Topics on log and Coulomb gases

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Coulomb kernel

$$g(x) = \begin{cases} -|x| & d = 1 \\ -\log|x| & d = 2 \\ \frac{1}{|x|} & d = 3 \\ \frac{1}{|x|^{d-2}} & d \geq 3. \end{cases}$$

Fundamental solution of Laplacian

$$-\Delta g = c_d \delta_0 \quad \text{(in the sense of distributions)}$$

→ solution g = Coulomb kernel → solve Poisson's equation.

Also consider $g = -\log|x|$ for $d = 1$, **log gas**

One-component Coulomb gas / plasma

- $\mathrm{d} \geq 1$, $N \geq 1$
- $\mathrm{X}_N := (x_1, \ldots, x_N)$ positions of point particles in $\mathbb{R}^{\mathrm{d}}$ with same charge $+1$.
- V **confining potential**, smooth and large at ∞
- Total energy of the system in state X_N

$$\mathrm{H}_N(\mathrm{X}_N) := \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \mathrm{g}(x_i - x_j) + \sum_{i=1}^{N} N \cdot V(x_i).$$

- (Canonical) **Gibbs measure**

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\beta \mathrm{H}_N(\mathrm{X}_N)\right) dx_1 \ldots x_N$$

$Z_{N,\beta}$= **partition function**

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$Z_{N,\beta}$= **partition function**

Motivations / history

- In RMT
  - Ginibre ensemble: random $N \times N$ with complex iid Gaussian entries. Law of eigenvalues is

$$\propto \exp\left(\sum_{1 \le i \ne j \le N} \log(x_i - x_j) + N \sum_{i=1}^{N} |x_i|^2 \right)$$

    $=$ a 2D Coulomb gas at $\beta = 2$ (Dyson, Mehta, Wigner)
  - GOE and GUE: law of eigenvalues is a 1D log gas with $V(x) = |x|^2$, $\beta = 1, 2$.
  - RMT model for 1D log gas / β-ensemble **for all** β Dumitriu-Edelman.
- in quantum mechanics: fractional Hall effet via the "plasma analogy" Laughlin $\leftrightarrow$ 2D log gas
- other 1D quantum mechanics models, self-avoiding paths in probability, see [Forrester '10] $\leftrightarrow$ 1D log gas

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- in statistical physics: plasmas, astrophysics $\leftrightarrow$ $d \geq 2$ classical Coulomb gas
  [Lieb-Lebowitz '72,Lieb-Narnhofer '75, Penrose-Smith '72, Sari-Merlini '76, Kiessling-Spohn '99, Alastuey-Jancovici '81, Jancovici-Lebowitz-Manificat' 93...]
- $d = 2$ logarithmic, "two-component plasma": particles of $\pm$ charges $\rightsquigarrow$ **theoretical physics** models (XY, sine-Gordon, Kosterlitz-Thouless)
  [Gunson-Panta '77, Frohlich-Spencer '81, Leblé-S-Zeitouni '17]

Two technical challenges:

1. Singularity at the origin, and particles living in the continuum.
2. **Long-range interaction**.

$$\int_0^{+\infty} g(r) r^{d-1} dr = +\infty.$$

   2.1 $\rightarrow$ The effect of one particle at 0 is felt everywhere in the system.
   2.2 $\rightarrow$ Interaction energy is **not spatially additive** (even up to a small error).

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Global behavior

[Recall $\mathrm{H}_N = \frac{1}{2}\sum_{i\neq j} \mathrm{g}(x_i - x_j) + N\sum_i V(x_i)$]

Limit of empirical measure

$$\hat{\mu_N} := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} ?$$

μ_V = **Frostman equilibrium measure** is the unique minimizer among probabilities of

$$\mathcal{E}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{g}(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x).$$

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Equilibrium measure

Euler-Lagrange equations associated to the minimization problem show that:

$$\mu_V = \left(\frac{1}{4\pi} \Delta V \right) \mathbb{1}_\Sigma .$$

- Finding Σ is challenging.
- If $V(x) = |x|^2$, Coulomb case, then

$$\mu_V = \frac{1}{c_d} \mathbb{1}_{B_1} \ (\textbf{circle law})$$

- $d = 1$, $\mathrm{g} = -\log |x|$, $V(x) = x^2$ then

$$\mu_V(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\, \mathbb{1}_{|x|<2} \ (\textbf{semicircle law})$$

