Fredholm Determinant Solutions of the Painlevé II Hierarchy and Gap Probabilities of Determinantal Point Processes

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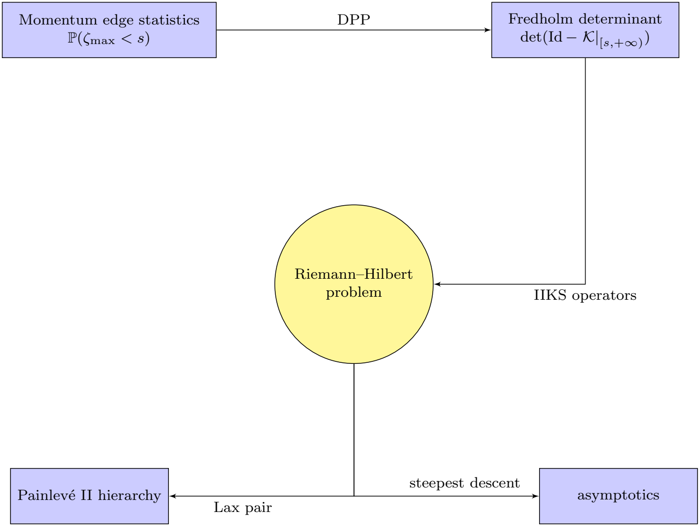
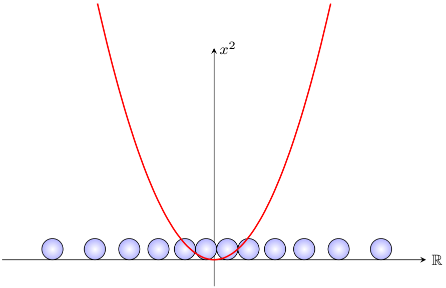
Overview


Momentum edge statistics
$\mathbb{P}(\zeta_{\max} < s)$
DPP
Fredholm determinant
$\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$
Riemann–Hilbert problem
IIKS operators
Painlevé II hierarchy
Lax pair
steepest descent
asymptotics

Motivation (LeDoussal, Majumdar, Schehr, '18)

Joint statistics of the momenta $\{p_i\}$ of N non-interacting fermions in 1-dimension and of the largest one $p_{\max}$ in a trap

$$V(x) \sim x^{2n}, \qquad n \geq 1$$



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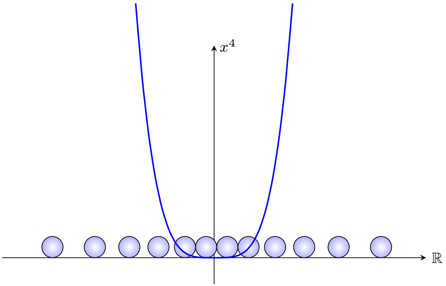

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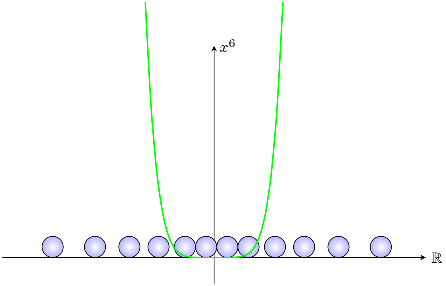
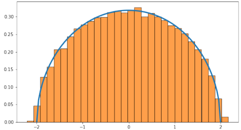


Harmonic trap ($n = 1$)

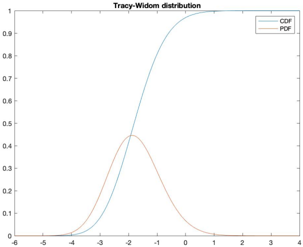
There is a one-to-one correspondence with the eigenvalues of GUE:

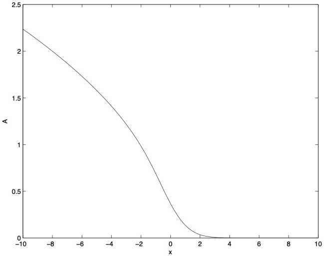
- macroscopic density = Wigner semicircle law;



- edge behaviour $\rho(p) \sim (p_{\max} - p)^{\frac{1}{2}}$, Tracy–Widom distribution.

$$F_2(x) = \exp\left\{ -\int_s^{+\infty} (s-x) q^2(s) \mathrm{d}s \right\}$$





Flat trap ($n > 1$)

- macroscopic density related to matrix models with multicritical points (Di Francesco, Ginsparg, Zinn-Justin, '95; Brézin, Kazakov, Eynard, ...);
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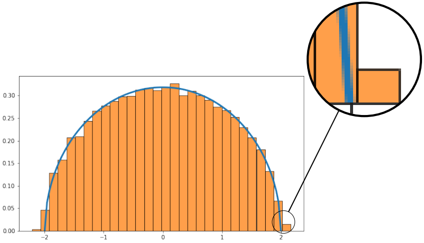
Statistical setup

- The set of momenta of the N fermions is a *point process* on $\mathbb{R}$.
- (**gap probability**) The distribution of the largest momentum is $\mathbb{P}\left(p_{\max} < s\right)$



The same arguments hold in the limit as $N \to \infty$:

$$\begin{array}{c} \text{(generalized) TW distribution} \\ = \mathbb{P}(\zeta_{\max} < s) \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \text{infinitesimal oscillations} \\ \text{at the edge of the spectrum} \end{array}$$

Theorem (LeDoussal, Majumdar, Schehr, '18)

The set of momenta $\{p_i\}_{i=1,\dots,N}$ and the rescaled fermion momenta near the edge $\zeta_{\max}$ are both Determinantal Point Processes.

