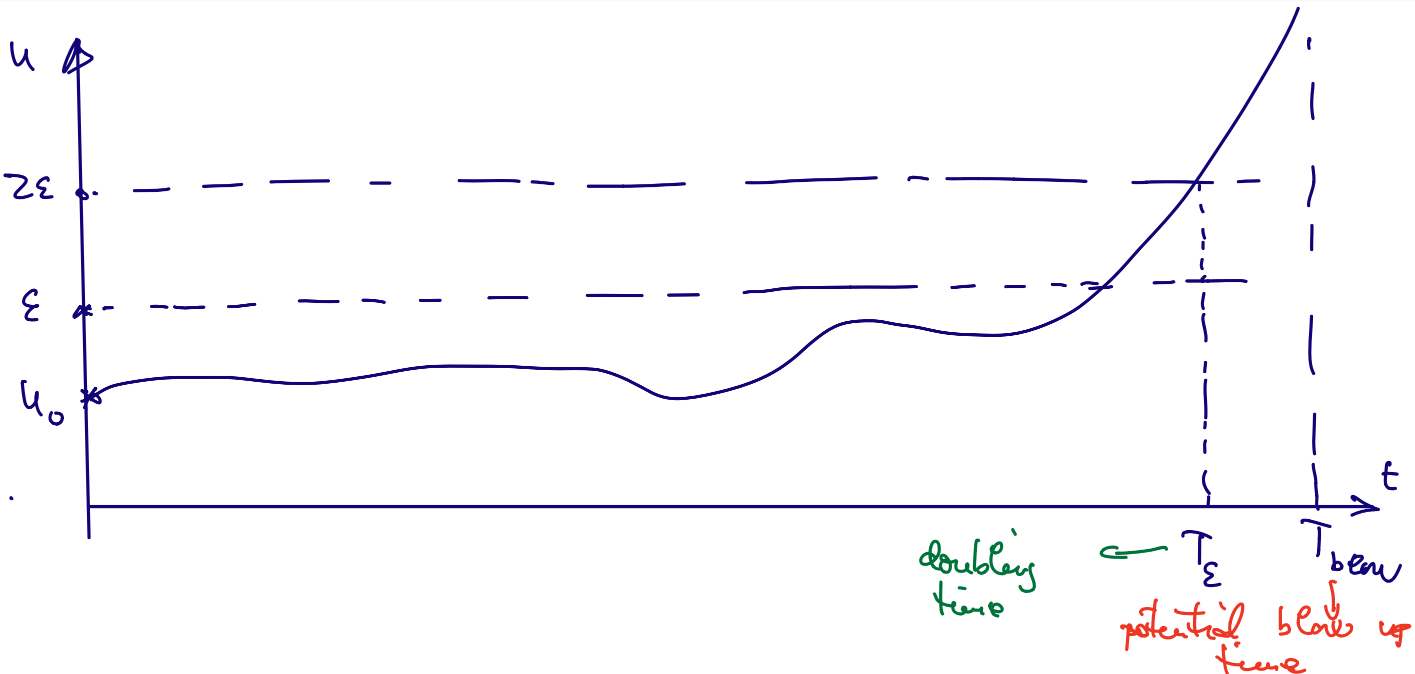
b) Capillary waves are locally well-posed for $(\mathbf{W}_0, R_0) \in H^{2+\sigma} \times H^{\frac{3}{2}+\sigma}$.

2-d gravity waves:

σ	result	method	year
$-1/2$	scaling		never
4	Wu	energy estimates	'99
ϵ	Alazard-Burq-Zuily	energy estimates (EE)	'11
0	Hunter-Ifrim-T.	cubic energy estimates	'14
$-1/24$	Alazard-Burq-Zuily	EE+Strichartz	'15
$-1/12$	Ai	EE +Strichartz	'17
$-1/8$	Ai	EE +lossless Strichartz	'18
$-1/4$	Ai-Ifrim-T.	balanced energy estimates	'19
$-3/8$	Ai-Ifrim-T.	balanced EE + Strichartz	ongoing

Long time solutions

Question: Given initial data of size $\epsilon \ll 1$, find optimal bound T_ϵ on lifespan of solutions.



Rules of the game: Balance dispersive decay with growth caused by nonlinear interactions.