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Comments

- The convergence $\widehat{\mu}_N \to \mu_V$ holds at speed βN^2, in the sense of a **Large Deviations Principle**: [Petz-Hiai '98, Ben Arous-Guionnet '97, Ben Arous -Zeitouni '98...]

$$\mathbb{P}_{N,\beta}\left(\widehat{\mu}_N \in B(\mu,\epsilon)\right) \simeq \exp\left(-\beta N^2(\mathcal{E}(\mu) - \mathcal{E}(\mu_V))\right),$$

- The support and the density depend strongly on V, but not on β!
- Could take β small (high temperature) as long as $N\beta \to +\infty$.
- **Global scale**: system of N particles in Σ compact, scalelength ~ 1.
- **Local/micro scale**: finite number of particles, scale $\sim N^{-1/d}$.
- Mesoscopic scales: between $N^{-1/d}$ and 1.

Questions

We know $\widehat{\mu}_N \to \mu_V$ at speed βN^2. What's next?

Fluctuations

For $\varphi : \mathbb{R}^d \to \mathbb{R}$ test function:

- Measure the size of $\widehat{\mu}_N - \mu_V$ in a dual sense.

$$\text{size of} \int \varphi(x)\,(d\widehat{\mu}_N(x) - d\mu_V(x)) \text{ ?}$$

- What if φ is **smooth** and lives at some mesoscopic scale?
- What if φ is the **indicator function** of a mesoscopic domain?

Local arrangement of points

Pick $\bar{x}$ inside Σ and zoom in by a factor $N^{1/\mathrm{d}}$ around $\bar{x}$?.

- What do we see? At the limit $N \to \infty$ a point process?
- Does it depend on β?
- How much does it depend on μ_V (**universality**)?
- Can we characterize the local arrangement in a variational way?
- Is there a phase-transition as β changes?
- Describe the $\beta \to 0$ and $\beta \to \infty$ limits?

Free energy expansions

Asymptotics of **free energy** $-\frac{1}{\beta}\log Z_{N,\beta}$ as $N \to \infty$?
Easy:

$$-\frac{1}{\beta}\log Z_{N,\beta} \sim N^2\mathcal{E}(\mu_V) + o(N^2)$$

Next order terms?

Link with fluctuations: Laplace transform of linear statistics

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp(tN\sum_{i=1}^{N}\varphi(x_i))\right]$$

$$= \frac{1}{Z_{N,\beta}}\int \exp\left(-\beta\sum_{i\neq j} g(x_i - x_j) + N\sum_{i=1}^{N} V(x_i) + tN\sum_{i=1}^{N}\varphi(x_i)\right) dX_N$$

$$= \frac{Z_{N,\beta}(V+t\varphi)}{Z_{N,\beta}(V)}$$

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$$=\frac{Z_{N,\beta}(V+t\varphi)}{Z_{N,\beta}(V)}$$

1d log-gas: fluctuations

Theorem (CLT for fluctuations)

Let $\beta > 0$. Take φ smooth enough, assume V is nice. Then:

$$\sum_{i=1}^{N} \varphi(x_i) - N \int \varphi(x) d\mu_V(x) = N \int \varphi(x) \left(d\widehat{\mu}_N(x) - d\mu_V(x) \right)$$

has a Gaussian limit.
True at mesoscopic scales i.e. $\varphi = \bar{\varphi}(x/\ell)$ for some $\ell >> 1/N$.

No $\frac{1}{\sqrt{N}}$ normalization!
Johansson, Borot-Guionnet, Bourgade-Erdös-Yau, Bekerman-Lodhia, M. Shcherbina, Borot-Guionnet, Bekerman-Leblé-S

Theorem (Expansion of free energy to all orders)

$$-\frac{1}{\beta} \log Z_{N,\beta} = N^2 \mathcal{E}(\mu_V) + N \log N + A_\beta N + B_\beta + \frac{C_\beta}{N} + \dots$$

Shcherbina, Borot-Guionnet

1d log-gas: existence of limiting point processes

Theorem (Limiting point process)

Take V quadratic, $d\mu_V(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$ (semi-circle) and $\Sigma = [-2, 2]$. Consider the zoomed point configuration:

$$\sum_{i=1}^{N} \delta_{N(x_i - \bar{x})}$$

- *If $\bar{x} = \pm 2$, limiting point process Airy-β*
- *If $\bar{x}$ is inside $(-2, 2)$, limiting point process Sine-β.*

Ramírez-Rider-Virág (edge), Valkó-Virág & Killip-Stoiciu (bulk).
CLT for linear statistics of Sine-β Leblé

Theorem (Universality)

The local statistics depend on V only through a rescaling by the mean density μ_V.