The n-point correlation function reads

$$\mathfrak{p}(x_1,\dots,x_n) = \det \begin{bmatrix} K(x_1,x_1) & K(x_1,x_2) & \dots & K(x_1,x_n) \\ K(x_2,x_1) & K(x_2,x_2) & & \\ \vdots & & \ddots & \\ K(x_n,x_1) & & \dots & K(x_n,x_n) \end{bmatrix}$$

for some kernel $K(x,y)$ – containing all the statistical information –.

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for some kernel $K(x,y)$ – containing all the statistical information –.

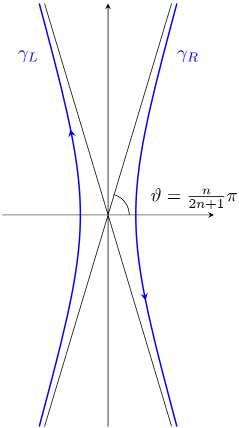
In particular, the gap probability over $I = (s, +\infty) \subset \mathbb{R}$ becomes:

$$\mathbb{P}\left(\zeta_{\max} < s\right) = \det\left(\mathrm{Id} - \mathcal{K}|_{(s,+\infty)}\right)$$

the *Fredholm determinant* of an integral operator with *universal* kernel

$$K(x,y) = \int_{\gamma_R} \frac{\mathrm{d}\mu}{2\pi \mathrm{i}} \int_{\gamma_L} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} \frac{\mathrm{e}^{(-1)^{n+1}\left(\frac{\mu^{2n+1}}{2n+1} - \frac{\lambda^{2n+1}}{2n+1}\right) - x\mu + y\lambda}}{\lambda - \mu},$$

which depends on the order of the potential $V(x) = x^{2n}$.



$$\mathbb{P}\left(\zeta_{\max} < s\right) = \det\left(\mathrm{Id} - \mathcal{K}|_{(s,+\infty)}\right)$$

Remark

- the largest point $\zeta_{\max}$ in the process exists almost surely, since the Fredholm determinant is well-defined;
- in general, we can analyze the quantity

$$F(s;\varrho) := \det(\mathrm{Id} - \varrho\mathcal{K}|_{[s,+\infty)}), \qquad \varrho \in (0,1],$$

  which represents the probability distribution of the largest particle $\zeta_{\max}^{(\varrho)}$ in the associated *thinned* process, which is obtained from the original process by removing each of the particles independently with probability $1-\varrho$.

The higher-order Airy kernel

$$K(x,y) = \int_{\gamma_R} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \int_{\gamma_L} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \frac{\mathrm{e}^{(-1)^{n+1}\left(\frac{\mu^{2n+1}}{2n+1} - \frac{\lambda^{2n+1}}{2n+1}\right) - x\mu + y\lambda}}{\lambda - \mu}$$
$$= \int_0^\infty \mathrm{Ai}_{2n+1}(x+t)\mathrm{Ai}_{2n+1}(y+t)\,\mathrm{d}t$$

where

$$\mathrm{Ai}_{2n+1}(x) = \int_{\gamma_R} e^{\frac{(-1)^{n+1}\mu}{2n+1} - x\mu} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}$$

solves $\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\phi(x) = (-1)^{n+1}x\phi(x)$ (Kohno, '79).

For $n = 1$ we recover the Airy kernel:

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}$$

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Our results

More generally, we will be interested in the following Fredholm determinant:

$$\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$$

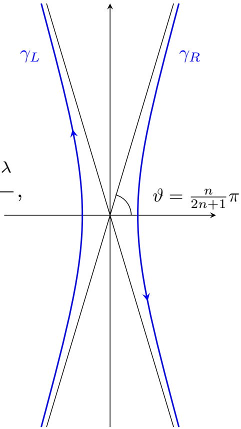
with kernel

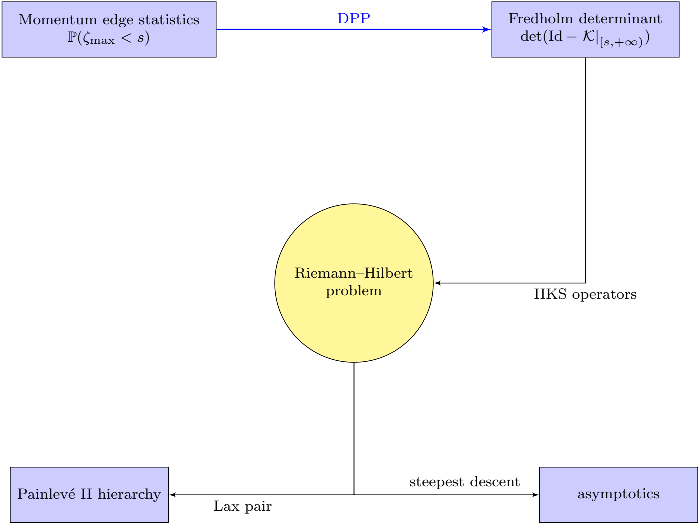
$$K(x,y) = \int_{\gamma_R} \frac{\mathrm{d}\mu}{2\pi i} \int_{\gamma_L} \frac{\mathrm{d}\lambda}{2\pi i} \frac{\mathrm{e}^{(-1)^{n+1}\left(p_{2n+1}(\mu) - p_{2n+1}(\lambda)\right) - x\mu + y\lambda}}{\lambda - \mu},$$

where $x, y \in \mathbb{R}$, $n \in \mathbb{N}$ and

$$p_{2n+1}(x) = \frac{x^{2n+1}}{2n+1} + \sum_{j=1}^{n-1} \frac{\tau_j}{2j+1} x^{2j+1},$$

with $\tau_1, \ldots, \tau_{n-1} \in \mathbb{R}$.

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Tracy-Widom-like identity

Theorem (Cafasso, Claeys, G., '19)

Let $n \in \mathbb{N}$, $\tau_1, \dots, \tau_{n-1} \in \mathbb{R}$ and let $F(s) = \det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$, then

$$F(s) = \exp\left\{ -\int_s^{+\infty} (x-s)\, q^2\left((-1)^{n+1}x\right) \mathrm{d}x \right\},$$

where $q(s) = q(s; \tau_1, \dots, \tau_{n-1})$ is the solution to the equation of order $2n$ in the **Painlevé II hierarchy** *with prescribed asymptotics.*

Note: the same result holds also for $F(s; \varrho) := \det(\mathrm{Id} - \varrho\,\mathcal{K}|_{[s,+\infty)})$, $\varrho \in (0, 1]$.

A little detour: Painlevé equations and their hierarchy

In the quest for classifying 2nd-order ODEs of the type

$$q'' = \mathcal{R}\left(s, q, q'\right)$$

($\mathcal{R}$ a rational function) whose solutions have no movable singularities other than poles (*Painlevé property*), Painlevé (1902) and Gambier (1910) found 50 different classes.

Only 6 of them are irreducible (i.e. their solutions cannot be expressed in terms of elementary functions or previously known transcendental functions):

$$q'' = 2q^3 + sq + \alpha, \qquad \alpha \in \mathbb{R} \quad \text{(Painlevé II)}.$$

Furthermore, the Painlevé equations are integrable systems (Lax pair!).
In fact, they are connected to solutions of integrable non-linear PDEs (Ablowitz–Segur, '77).

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The Painlevé II hierarchy

The Painlevé II hierarchy (Flaschka, Newell, '80) is a sequence of ODEs obtained from the equations of the KdV hierarchy via self-similar reduction.

The n-th member of the Painlevé II hierarchy is an equation for $q = q(s)$ defined as follows:

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} + 2q\right) \mathcal{L}_n[q_s - q^2] + \sum_{\ell=1}^{n-1} \tau_\ell \left(\frac{\mathrm{d}}{\mathrm{d}s} + 2q\right) \mathcal{L}_\ell[q_s - q^2] = sq - \alpha, \qquad n \geq 1,$$

where $\alpha, \tau_1, \ldots, \tau_{n-1} \in \mathbb{R}$ and the operators $\{\mathcal{L}_n\}_{n\geq 0}$ are the Lenard operators defined recursively by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}_{j+1}f = \left(\frac{\mathrm{d}^3}{\mathrm{d}s^3} + 4f\frac{\mathrm{d}}{\mathrm{d}s} + 2f_s\right)\mathcal{L}_j f, \qquad \mathcal{L}_0 f = \frac{1}{2}, \qquad \mathcal{L}_j 1 = 0, \quad j \geq 1.$$

The first members of the hierarchy are

$n = 1$

$$q'' - 2q^3 = sq - \alpha \qquad \text{(Painlevé II)}$$

$n = 2$

$$q'''' - 10q(q')^2 - 10q^2q'' + 6q^5 + \tau_1(q'' - 2q^3) = sq - \alpha$$

$n = 3$

$$q'''''' - 14q^2q'''' - 56qq'q''' - 70(q')^2q'' - 42q(q'')^2 + 70q^4q'' + 140q^3(q')^2 - 20q^7 \\ + \tau_2(q'''' - 10q(q')^2 - 10q^2q'' + 6q^5) + \tau_1(q'' - 2q^3) = sq - \alpha,$$

We will be interested in the *homogeneous* Painlevé II hierarchy ($\alpha = 0$).

