Vortex filament dynamics

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Plan of the talk

- Vortex filaments dynamics
  - Formal approach
  - Some rigorous recent results
  - Other related models
- Binormal flow model methods of study
  - Dispersive methods
  - Other methods
- Singularites formation for the binormal flow model
  - One singularity (self-similar solutions and perturbations)
  - Several singularities (periodic and non-periodic cases)

A model for one vortex filament dynamics

Vortex filaments appear in 3-D fluids when vorticity is large and concentrated in a thin tube around a curve in $\mathbb{R}^3$.

"Vortex motions are sinews and muscles of fluid motions" (Küchement 64, Saffman-Baker 79, Moffatt-Kida-Ohkitani 94,...).

Considering the vorticity to be a singular measure on a curve in $\mathbb{R}^3$ is too singular for Euler's equations $\rightsquigarrow$ need to smooth the data or the equation.

The binormal flow (or LIA or VFE) is the oldest, simpler and richer formally derived model for one vortex filament dynamics
(Da Rios 1906, Levi-Civita 32, Murakami-Takahashi-Ukita-Fujiwara 37, Arms-Hama 65, Callegari-Ting 78; Jerrard-Seis 17).

Sketch of the formal derivation

Suppose the vorticity $\omega(t)$ is a singular vectorial measure located on a curve $\chi(t)$ in $\mathbb{R}^3$, i.e. $\omega_\chi(t,x) = \Gamma\,\chi_x(t,x)\delta_{\chi(t,x)}$.

For getting the dynamics of a vortex filament we look at the fluid velocity near the filament by using the Biot-Savart law ($u = (-\Delta)^{-1}(\nabla \times \omega)$):

$$u(t,x) = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{(x-\chi(t,s)) \times \chi_s(t,s)}{|x-\chi(t,s)|^3}\,ds.$$

- given t suppose $\chi(t,0) = (0,0,0)$, $\chi_s(t,0) = C(0,0,1)$; to approximate $\chi_t(t,0)$ we look at $u(t,(x_1,x_2,0))$ for $\epsilon = (x_1^2 + x_2^2)^{\frac{1}{2}}$,
- take into account only the local contributions $s \in [-L, L]$,
- perform a Taylor development of $\chi(t,s)$ around $s = 0$:

Sketch of the formal derivation

$$u(t,(x_1,x_2,0)) \approx \frac{\Gamma}{4\pi}\int_{-L}^{L} \frac{((x_1,x_2,0) - s\chi_s(t,0) - \frac{s^2}{2}\chi_{ss}(t,0)) \times (\chi_s(t,0) + s\chi_{ss}(t,0))}{|(x_1,x_2,-Cs)|^3}ds$$

$$= \frac{\Gamma}{4\pi C^3}\frac{(-x_2,x_1,0)}{\epsilon^2}\int_{-LC/\epsilon}^{LC/\epsilon}\frac{d\tilde{s}}{(1+\tilde{s}^2)^{\frac{3}{2}}} + \frac{\Gamma}{4\pi}(x_1,x_2,0)\times\chi_{ss}(t,0)\int_{-L}^{L}\frac{s}{|\epsilon^2+C^2s^2|^{\frac{3}{2}}}ds$$

$$-\frac{\Gamma}{8\pi C^3}\chi_s(t,0)\times\chi_{ss}(t,0)\int_{-LC/\epsilon}^{LC/\epsilon}\frac{\tilde{s}^2}{|1+\tilde{s}^2|^{\frac{3}{2}}}d\tilde{s}.$$

- the first term will be neglected as it is the standard spinning around a still straight vortex filament,
- the second term vanishes by parity,
- the last term diverges logarithmically.

