Fluctuations of the characteristic polynomial of random Jacobi matrices

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joint work with R. Butez & O. Zeitouni

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Gaussian Ensembles

Let X a GOE or GUE matrix ($\beta = 1, 2$),

$$X \propto e^{-\frac{n\beta}{4}\mathrm{tr}H^2} d\ell_n^{(\beta)}(H),$$

where $\ell_n^{(\beta)}$ is the Lebesgue measure on the space of symmetric ($\beta = 1$) or Hermitian matrices ($\beta = 2$).

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(Wigner) $$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \underset{n\to+\infty}{\Rightarrow} \mu_\sigma,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X and

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How does μ_n fluctuate around μ_σ?

Central limit theorems

The linear statistics associated to $f : \mathbb{R} \to \mathbb{R}$ is the observable

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(Guionnet-Zeitouni): If f is a L-Lipschitz function, for any $t > 0$,

$$\mathbb{P}\Big(\int f d(\mu_n - \mathbb{E}\mu_n) > t \Big) \leq e^{-\frac{Cn^2 t^2}{L^2}}.$$

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For Lipschitz functions,

$$n \int f d(\mu_n - \mathbb{E}\mu_n) \text{ is tight.} \tag{1}$$

In sharp contrast with the i.i.d $(X_i)_i$ case where for $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$, and f bounded,

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(1) is the manifestation of cancellations happening due to the rigidity of the eigenvalues.

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(Johansson, 98'): For a smooth function f,

$$n \int f d(\mu_n - \mathbb{E}\mu_n) \underset{n\to+\infty}{\Rightarrow} \mathcal{N}(0, \sigma(f)^2),$$

Variance structure:

$$\sigma(f)^2 = \frac{1}{4\pi^2} \int_{-2}^{2} \frac{f(t)}{\sqrt{4-t^2}} \Big(\int_{-2}^{2} \frac{f'(s)\sqrt{4-s^2}}{t-s} ds \Big) dt.$$

Expression of the centering:

$$n \int f d\mathbb{E}\mu_n = n \int f d\mu_\sigma + \Big(\frac{2}{\beta} - 1\Big) \int f d\nu_\beta + o(1),$$

with ν_β a signed measure supported on $[-2, 2]$.

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For $z \in (-2, 2)$, $f = \log|z - .|$, what are the fluctuations of

$$n\int f d\mu_n = \log|\det(zI_n - X)|, \ X \text{ GUE matrix?}$$

Variance of the log determinant: heuristics

As a matter of fact

$$\operatorname{Var}(\log|\phi_n(z)|) \asymp \log n.$$

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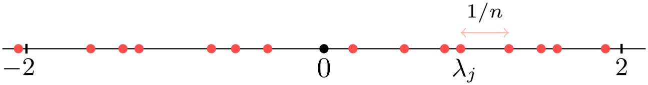
$\log|\phi_n(0)| = \sum_{i=1}^{n} \log|\lambda_i|$



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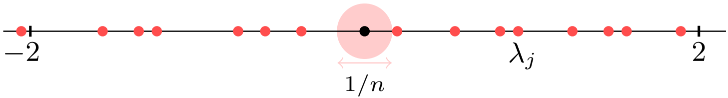
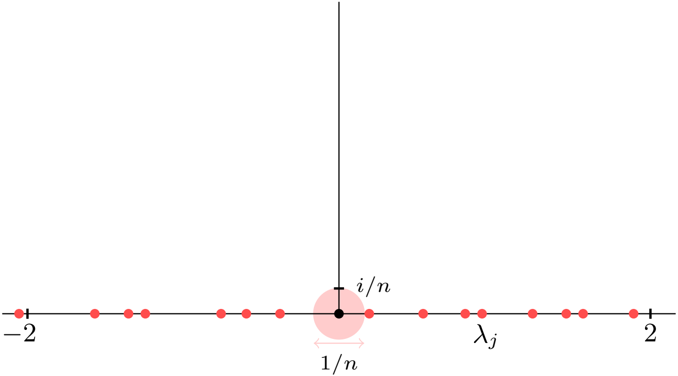


Typically no eigenvalues in an interval of length $1/n$ around 0.