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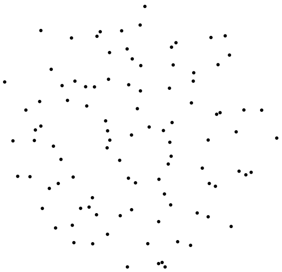
What about Coulomb gases (in $d \geq 2$)?

Simulation of 2D log gas for $V(x) = |x|^2$



$g = -\log$, $V = |x|^2$, 100 points, $\beta \in [0.7, 400]$ (simul: Thomas Leblé)

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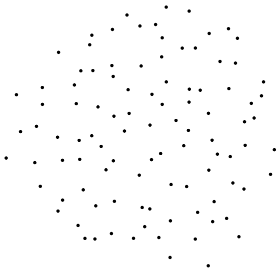

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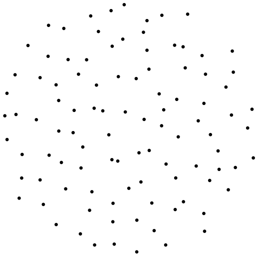

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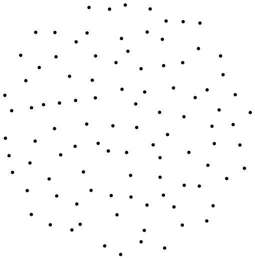

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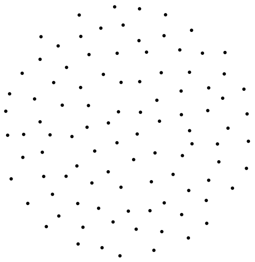

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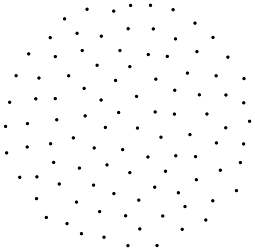

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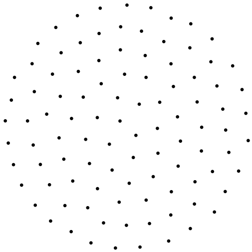

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Numerical observations

- The local behavior depends strongly on β. Order increases with β.
- The local behavior depends on μ_V **only through a scaling** (universality).
- For $d = 2, 3$, a phase transition (?) happens at finite β (150?) (computational physics literature in the 80's: Choquard-Clerouin, Alastuey-Jancovici, Caillol-Levesque-Weis-Hansen).
- As $\beta \to \infty$, for $d = 2$, the points arrange themselves **on a triangular lattice** (Wigner crystal, $\sim$ Abrikosov lattice in superconductivity).

Proofs?

No proof of phase transition, no proof of Abrikosov conjecture. No good order parameter. No universality for general β...

The case of the Ginibre ensemble $d = 2$, $\beta = 2$, $V = |x|^2$

It is *determinantal* i.e. the k-point correlation function can be computed as $k \times k$ determinants.

- CLT for fluctuations at all scales Rider-Virág, Ameur-Hedenmalm-Makarov, Shirai
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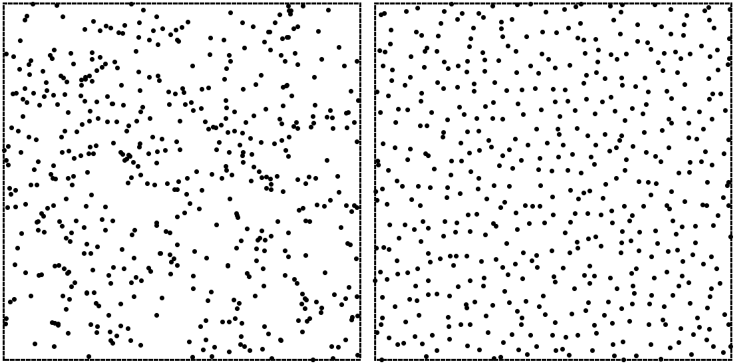

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- Many quantitites can be estimated, or even explicitly computed.
- **Number-rigidity** of the Ginibre p.p. : the knowledge of the (infinite) configuration outside a ball determines the number of points inside almost surely. Ghosh-Peres
- **Hyperuniformity** (à la Torquato):

$$\mathrm{Var}\left[\#\text{ points in disk } D(0,R)\right] \sim R \quad \text{as } R \to \infty \quad \text{Shirai}$$

What about general β? $d \geq 3$?