$\rightsquigarrow$ rescaling the time variable, the dynamics of the vortex is governed by

$$\chi_t = \frac{\chi_x \times \chi_{xx}}{|\chi_x|^3}.$$

Some smooth BF dynamics coherent with fluid mechanics

Type	BF	Euler	Nature
Line	$(0,0,x)$	use of 2D solutions	Tornados
Circle	$(\cos x, \sin x, t)$	Fraenkel -Berger 74 Ambrosetti-Struwe 89 Benedetto-Caglioti–Marchioro 00	Smoke rings
Helix	$(\cos(x-\frac{t}{2\sqrt{2}}),$ $\sin(x-\frac{t}{2\sqrt{2}}), x+\frac{t}{2\sqrt{2}})$	Levy-Forsdyke 28 Dávila-Del Pino-Musso-Wei 20	Eden 1911
TW on helices	Hasimoto 72	?	Hopfinger -Broward 81

Jerrard-Seis's result (2017)

$$\left.\begin{array}{c} \chi(t) \text{ smooth closed curve of length } L,\ \forall t \in [0,T] \\ u_\epsilon \text{ conservative weak Euler}_\epsilon \text{ solution },\ \forall \epsilon \in (0, \frac{L}{2}) \\ \frac{1}{2}\int |u_\epsilon(0)|^2 \leq \frac{|\log \epsilon|}{4\pi} L + C \\ \|\omega_\epsilon(t) - \omega_{\chi(t)}\|_F \leq L\epsilon,\ \forall t \in [0,T] \end{array}\right\} \Longrightarrow \chi \text{ solves BF.}$$

(where $\|\mu\|_F = \sup\{\langle \mu, \psi\rangle, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^3), \|\nabla \times \psi\|_{L^\infty} \leq 1\}$ is the flat-norm of vector-valued Radon measures on $\mathbb{R}^3$)

The proof is driven by the Hamilton-Poisson structures of Euler and BF:

concentration of ω_ϵ in $\mathcal{V}_\epsilon(\chi(t))$ and $\frac{4\pi}{|\log \epsilon|} H_E(\omega_\epsilon(t)) - H_B(\chi(t))$ small

$\rightsquigarrow \{\frac{4\pi}{|\log \epsilon|} H_E, \int_{\mathbb{R}^3} \psi \cdot\}(\omega_\epsilon(t)) - \{H_B, \int_\cdot \psi \cdot\}(\chi(t))$ small, $\forall \psi \in \mathcal{C}_c^2(\mathbb{R}^3, \mathbb{R}^3)$

i.e. $\frac{4\pi}{|\log \epsilon|} \int \nabla(\nabla \times \psi) : u_\epsilon \otimes u_\epsilon - \int_\chi \nabla(\nabla \times \psi) : (I - \chi_x \otimes \chi_x)$ small.

At the dynamical level $\partial_t \langle \omega_\epsilon(t), \psi\rangle - \partial_t \langle \omega_{\chi(t)}, \psi\rangle$ small, provided χ solves BF

$\rightsquigarrow$ use BF stability properties from Jerrard-Smets 15.

Some other very recent results

- Navier-Stokes: on small times one vortex filament evolves as Lamb-Oseen vortex around the initial curve $\rightsquigarrow$ large self-similar solutions. The proof uses a heat ansatz in self-similar variables in the plane normal to the curve (Bedrossian-Germain-Harrop-Griffiths 18).

- Euler for particular cases with reductions: helices and leap-frogging in axysymmetry without swirl. The proofs use Ettinger-Titi 09 helical symmetry reduction and resp. local Fermi coordinates, a dynamical ansatz and the gluing-method. (Dávila-Del Pino-Musso-Wei 20).

- Superfluids: one almost-parallel to the $z-$axis quantized vortex filament in Bose condensate in the incompressible limit evolves by a linearized version of BF. The concentration of the Jacobian $\frac{1}{2}\text{curl}(u_\epsilon \wedge \nabla u_\epsilon)$ is tracked by dynamical arguments combined with variational constraints in terms of the modulated Ginzburg-Landau energy and the one of the curve equation (Jerrard-Smets 20).

Models related to the binormal flow

- + axial flow (Moore-Saffman 72, Fukumoto-Miyazaki 88):

$$\chi_t = \chi_x \times \chi_{xx} + \alpha(\chi_{xxx} + \frac{3}{2}\chi_{xx} \times (\chi_x \times \chi_{xx}))$$

 leads to Hirota 73 equation (complex mKdV+cubic NLS):

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^2\psi - i\alpha(\psi_{xxx} + \frac{3}{2}|\psi|^2\psi_x) = 0.$$

- + self-streching (Klein-Majda 91): perturbation of BF by a nonlocal operator, leads to 1-D cubic NLS with a nonlocal term defined by the Fourier multiplier $-\xi^2 \log|\xi| + (\frac{1}{2} - \gamma)\xi^2$, with γ the Euler's constant.