Variance of the log determinant: heuristics

$\log|\phi_n(0)| = \sum_{i=1}^n \log|\lambda_i| = \log|\phi_n(i/n)| + O(1)$ (variance-wise).



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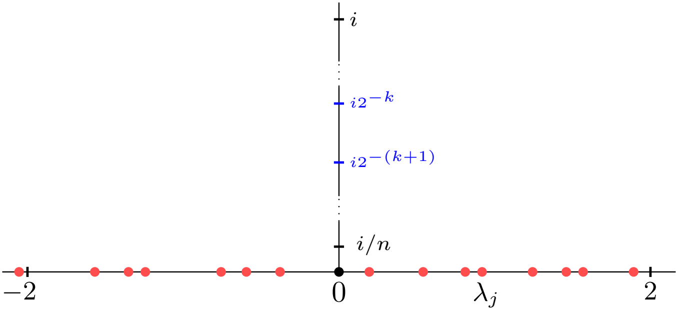
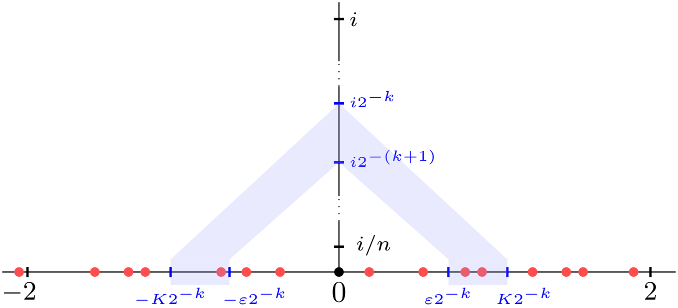
Variance of the log determinant: heuristics

$$\log|\phi_n(i/n)| = \sum_{k=0}^{N} \left(\log|\phi_n(i2^{-(k+1)})| - \log|\phi(i2^{-k})| \right) + \log|\phi_n(i)|.$$



Variance of the log determinant: heuristics

$$\log|\phi_n(i2^{-(k+1)})| - \log|\phi(i2^{-k})| = \int f_k d\mu_n,\ \sigma(f_k)^2 \asymp 1.$$



Variance of the log determinant: heuristics

With $N \asymp \log n$,

$$\log|\phi_n(i/n)| = \log|\phi_n(i)| + \sum_{k=0}^{N-1} \left(\log|\phi_n(i2^{-(k+1)})| - \log|\phi(i2^{-k})| \right)$$

For a fixed k,

$$\text{(Johansson)} \quad \operatorname{Var}\left(\log\left|\frac{\phi_n(i2^{-(k+1)})}{\phi_n(i2^{-k})}\right| \right) \sim \sigma(f_k)^2 \asymp 1.$$

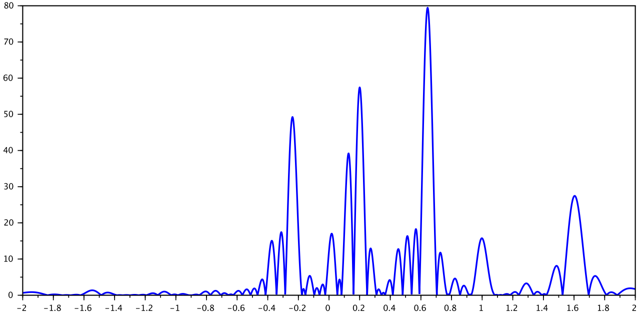
One expect the cross terms to vanish, which gives

$$\operatorname{Var}\Big(\log|\phi_n(i/n)| \Big) \asymp \log n,$$

and thus $\operatorname{Var}(\log|\phi_n(0)|) \asymp \log n$.