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Back to our Fredholm Determinant

$$q^2\big((-1)^{n+1}s\big) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)}),$$

- from the Physical point of view, we provide a rigorous description of the statistics of the fluctuations of the largest momentum of a collection of fermions in a “flat trap”;
- from the Mathematical point of view, we construct a family of solutions to the Painlevé II hierarchy in terms of Fredholm determinants (which can be numerically computed with high accuracy; Bornemann, ’10).

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Remark

- the occurrence of the Painlevé-II hierarchy for the monic potential was first established for selected values of n in the arXiv version of LeDoussal, Majumdar, Schehr's paper.
- this family of solutions contains natural generalizations of the Hastings–McLeod ($\rho = 1$) and the Ablowitz–Segur ($\rho < 1$) solutions to the Painlevé II equation.
- generalizations of the TW-distribution were also obtained by Claeys, Its, Krasovsky ('10) for extreme eigenvalues of unitary random matrices with critical edge points, but $\alpha = \frac{1}{2}$.

Treasure map

1. establish the equality

$$\begin{array}{rcl} \text{gap probability} & \Leftrightarrow & \text{Fredholm determinant of IIKS integral operator} \\ \mathbb{P}(\zeta_{\max} < s) & = & \det\left(\mathrm{Id} - \mathcal{L}_s\right) \end{array}$$

2. build up the corresponding

RH problem

3. prove the link

RH problem $\Leftrightarrow$ Malgrange-Bertola τ-function;

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The proof

We follow a general method introduced by Bertola-Cafasso ('11).

Proposition

Via a *conjugation with a Fourier-type transform*, we obtain

$$\det\left(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)}\right) = \det\left(\mathrm{Id} - \mathcal{L}_s\right),$$

where $\mathcal{L}_s$ is an integral operator in the sense of Its-Izergin-Korepin-Slavnov ('90) with kernel of the form

$$L_s(\lambda,\mu) = \frac{\mathbf{f}(\lambda)^\top \mathbf{g}(\mu)}{\lambda - \mu}, \qquad \lambda, \mu \in \gamma_L \cup \gamma_R$$

$\mathbf{f}(\lambda), \mathbf{g}(\lambda) \in \mathbb{C}^{2\times 1}$, with regularity condition $\mathbf{f}(\lambda)^T \mathbf{g}(\lambda) = 0$.

Note: the original operator $\mathcal{K}$ has IIKS kernel, but it's not practical.

A Riemann–Hilbert problem

The IIKS operator $\mathcal{L}_s$ naturally carries an associated **Riemann–Hilbert problem**:

RH problem

Find a matrix-valued function $\mathbf{\Gamma}(z) \in \mathbb{C}^{2\times 2}$ *such that*

(a) $\mathbf{\Gamma}$ *is analytic on* $\mathbb{C} \setminus (\gamma_L \cup \gamma_R)$

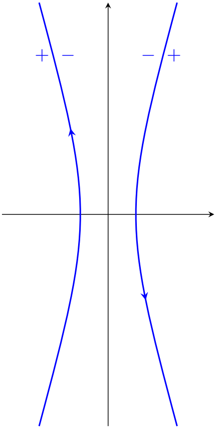
(b) $\mathbf{\Gamma}$ *has continuous boundary values* $\mathbf{\Gamma}_\pm$ *as* $\zeta \in \gamma_L \cup \gamma_R$ *is approached from the left (+) or right (−) side, and they are related by*

$$\mathbf{\Gamma}_+(\zeta) = \mathbf{\Gamma}_-(\zeta) \underbrace{\left[\boldsymbol{I} - 2\pi \mathrm{i} \mathbf{f}(\zeta) \mathbf{g}^\top(\zeta) \right]}_{\textit{jump } \boldsymbol{J}} \qquad \zeta \in \gamma_L \cup \gamma_R,$$

(c) *there exists a matrix* $\mathbf{\Gamma}_1$ *independent of* ζ *(but depending on* n, τ_j *and* s*) such that* $\mathbf{\Gamma}$ *satisfies*

$$\mathbf{\Gamma}(\zeta) = \boldsymbol{I} + \frac{\mathbf{\Gamma}_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty.$$

The jump condition:



$$\mathbf{\Gamma}_+(\zeta) = \mathbf{\Gamma}_-(\zeta)\begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-(-1)^{n+1}p_{2n+1}(\zeta)+s\zeta} & 1 \end{pmatrix}$$

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Why do we care about the RHP?