- N vortex filaments nearly parallel (Zakharov 88, Klein-Majda-Damodaran 95): parametrized by $(x_j(t,z), y_j(t,z), z)$, of circulation Γ_j, the evolution of $\Psi_j(t,z) = x_j(t,z) + iy_j(t,z)$ is modeled by the linear Schrödinger+2D point vortex system:

$$i\partial_t\Psi_j + \Gamma_j\partial_z^2\Psi_j + \sum_{k\neq j}\Gamma_k\frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0,\ 1 \leq j \leq N.$$

 Rigorous proof for superfluids (Jerrard-Smets 20).

BF $\rightsquigarrow$ Schrödinger map

Let $\chi(t)$ be a $\mathbb{R}^3$-curve parametrized by arclength x, solution of BF:

$$\chi_t = \chi_x \times \chi_{xx}$$

$$\Big\downarrow \partial_x$$

Its tangent vector $T(t,x) = \partial_x \chi(t,x)$ solves the 1D Schrödinger map to $\mathbb{S}^2$, that is the classical continuous Heisenberg model in ferromagnetism:

$$T_t = T \times T_{xx}$$

Schrödinger map $\rightsquigarrow$ 1D cubic NLS

Let T be a solution of the Schrödinger map. By using Frenet's system:

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix}_x = \begin{pmatrix} 0 & c & 0 \\ -c & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix},$$

$\overset{T_{tx}=T_{xt}}{\rightsquigarrow}$ the filament function $c(t,x)e^{i\int_0^x \tau(t,s)ds}$ solves the focusing 1D cubic NLS (Hasimoto 72 $\approx$ Madelung^{-1}).

In order to avoid issues related to vanishing curvature, use Bishop parallel frames ('75 "There are more than one way to frame a curve"):

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_x = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix},$$

$\rightsquigarrow$ the filament function $\alpha(t,x) + i\beta(t,x)$ solves the 1D cubic NLS.

1D cubic NLS $\rightsquigarrow$ BF: Hasimoto's recipe

Given u solution of $iu_t + u_{xx} + (|u|^2 - A(t))u = 0$, a point $P \in \mathbb{R}^3$, an o.n.b. (v_1, v_2, v_3) of $\mathbb{R}^3$, $(t_0, x_0) \in \mathbb{R}^2$ construct a BF solution as follows:

initial data at (t_0, x_0)	|	$(T, N) = (v_1, v_2 + iv_3)$
evolution at fixed $x = x_0$	|	$T_t = \Im \overline{u_x} N, \quad N_t = -i\psi_x T + i(\|u\|^2 - A(t))N$
evolution at fixed t	|	$T_x = \Re(\overline{u} N), \quad N_x = -uT$

$\downarrow$

$T(t,x)$ solves the Schrödinger map

initial data at (t_0, x_0)	|	$\chi(t_0, x_0) = P$
evolution at fixed $x = x_0$	|	$\chi_t = T \times T_x$
evolution at fixed t	|	$\chi_x = T$

$\downarrow$

$\chi(t,x)$ solves BF.

Describing $\chi(t)$ geometrically might be difficult.

Note that $u(t,x)e^{i\Phi(t)}$ yields $(T(t,x), N(t,x)e^{i\Phi(t)})$, so again $\chi(t,x)$.

Some general existence results

- BF well-posedness for (c, τ) in high Sobolev spaces (Hasimoto 72, Nishiyama-Tani 94-97, Koiso 97-08).
- Uniqueness for the Schrödinger map with $T \in H^2$ or some $T \in H^1$ (Chang-Shatah-Uhlenbeck 00, McGahagan 04, Nahmod-Shatah-Vega-Zeng 06).
- BF well-posedness for curves with one corner and curvature in weighted space (B.-Vega 09-15).

Other methods

- Geometric methods: global existence of weak BF solutions for integral currents in the sense of Federer 69, weak-strong uniqueness (Jerrard-Smets 15).
- Integrable system methods: linear stability properties, construction of solutions, knotted curves analysis (Calini, Lafortune, Ivey 94–, Grinevich-Schmidt 00, Ricca, Barenghi 90–).
- Probabilistic methods: global existence for geometric rough paths in the sense of Lyons 98 (Gubinelli&Co 05-13).