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What about general β? $d \geq 3$?

A few positive results

CLT for smooth linear statistics in 2D log / Coulomb case

Theorem

Assume $d = 2$, $\beta > 0$ *arbitrary fixed,* $V \in C^{3,1}$. *Let* $\varphi \in C_c^{2,1}(\Sigma)$ *Then*

$$\sum_{i=1}^{N} \varphi(x_i) - N \int_\Sigma \varphi \, d\mu_V$$

converges in law as $N \to \infty$ *to a Gaussian distribution with*

$$mean = \frac{1}{2\pi}\left(\frac{1}{\beta} - \frac{1}{4}\right)\int \Delta\varphi \log \Delta V \qquad var = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2} |\nabla\varphi|^2.$$

$\rightsquigarrow \Delta^{-1}\left(\sum_{i=1}^N \delta_{x_i} - N\mu_V\right)$ *converges to the Gaussian Free Field.*
The result can be localized with φ *supported on all mesoscales* $\ell >> N^{-1/2}$.

Leblé-S, Bauerschmidt-Bourgade-Nikula-Yau, S, case of φ overlapping $\partial\Omega$ in Leblé-S.

Local laws in any dimension

Theorem (Armstrong-S. '20)

- Control in exponential moments of energy and of fluctuations of (nonsmooth) linear statistics in boxes, down to a **temperature-dependent minimal scale** $\simeq N^{-1/d}\max(1, \beta^{-1/2})$
- Free energy expansion to next order (existence of thermodynamic limit) in the case of uniform μ_V

(can couple β and N)

Corollary

For fixed β, bound on the number of points in microscopic boxes $\rightsquigarrow$ existence of a limit point process after subsequence.

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A Large Deviations Principle for limiting point processes

Theorem (Leblé-S, '17, Armstrong-S '20)

For Coulomb or log (or Riesz interactions $|x|^{-s}$, $d-2 \le s < d$*), there is an LDP for the "empirical field", averaged at any mesoscale after zoom by* $(\mu_V(x)N)^{1/d}$ *around* x*, at speed* $N^{1+\frac{s}{d}}$ *with rate function* $\mathcal{F}_\beta - \min \mathcal{F}_\beta$,

$$\mathcal{F}_\beta(P) := \beta \mathbb{W}(P) + \mathrm{ent}[P|\Pi] \qquad \Pi = \textit{Poisson 1}$$

$\mathbb{W}$ = *Coulomb renormalized energy for an infinite point configuration (jellium)*

$\rightsquigarrow$ The Gibbs measure concentrates asymptotically on point processes which minimize $\mathcal{F}_\beta$

- competition between **energy** and **relative entropy**
- $\beta \ll 1$ entropy dominates $\rightsquigarrow$ convergence to Poisson point process
- $\beta >> 1$ convergence to minimizers of $\mathbb{W}$

Corollary

Variational characterization of Sine-β and Ginibre (minimize $\beta\mathbb{W} + \mathrm{ent}$).

The **jellium energy** $\mathbb{W}$ (defined in [Sandier-S '12, Rougerie-S '16, Petrache-S '17]) seems to favor crystalline configurations in low dimensions

- In dimension $d = 1$, the minimum of $\mathbb{W}$ over all possible configurations is achieved for the **lattice** $\mathbb{Z}$.
- In dimension $d = 8$ the minimum of $\mathbb{W}$ is achieved by the E_8 lattice and in dimension $d = 24$ by the Leech lattice: consequence (by [Petrache-S '19] of the **Cohn-Kumar conjecture** proven in [Cohn-Kumar-Miller-Radchenko-Viazovska '19]
- the Cohn-Kumar in dimension 2 implies that $\min \mathbb{W}$ is achieved at the **triangular lattice**, but remains **open**

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The **jellium energy** $\mathbb{W}$ (defined in [Sandier-S '12, Rougerie-S '16, Petrache-S '17]) seems to favor crystalline configurations in low dimensions