The field of the characteristic polynomial

Let X a GUE matrix and $\phi_n(z) = \det(zI_n - X)$.



$$z \in [-2, 2] \mapsto |\phi_n(z)| e^{-\mathbb{E} \log |\phi_n(z)|}, \; n = 60$$

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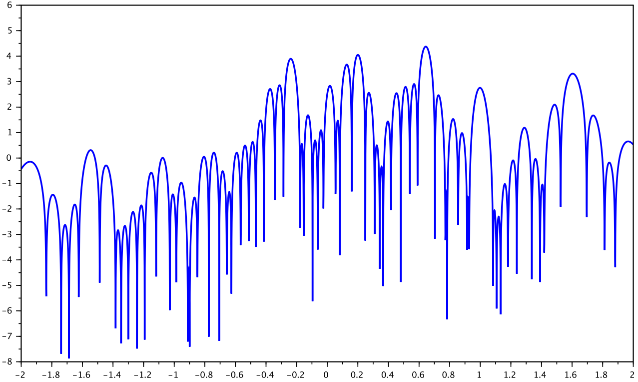

$$z \in [-2, 2] \mapsto \log|\phi_n(z)| - \mathbb{E}\log|\phi_n(z)|,\ n = 60$$

The maximum modulus of the characteristic polynomial

Conjecture (Fydororov- Simm, 15'):

$$\max_{z\in[-2,2]} \big(\log|\phi_n(z)| - \mathbb{E}\log|\phi_n(z)|\big) - \log n + \frac{3}{4}\log\log n \underset{n\to+\infty}{\Rightarrow} \xi,$$

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If $X_1,\dots,X_n$ are i.i.d $\mathcal{N}(0,\frac{1}{2}\log n)$, then w.h.p

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- (Chahabi-Madaule-Najnudel, 18') proved the tightness for the Circular β Ensemble.

Gaussian β-Ensembles

Let $\beta > 0$. The probability measure $\mathbb{P}_{n,\beta}$ on $\mathbb{R}^n$,

$$d\mathbb{P}_{n,\beta} = Z_{n,\beta}^{-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{n\beta}{4}\sum_{i=1}^n \lambda_i^2} \prod_{i=1}^n d\lambda_i.$$

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- GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$).
- random Jacobi matrix with independent parameters $(a_i), (b_j)$

$$\frac{1}{\sqrt{n}} \begin{pmatrix} b_n & a_{n-1} & & \\ a_{n-1} & \ddots & \ddots & \\ & \ddots & \ddots & a_1 \\ & & a_1 & b_1 \end{pmatrix}, \quad \begin{array}{l} b_j \sim \mathcal{N}(0,1) \\ a_i \sim \mathcal{X}_{j\beta}. \end{array}$$

Random Jacobi matrices

$$J_n = \begin{pmatrix} b_n & a_{n-1} & & \\ a_{n-1} & \ddots & \ddots & \\ & \ddots & \ddots & a_1 \\ & & a_1 & b_1 \end{pmatrix}.$$

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Gaussian β-ensembles satisfy these assumptions with $v = 2/\beta$.

Fluctuations of the characteristic polynomial

(A.-Butez-Zeitouni, 20):

Let $z \in (-2,2) \setminus \{0\}$. Define $\phi_n = \log|\det(zI_n - J_n/\sqrt{n})|$.

$$\frac{\log|\phi_n(z)| - \log C_n(z)}{\sqrt{v \log n/2}} \underset{n \to +\infty}{\Rightarrow} \mathcal{N}(0,1).$$

where

$$\log C_n(z) = n \underbrace{\left(\frac{z^2}{4} - \frac{1}{2}\right)}_{\int \log(z-x)d\mu_\sigma(x)} + a_v \log n + O(1).$$

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- The case $z = 0$ exhibits special symmetries: see (Tao,Vu 11').
- (Bourgade-Mody-Pain): Multi-dimensional CLT for β-ensembles with generic potential.
- (Lambert-Paquette): CLT at the edge for Gaussian β-ensembles.