It can be proven (Bertola, '10 and Bertola, Cafasso, '11) that

$$\frac{\mathrm{d}}{\mathrm{d}s} \log F(s) = \int_{\gamma_R \cup \gamma_L} \mathrm{Tr}\left[\boldsymbol{\Gamma}_-^{-1}(\zeta) \boldsymbol{\Gamma}'_-(\zeta)(\partial_s \boldsymbol{J})(\zeta) \boldsymbol{J}^{-1}(\zeta) \right] \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}},$$

where $\boldsymbol{\Gamma}'$ is the derivative of $\boldsymbol{\Gamma}$ and $\boldsymbol{J}$ is the jump matrix.

Proposition

The Fredholm determinant $F(s)$ satisfies the differential identity

$$\frac{\mathrm{d}}{\mathrm{d}s} \log F(s) = \boldsymbol{\Gamma}_{1,11},$$

where $\boldsymbol{\Gamma}_1$ is the first coefficient in the expansion of the RHP solution $\boldsymbol{\Gamma}$ at infinity:

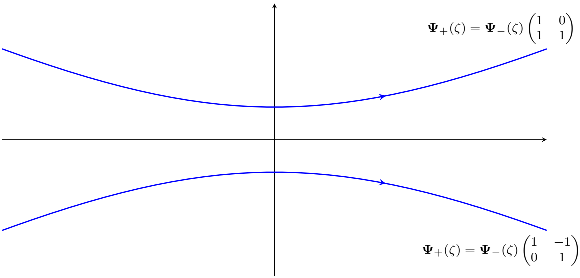
$$\boldsymbol{\Gamma}(\zeta) = \boldsymbol{I} + \frac{\boldsymbol{\Gamma}_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty.$$

Via an invertible transformation, we define

$$\boldsymbol{\Psi}(\zeta) := \boldsymbol{\sigma}_3\, \boldsymbol{\Gamma}(2\mathrm{i}\zeta)\, \boldsymbol{\sigma}_3\, \mathrm{e}^{\boldsymbol{T}_s(2\mathrm{i}\zeta)}$$

with $\boldsymbol{T}_s(\zeta) := \left(\frac{(-1)^{n+1}}{2} p_{2n+1}(\zeta) - \frac{1}{2} s\zeta\right) \boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The RHP for $\boldsymbol{\Psi}$ has constant jumps!



We can easily find $\boldsymbol{\Psi}$'s Lax pair

$$\boldsymbol{\Psi}_\lambda = \boldsymbol{A}\boldsymbol{\Psi}$$
$$\boldsymbol{\Psi}_s = \boldsymbol{B}\boldsymbol{\Psi}$$

which turns out to be same one as the one for the Painlevé II hierarchy in the case $\alpha = 0$, with Stokes parameters $s_1 = -s_{2n+1} = 1$, $s_2 = \ldots = s_{2n} = 0$.

The compatibility conditions

$$\boldsymbol{\Psi}_{s\lambda} = \boldsymbol{\Psi}_{\lambda s} \quad \Leftrightarrow \quad \partial_s \boldsymbol{A} - \partial_\lambda \boldsymbol{B} + [\boldsymbol{A}, \boldsymbol{B}] = 0$$

yields

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Psi_{1,11} = -2\mathrm{i}\left(\Psi_{1,12}\right)^2$$

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$$\mathbf{\Psi}_{s\lambda} = \mathbf{\Psi}_{\lambda s} \quad \Leftrightarrow \quad \partial_s \boldsymbol{A} - \partial_\lambda \boldsymbol{B} + [\boldsymbol{A}, \boldsymbol{B}] = 0$$

yields

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Psi_{1,11} = -2\mathrm{i}\left(\Psi_{1,12}\right)^2$$

$q((-1)^{n+1}s)$ solution to PII hierarchy

We can easily find $\boldsymbol{\Psi}$'s Lax pair

$$\boldsymbol{\Psi}_\lambda = \boldsymbol{A}\boldsymbol{\Psi}$$
$$\boldsymbol{\Psi}_s = \boldsymbol{B}\boldsymbol{\Psi}$$

which turns out to be same one as the one for the Painlevé II hierarchy in the case $\alpha = 0$, with Stokes parameters $s_1 = -s_{2n+1} = 1$, $s_2 = \ldots = s_{2n} = 0$.

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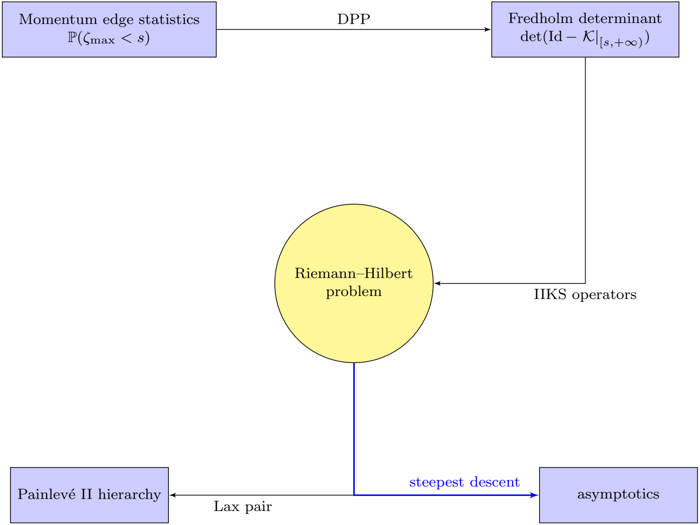
$$\begin{tikzcd} & \frac{\mathrm{d}}{\mathrm{d}s}\Psi_{1,11} = -2\mathrm{i}\left(\Psi_{1,12}\right)^2 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ -\frac{\mathrm{d}}{\mathrm{d}s}\log F(s) & & q((-1)^{n+1}s) \text{ solution to PII hierarchy} \end{tikzcd}$$