Dispersive equations in 1-d

Model linear problem:

$$iu_t = A(D_x)u, \qquad u(0) = u_0$$

Dispersion relation:

$$\xi \to a(\xi)$$

Characteristic set:

$$C = \{\tau + a(\xi) = 0\}$$

Grup velocity:

$$v_\xi = \partial_\xi a(\xi)$$

Linear scattering (if $a_{\xi\xi} \neq 0$)

$$u(t,x) \approx U(v)\frac{1}{\sqrt{t}}\, e^{it\phi(v)}, \qquad v = \frac{x}{t}$$

where ϕ solves an eikonal equation.
Strichartz estimates:

$$\|u\|_{L^4 L^\infty} \lesssim \|u_0\|_{L^2}$$

Bilinear interactions in dispersive flows

Model nonlinear linear problem:

$$iu_t = A(D_x)u + Q(u,u), \qquad u(0) = u_0$$

Characteristic set (a real valued):

$$C = \{\tau + a(\xi) = 0\}$$

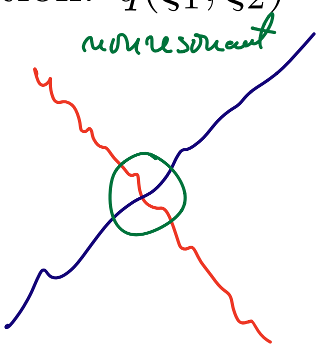
Grup velocity:

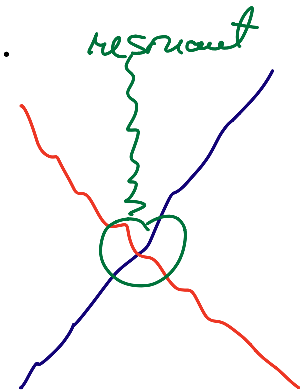
$$v_\xi = \partial_\xi a(\xi)$$

Resonant interactions:

$$(\xi_1, \tau_1) \in C, \quad (\xi_2, \tau_2) \in C \quad \longrightarrow \quad (\xi_1 + \xi_2, \tau_1 + \tau_2) \in C$$

Null condition: $q(\xi_1, \xi_2) = 0$ on resonant set.





Long time existence via energy estimates

Question: Obtain lifespan estimates for small data.

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan

$$T_\epsilon \approx \epsilon^{-1} \qquad \text{(quadratic lifespan)}$$

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan

$$T_\epsilon \approx \epsilon^{-2} \qquad \text{(cubic lifespan)}$$

This analysis neglects dispersion and resonance analysis ! e.g. Burgers

The normal form method (Shatah '85)

Transform an equation with a quadratic nonlinearity

$$iu_t = A(D_x)u + Q(u,u), \qquad u(0) = u_0$$

into one with a cubic one via a normal form transformation,

$$u \to v = u + B(u,u)$$

so that

$$iv_t = A(D_x)v + Q_3(u,u,u), \qquad u(0) = u_0$$

Algebraic computation:

$$b(\xi_1,\xi_2) = \frac{q(\xi_1,\xi_2)}{a(\xi_1) + a(\xi_2) - a(\xi_1+\xi_2)}$$

- works for nonresonant and null resonant interactions, but
- it is unbounded for quasilinear problems
- computations more involved for systems

Normal form methods for quasilinear pde's

1. Modified energy method (Hunter-Ifrim-T. '12-'14)

Issue: incompatible estimates

$$\text{Quasilinear:} \frac{d}{dt} E^Q(u) \lesssim \|u\| E^Q(u)$$

$$\text{Normal form:} \frac{d}{dt} E^{NF}(u) \lesssim \|u\|^2 E^{NF,1}(u)$$

Solution: Modify the energy functionals rather than the unknown,

$$\frac{d}{dt} E^{NL}(u) \lesssim \|u\|^2 E^{NL}(u)$$

where

$$E^{NL}(u) = E^Q(u) + cubic\ l.o.t., \qquad E^{NL}(u) = E^{NF}(u) + quartic$$

- works for quasilinear problems, also for more null interactions
- we provide an algorithm to compute these energies

Normal form methods for quasilinear pde's

2. Normal form flow method:

[Hunter-Ifrim ('12, Burgers-Hilbert), Ifrim (ongoing, WW)]

Replace unbounded NF

$$v = u + B(u, u)$$

with a bounded transformation

$$v = u + B(u, u) + higher$$

constructed via a Hamiltonian flow

$$w_t = B(w, w), \qquad w(0) = u, \quad w(1) = v$$

- provides a nonlinear, symplectic change of coordinates in the phase space
- most elegant, but problem specific
- other non-flow based transformations [Wu, Berti-Feola-Pusateri]

Normal form methods for quasilinear pde's

3. Paradiagonalization (Delort, Alazard-Delort '13) Combines a partial normal form with a paradifferential symmetrization.

Writing the nonlinear flow

$$u_t = F(u)$$

in a paradifferential form

$$u_t = T_{DF(u)} u + N(u)$$

one applies different tools to the terms on the right:

- use an invertible normal form to eliminate quadratic terms in $N(u)$.
- use a microlocal conjugation to (anti)symmetrize the paradifferential term

Thank you !