The three-terms recursion

The sequence $p_k = \det(z\sqrt{n} I_k - J_k)$ satisfies the recursion

$$\forall k \in \{1, \dots, n\},\ p_{k+1} = (z\sqrt{n} - b_{k+1})p_k - \underbrace{a_k^2}_{\simeq k} p_{k-1},$$

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Choose the normalization $\psi_k = p_k/\sqrt{k!}$. We get the new recursion:

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Equivalently, if $X_k = (\psi_k, \psi_{k-1})$,

$$X_{k+1} = T_k X_k, \quad T_k = \begin{pmatrix} z\sqrt{\frac{n}{k}} - \frac{b_k}{\sqrt{k}} & -\frac{a_{k-1}^2}{k} \\ 1 & 0 \end{pmatrix}.$$

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The dynamics will depend on the spectrum of the matrices T_k.

Spectrum of the transfer matrix

$$T_k = \begin{pmatrix} z\sqrt{\frac{n}{k}} - \frac{b_k}{\sqrt{k}} & -\frac{a_{k-1}^2}{k} \\ 1 & 0 \end{pmatrix}, \ \mathbb{E}T_k = \begin{pmatrix} z\sqrt{\frac{n}{k}} & -1 \\ 1 & 0 \end{pmatrix},$$

Note that $\det(\mathbb{E}T_k) = 1$.

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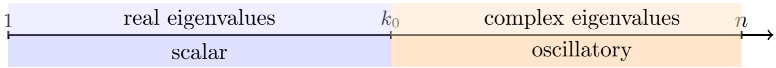

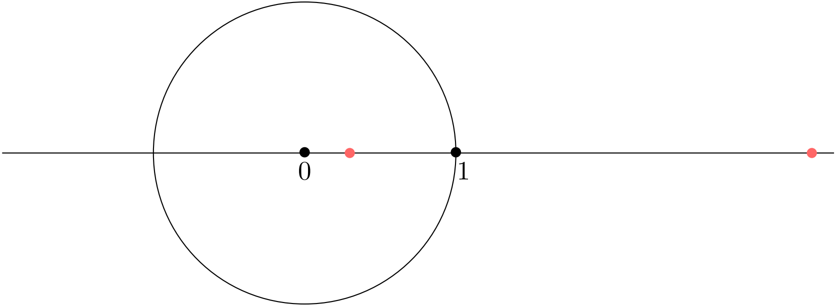

Spectrum of the transfer matrix

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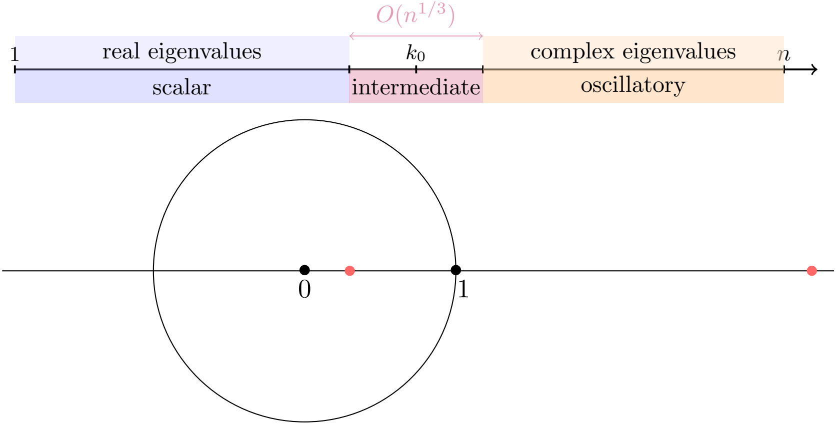

Scalar regime

For $k \leq k_0$, the recursion is

$$\forall k \in \{1, \ldots, n\}, \; \psi_{k+1} = \left(z\sqrt{\frac{n}{k}} - \frac{b_{k+1}}{\sqrt{k}} \right) \psi_k - \frac{a_k^2}{k} \psi_{k-1}.$$