Self-similar solutions and perturbations

Scaling for BF: $\lambda^{-1}\chi(\lambda^2 t,\lambda x) \rightsquigarrow$ self-similar sol. $\chi(t,x)=\sqrt{t}G(\frac{x}{\sqrt{t}})$

- form a family $\{\chi_a\}_{a\in\mathbb{R}^{+*}}$ characterized by $(c_a,\tau_a)(t,x)=\left(\frac{a}{\sqrt{t}},\frac{x}{2t}\right)$ ($\rightsquigarrow$ the filament function is $u_a(t,x)=a\frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$), used for the vortex dynamics in superfluids (Schwarz 85, Buttke 88, Lipniacki 02, Fonda&co 14), in ferromagnetism (Lakshmanan&co 76-81), and in modeling the heart collagen fibers (Peskin&Co 94–),

- $\chi_a(0)$ is a corner of angle θ s.t. $\sin(\frac{\theta}{2})=e^{-\pi\frac{a^2}{2}} \rightsquigarrow$ the formation of a corner for the curve corresponds to a Dirac mass for 1D cubic NLS, at the critical Sobolev and Fourier-Lebesgue regularity (Gutierrez-Rivas-Vega 03, fine analysis of the profile's ODE),

- the formation of a corner and its instantaneously smoothening is stable (B.-Vega 09-15, based on scattering techniques at NLS level in Hasimoto's recipe).

Coherence with the experiments

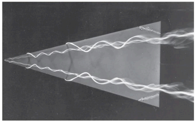


Figure 1. Tourbillons filamentaires dans un fluide rencontrant un obstacle triangulaire de type aile delta, Werlé, ONERA 63.

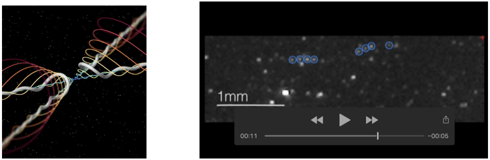


Figure 2. Direct observation of Kelvin waves excited by quantized vortex reconnection, Fonda&Co, PNAS 14

The periodic case of regular polygons

- corners interaction $\rightsquigarrow$ turbulent features appears for noncircular jets, as axis switching (experiments Todoya-Hussain 89, numerics Grinstein-De Vore 96).

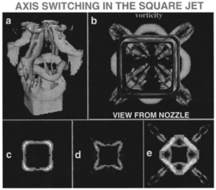


Figure 3. Grinstein-DeVore, Dynamics of coherent structures and transition to turbulence in free square jets, Phys. Fluids 96

- through BF a planar regular polygon with n sides is expected to evolve to skew polygons with nq sides at rational times of type $\frac{p}{q}$ (numerics Jerrard-Smets 15 and integration of the Frenet frame at rational times De la Hoz-Vega 15).
- trajectory of one polygon's corner linked with multifractal Riemann's type functions (numerics De la Hoz-Vega 15,18, De la Hoz-Kumar-Vega 20, see also Peskin-McQueen 94).

The non-periodic case of polygonal lines

Theorem (rough data evolution; singularity continuation) B.-Vega 18

Let $\chi_0(x)$ be an arclength parametrized polygonal line with corners located at $x = k \in \mathbb{Z}$, of angles θ_k s.t. $\sin(\frac{\theta_k}{2}) = e^{-\pi \frac{a_k^2}{2}}$ with $\{a_k\} \in l^{2,3}$. There exists a $\chi \in \mathcal{C}(\mathbb{R}, Lip) \cap \mathcal{C}(\mathbb{R}^*, \mathcal{C}^4)$ unique solution of BF on $\mathbb{R}^*$, solution in the weak sense on $\mathbb{R}$, with

$$|\chi(t,x) - \chi_0(x)| \leq C\sqrt{t}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

- The evolution can have an intermittent behaviour in the sense that at times $t = \frac{p}{q}$ the curvature of $\chi(t)$ displays concentrations near the locations $x \in \frac{1}{q}\mathbb{Z}$, and is almost straight segments between. This is based on a Talbot effect we prove at the NLS level.
- in Hasimoto's recipe at the NLS level we prove existence of solutions for $t > 0$ of type $\sum_{k\in\mathbb{Z}} e^{-i\frac{|\alpha_k|^2}{4\pi}\log\sqrt{t}}(\alpha_k + R_k(t))e^{it\Delta}\delta_k(x)$, with $\{R_k(t)\}$ decaying as t goes to zero.