Overview



Momentum edge statistics
$\mathbb{P}(\zeta_{\max} < s)$
DPP
Fredholm determinant
$\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$
IIKS operators
Riemann–Hilbert problem
Painlevé II hierarchy
Lax pair
steepest descent
asymptotics

Asymptotics as $s \to +\infty$

Theorem (Cafasso, Claeys, G., '19)

Let $n \in \mathbb{N}$, $\tau_1, \ldots, \tau_{n-1} \in \mathbb{R}$. There is a real solution $q(s) = q(s; \tau_1, \ldots, \tau_{n-1})$ to the equation of order $2n$ in the Painlevé II hierarchy which has no poles for real s, such that

$$q^2\big((-1)^{n+1}s\big) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} \log\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)}),$$

and with asymptotic behaviour

$$q((-1)^{n+1}s) = \mathcal{O}\left(e^{-Cs^{\frac{2n+1}{2n}}}\right), \qquad \text{as } s \to +\infty, \text{ for some } C > 0,$$

$$q((-1)^{n+1}s) \sim \left(\frac{n!^2}{(2n)!}|s|\right)^{\frac{1}{2n}}, \qquad \text{as } s \to -\infty.$$

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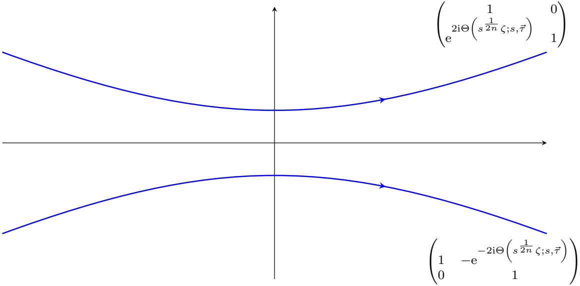
Sketch of the proof

Consider a rescaled and rotated version $\mathbf{\Xi}$ of our RHP $\mathbf{\Gamma}$:



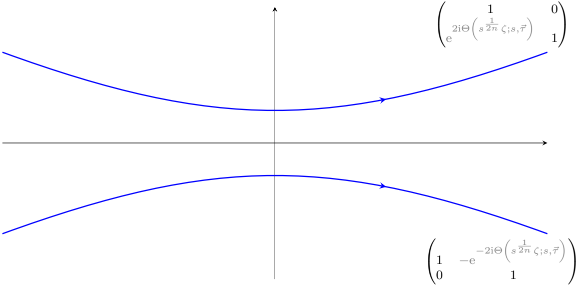
with $\Theta\left(s^{\frac{1}{2n}}\zeta; s, \vec{\tau}\right) = s^{\frac{2n+1}{2n}}\left(\frac{(2\zeta)^{2n+1}}{4n+2} + \zeta \right) + \mathcal{O}\left(s^{\frac{2n-1}{2n}}\zeta^{2n-1}\right).$

$$\Theta\left(s^{\frac{1}{2n}}\zeta; s, \vec{\tau}\right) = s^{\frac{2n+1}{2n}}\left(\boxed{\frac{(2\zeta)^{2n+1}}{4n+2} + \zeta}\right) + \mathcal{O}\left(s^{\frac{2n-1}{2n}}\zeta^{2n-1}\right)$$

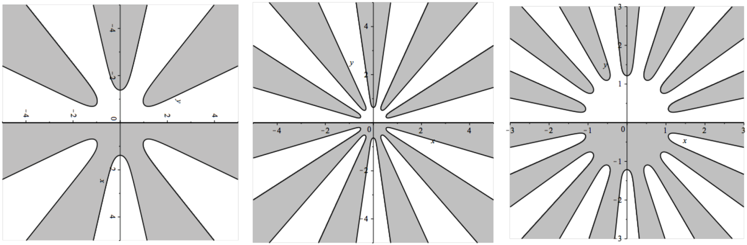
We would like to smartly choose the contours in such a way that

$$\operatorname{Im}\left[\frac{(2\zeta)^{2n+1}}{4n+2} + \zeta\right] > 0,\ \zeta \in \gamma_U, \qquad \operatorname{Im}\left[\frac{(2\zeta)^{2n+1}}{4n+2} + \zeta\right] < 0,\ \zeta \in \gamma_D.$$

so that the off-diagonal terms in the jumps $e^{\pm 2i\Theta} \to 0$ as $s \to +\infty$.