- In dimension $d = 1$, the minimum of $\mathbb{W}$ over all possible configurations is achieved for the **lattice** $\mathbb{Z}$.
- In dimension $d = 8$ the minimum of $\mathbb{W}$ is achieved by the E_8 lattice and in dimension $d = 24$ by the Leech lattice: consequence (by [Petrache-S '19] of the **Cohn-Kumar conjecture** proven in [Cohn-Kumar-Miller-Radchenko-Viazovska '19]
- the Cohn-Kumar in dimension 2 implies that $\min \mathbb{W}$ is achieved at the **triangular lattice**, but remains **open**

Free energy expansion with a rate (Coulomb any d)

Theorem (Leblé-S '17, S '20)

Let $s = d - 2$.

$$\log Z_{N,\beta} = -\beta N^2 \mathcal{E}(\mu_V) + \left(\frac{\beta}{4} N \log N\right) \mathbb{1}_{d=2} - N\left(1 - \frac{\beta}{4}\right)\left(\int \mu_V \log \mu_V\right) \mathbb{1}_{d=2} - N^{1+\frac{s}{d}} \int f_d(\beta \mu_V^{s/d}) d\mu_V + \beta O(N^{1+\frac{s}{d}-\varepsilon})$$

with $\varepsilon = \frac{1}{2d}$ *and*

$$f_d(\beta) = \min_{\text{stationary p.p.}} \beta \mathbb{W} + \mathrm{ent}(\cdot|\Pi)$$

Analogous result for log and Riesz interactions.

To be compared with [Borot-Guionnet '13, Shcherbina '13] ($d = 1$, log), [Wiegmann-Zabrodin '09] ($d = 2$, log) (formal)

Main ingredients

- The **electric approach**
- The **screening procedure** $\rightsquigarrow$ almost **additivity** of the (free) energy over boxes
- A **bootstrap on scales** for local laws + free energy expansion, which allow to perform the screening down to smaller and smaller scales, and improve local laws + free energy expansion, etc
- **Transport approach** for the CLT

The electric approach

- Exact **splitting** of the energy

$$\mathrm{H}_N(X_N) = N^2\mathcal{E}(\mu_V) + N\sum_{i=1}^{N}\zeta_V(x_i)$$

$$\underbrace{+\frac{1}{2}\iint_{\triangle^c} \mathrm{g}(x-y)d\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_V\right)(x)d\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_V\right)(y)}_{F(X_N,\mu_V)}$$

- Define the **electric potential**

$$h(x) = \int \mathrm{g}(x-y)\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_V\right)(y)$$

- **use Coulomb**

$$-\Delta h = c_d\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_V\right) \qquad !$$

- after integration by parts

$$F(X_N, \mu_V) = \frac{1}{c_d} \int_{\mathbb{R}^d} |\nabla h|^2$$

(except needs to be **renormalized** via truncations because of the self-interaction removal...)

- the energy becomes **local** in the electric potential h. Can hope to compute it additively over boxes (despite long range nature etc)
- Boundary conditions for solving h over a box will be important: use both Neumann and Dirichlet boundary conditions for solving provides **sub/superadditive** energy quantities
- Screening procedure allows to compare the two and show they are close (up to a modification of the configuration) hence almost additivity

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- After blow-up at suitable scale + limit $N \to \infty$

$$-\Delta h = \sum_{infinite} \delta_p - 1 \qquad \textbf{(jellium)}$$

- formally

$$\mathbb{W} = \lim_{R\to\infty} \fint_{\square_R} |\nabla h|^2$$

Method of proof for the CLT

- Evaluate

$$\frac{Z(V_t)}{Z(V)}$$

where $V_t := V + t\varphi$, equilibrium measure μ_{V_t}, $t = \frac{\tau}{N}$

- use map Φ_t that transports μ_V to μ_{V_t}, $\Phi_t \simeq I + t\psi$. By using change of variables $y_i = \Phi_t(x_i)$, we are led to compute

$$\mathbb{E}_{\mathbb{P}_{N,\beta}} \left(F_N(\Phi_t(X_N), \Phi_t \# \mu_V) - F_N(X_N, \mu_V) \right)$$

(replaces "loop equations" / Dyson-Schwinger)

- use linearization in t small for the rhs + expansion of $\log Z_{N,\beta}$ with a rate to evaluate this with $o(1)$ error when $t = \tau/N$.

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