Theorem (a finite energy framework) B.-Vega 19

Let χ be the BF evolution of a polygonal line from the previous theorem, and T its tangent vector. For $t>0$ we conserve

$$\Xi(t) := \lim_{n\to\infty}\int_n^{n+1} |\widehat{T_x}(t,\xi)|^2 d\xi = 4\pi\sum_k a_k^2.$$

At $t=0$ when singularities are created for BF we get a jump discontinuity of $\Xi(t)$:

$$\forall n, \quad \Xi(0) = \int_n^{n+1} |\widehat{T_x}(0,\xi)|^2 d\xi = 4\sum_k(1-e^{-\pi a_k^2}) \neq 4\pi\sum_k a_k^2 = \Xi(t).$$

- T_x describes the variations of the direction of the vorticity $\rightsquigarrow$ Constantin-Fefferman-Majda's criterium.

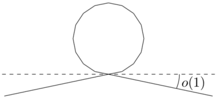
Refined analysis for some families of polygonal lines

Let $n \in \mathbb{N}^*$, $\nu \in]0,1]$, $\Theta > 0$. We consider polygonal lines $\chi_n(0)$ with finite but many corners of located at $j \in \mathbb{Z}$ with $|j| \leq n^\nu$, of same torsion ω_0 and angles θ_n such that

$$\theta_n = \pi - \frac{\Theta}{n} + o(\frac{1}{n}),$$

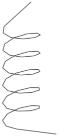
$\chi_n(0,0) = 0_{\mathbb{R}^3}$ and $\chi_n(0,[-1,1])$ symmetric wrt the $YZ-$plane.

$\chi_n(0)$ planar approximation of a line



$\chi_n(0)$ planar approximation of a (multi-)loop

$\chi_n(0)$ non-planar approximation of a line



$\chi_n(0)$ approximation of multi-turns of helices

Theorem (a Frish-Parisi multifractal behaviour) B.-Vega 20

For the previous solutions with torsion $\omega_0 \in \pi\mathbb{Q}$ we have the following description of the trajectory of the corner $\chi_n(t,0)$, uniformly on $(0,T)$:

$$n\,\chi_n(t,0) - (0, \Re(\tilde{\mathfrak{R}}(t)), \Im(\tilde{\mathfrak{R}}(t))) \overset{n\to\infty}{\longrightarrow} 0.$$

The function $\tilde{\mathfrak{R}}$ is multifractal, and its spectrum of singularities $d_{\tilde{\mathfrak{R}}}$ satisfies the multifractal formalism of Frisch-Parisi:

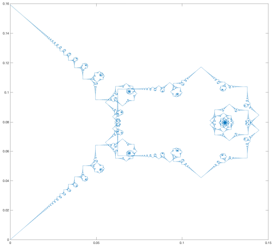
$$d_{\tilde{\mathfrak{R}}}(\beta) := \dim_{\mathcal{H}}\{t, \tilde{\mathfrak{R}} \in \mathcal{C}^{\beta}(t)\} = \inf_p(\beta p - \eta_{\tilde{\mathfrak{R}}}(p) + 1),$$

$$\eta_{\tilde{\mathfrak{R}}}(p) := \sup\{s,\ \tilde{\mathfrak{R}} \in B_p^{\frac{s}{p},\infty}\},$$

a model for predicting the structure function exponents in turbulent flows.

In the torsion-free case $\tilde{\mathfrak{R}}(t) = -\Theta\frac{\mathfrak{R}(4\pi^2 t)}{4\pi^2}$, where $\mathfrak{R}(t) = \sum_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$ is a complex version of Riemann's non-differentiable function.

- Graph on $[0, 2\pi]$ of Riemann's function $\mathfrak{R}(t) = \sum_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$:



- $\mathfrak{R}$ satisfies the multifractal formalism of Frisch-Parisi (Jaffard 96) is intermittent (Boritchev-Eceizabarrena-Da Rocha 19), its graph has no tangents and has Haussdorf dimension $\leq \frac{4}{3}$(Eceizabarrena 19),

- The theorem gives a non-obvious non-linear geometric interpretation for Riemman's function.