We can actually do so:



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure: The white areas are the zones where Im > 0 while the grey areas are the zones where Im < 0.

This way, the jump matrices are asymptotically equal to the identity matrix:

$$\boldsymbol{\Xi}_+(\zeta) = \boldsymbol{\Xi}_-(\zeta)\left(\boldsymbol{I} + \mathcal{O}\left(\frac{\mathrm{e}^{-Cs^{\frac{2n+1}{2n}}}}{\zeta^2}\right)\right), \quad \zeta \in \gamma_U \cup \gamma_D.$$

By the Small Norm Theorem, we can infer

$$\boldsymbol{\Xi}(\zeta) = \boldsymbol{I} + \mathcal{O}\left(\frac{\mathrm{e}^{-Cs^{\frac{2n+1}{2n}}}}{\zeta}\right) \qquad \text{as } s \to +\infty,$$

for some $C > 0$.

Getting back to q: as $s \to +\infty$,

$$q\big((-1)^{n+1}s\big) \sim \Xi_{1,12}(s) = \mathcal{O}\left(\mathrm{e}^{-Cs^{\frac{2n+1}{2n}}}\right).$$

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Finer asymptotics for higher order Airy

From the integral formula

$$\boldsymbol{\Xi}(\zeta) = \boldsymbol{I} + \int_{\gamma_U \cup \gamma_D} \frac{\boldsymbol{\Xi}_-(w)\left[\boldsymbol{I} - \boldsymbol{J}(w, s; \vec{\tau})\right]}{w - \zeta} \frac{\mathrm{d}w}{2\pi \mathrm{i}},$$

and its asymptotic expansion in ζ, we obtain

Corollary

If $\tau_1 = \ldots = \tau_{n-1} = 0$,

$$q\left((-1)^{n+1} s\right) = \mathrm{Ai}_{2n+1}(s)(1 + o(1)), \qquad \text{as } s \to +\infty.$$

(we suspect this holds also for general $\tau_1, \ldots, \tau_{n-1} \neq 0$)

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(we suspect this holds also for general $\tau_1, \ldots, \tau_{n-1} \neq 0$)

Asymptotics as $s \to -\infty$

Theorem (Cafasso, Claeys, G., '19)

Let $n \in \mathbb{N}$, $\tau_1, \dots, \tau_{n-1} \in \mathbb{R}$, and let $F(s) = \det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$. There is a real solution $q(s) = q(s; \tau_1, \dots, \tau_{n-1})$ to the equation of order $2n$ in the Painlevé II hierarchy which has no poles for real s, such that

$$q^2\left((-1)^{n+1}s\right) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log F(s),$$

and with asymptotic behaviour

$$q((-1)^{n+1}s) = \mathcal{O}\left(e^{-Cs^{\frac{2n+1}{2n}}}\right), \qquad \text{as } s \to +\infty, \text{ for some } C > 0,$$

$$\boxed{q((-1)^{n+1}s) \sim \left(\frac{n!^2}{(2n)!}|s|\right)^{\frac{1}{2n}}, \qquad \text{as } s \to -\infty.}$$

Large gap asymptotics for the pure potential

In the case $\tau_1 = \ldots = \tau_{n-1} = 0$ (higher-order Airy)

Theorem (Cafasso, Claeys, G., '19)

There exists a constant $C > 0$, possibly depending on n, such that

$$\log F(s) = -\frac{n^2}{(n+1)(2n+1)} \binom{2n}{n}^{-\frac{1}{n}} |s|^{2+\frac{1}{n}} + c \log|s| + \log C + o(1),$$

as $s \to -\infty$,with $c = -\frac{1}{8}$ if $n = 1$ and $c = -\frac{1}{2}$ otherwise. Moreover, the asymptotics can be improved to

$$q((-1)^{n+1}s) = \left(\frac{n!^2}{(2n)!}|s|\right)^{\frac{1}{2n}} + \frac{c}{2}\left(\frac{(2n)!}{n!^2}\right)^{\frac{1}{2n}} |s|^{-2-\frac{1}{2n}} + \mathcal{O}\left(|s|^{-2-\frac{1}{n}}\right),$$

as $s \to -\infty$.

Note: our method does not allow to evaluate the overall constant C. In the Airy case $n = 1$, it was proved that $C = \mathrm{e}^{\frac{1}{24}\log 2 + \zeta'(-1)}$, where ζ' is the derivative of the Riemann ζ function (Deift, Its, Krasovsky, 2008).