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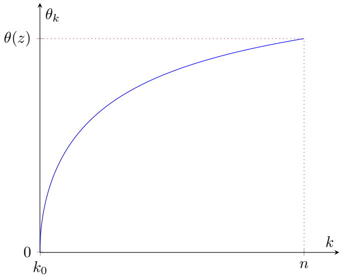
$$\begin{aligned} Y_{k+1} &= (R_{\theta_{k+1}} + W_{k+1}) Q_{k+1} Q_k^{-1} Y_k \\ &\simeq (R_{\theta_{k+1}} + \underbrace{W_{k+1}}_{\text{noise}} + \underbrace{\Delta_{k+1}}_{\text{drift}}) Y_k. \end{aligned}$$

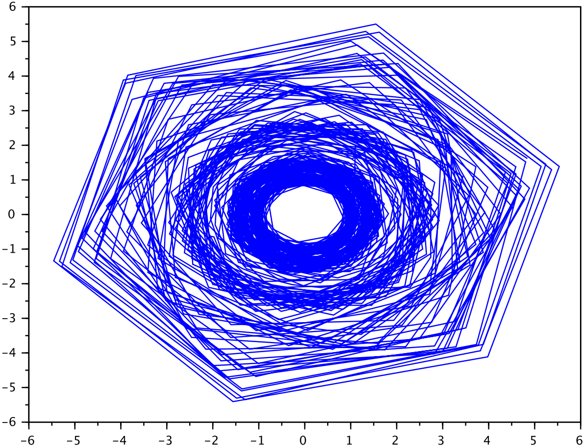


$$\mathbb{E}\|W_k\|^2 \asymp \frac{1}{k - k_0}$$

$$\|\Delta_k\| \asymp \frac{1}{k - k_0}$$

R_{θ_k} faster and faster

Oscillatory regime



A trajectory of Y_k, $k_0 \leq k \leq n$, $n = 2000$

Analysis of the recursion on a section

Output from the scalar regime: Initial condition of the oscillatory regime Y_{k_0} is close to the direction $(1, 0)$.

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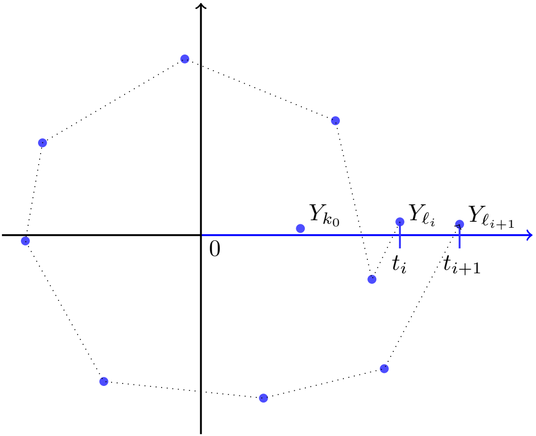
We define "return times" of the recursion in the direction $(1,0)$

$$k_0 \leq \ell_1 \leq \ldots \leq \ell_m.$$



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This allows us to represent $\log \|Y_n\|$ as a drifted martingale, and the CLT follows.

Fluctuations of the characteristic polynomial

(A.-Butez-Zeitouni, 20):

Let $z \in (-2,2) \setminus \{0\}$. Define $\phi_n = \log|\det(zI_n - J_n/\sqrt{n})|$.

$$\frac{\log|\phi_n(z)| - \log C_n(z)}{\sqrt{v \log n/2}} \underset{n \to +\infty}{\Rightarrow} \mathcal{N}(0,1).$$

where

$$\log C_n(z) = n \underbrace{\left(\frac{z^2}{4} - \frac{1}{2}\right)}_{\int \log(z-x) d\mu_\sigma(x)} + a_v \log n + O(1).$$

- The case $z = 0$ exhibits special symmetries: see (Tao,Vu 11').
- (Bourgade-Mody-Pain): Multi-dimensional CLT for β-ensembles with generic potential.
- (Lambert-Paquette): CLT at the edge for Gaussian β-ensembles.

Further questions

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- Multi-dimensional CLT (work in progress)

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- Can one strengthen the CLT to get a control on exponential moments? $\rightarrow$ key to obtaining estimate on the maximum

$$\max_{z \in [-2,2]} \left(\log |\phi_n(z)| - \mathbb{E} \log |\phi_n(z)| \right).$$

Thank you !