In the general case with $\tau_1, \dots, \tau_{n-1} \in \mathbb{R}$ consider

$$\lambda(z) := \sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k z^{2k}, \qquad \tilde{\lambda}(z) := \sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k z^{k},$$

and define $\theta_1, \dots, \theta_{2n}$ and $\theta_1^{[2]}, \dots, \theta_{2n}^{[2]}$ as follows:

$$\theta_i(\tau_1, \dots, \tau_n) \equiv \theta_i := \begin{cases} \binom{2n}{n}^{-\frac{1}{2n}} & i = 0, \\ \frac{1}{2i-1} \operatorname*{res}_{z=\infty} \lambda^{\frac{2i-1}{2n}}(z), & i \geq 1, \end{cases}$$

where the residue at infinity is minus the coefficient of z^{-1} in the large z expansion of the branch of $\lambda^{\frac{2i-1}{2n}}(z)$ which is positive for large $z > 0$, and similarly

$$\theta_i^{[2]}(\tau_1, \dots, \tau_n) \equiv \theta_i^{[2]} := \begin{cases} \theta_0^2, & i = 0, \\ \frac{\tau_{n-1}}{4n-2}, & i = 1, \\ \frac{1}{i-1} \operatorname*{res}_{z=\infty} \tilde{\lambda}^{\frac{i-1}{n}}(z), & i \geq 2. \end{cases}$$

Large gap asymptotics for generic potential

Theorem (Cafasso, Claeys, G., '19)

Let $n \in \mathbb{N}$, $\tau_1, \dots, \tau_{n-1} \in \mathbb{R}$. As $s \to -\infty$, there exists a constant $C = C(n; \boldsymbol{\tau}) > 0$, such that we have the asymptotics

$$\log F(s) = -\sum_{\substack{j=0 \\ j \neq n+1}}^{2n} \frac{n^2}{(n+1-j)(2n+1-j)} \theta_j^{[2]} |s|^{\frac{2n-j+1}{n}} + c \log|s| + \log C + o(1),$$

with $c = -\frac{1}{8}$ if $n = 1$ and $c = -\frac{1}{2}$ otherwise. Moreover, the asymptotics can be improved to

$$q((-1)^{n+1}s) = \sum_{i=0}^{2n} \theta_i |s|^{\frac{1}{2n} - \frac{i}{n}} + \frac{c}{2\theta_0} |s|^{-2-\frac{1}{2n}} + \mathcal{O}\left(|s|^{-2-\frac{1}{n}}\right), \qquad as\ s \to -\infty.$$

The nitty gritty

$n = 1$ There are no parameters of deformation τ_i and our result gives: as $s \to -\infty$

$$\log F(s) = -\frac{|s|^3}{12} - \frac{1}{8}\log|s| + \log C + o(1) \qquad \text{(Tracy–Widom distribution).}$$

$n = 2$ In the first non-trivial case, we obtain as $s \to -\infty$

$$\log F(s) = -\frac{2}{45}\sqrt{6}|s|^{5/2} - \frac{1}{12}\tau_1|s|^2 - \frac{\sqrt{6}}{54}\tau_1^2|s|^{3/2} - \frac{\sqrt{6}}{432}\tau_1^4|s|^{1/2} - \frac{1}{2}\log|s| + \log C + o(1).$$

$n = 3$ We have two deformation parameters τ_1, τ_2: as $s \to -\infty$

$$\begin{aligned}\log F(s) = &-\frac{9}{560}20^{\frac{2}{3}}|s|^{7/3} - \frac{1}{20}\tau_2|s|^2 + \frac{3\sqrt[3]{20}}{1000}\left(10\tau_1 - 3\tau_2^2\right)|s|^{5/3}\\ &+ \frac{3}{2000}20^{\frac{2}{3}}\tau_2\left(5\tau_1 - \tau_2^2\right)|s|^{4/3} - \frac{\sqrt[3]{20}}{5000}\tau_2\left(50\tau_1^2 - 25\tau_2^2\tau_1 + 3\tau_2^4\right)|s|^{2/3}\\ &+ \frac{20^{\frac{2}{3}}}{900000}\left(1000\tau_1^3 - 1800\tau_2^2\tau_1^2 + 630\tau_2^4\tau_1 - 63\tau_2^6\right)|s|^{1/3}\\ &- \frac{1}{2}\log|s| + \log C + o(1)\end{aligned}$$

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Sketch of the proof

Like in the $s \to +\infty$, we would love to have a RHP with jumps of the type $\boldsymbol{J}(\zeta) = \boldsymbol{I} + \delta \boldsymbol{J}(\zeta)$, however, the phases Θ in the jumps diverge as $s \to -\infty$.

We will recur to the *steepest descent method* (Deift, Zhou, '92).

The strategy is to apply a sequence of invertible transformations

$$\boldsymbol{\Gamma} \mapsto \underbrace{\boldsymbol{\Psi} \mapsto \ldots \mapsto \boldsymbol{S}}_{g\text{-function}} \mapsto \boldsymbol{R}$$

in such away that, within the regime $s \ll -1$, the final RHP

$$\boldsymbol{R}(\zeta) = \boldsymbol{S}(\zeta)\boldsymbol{\Omega}^{-1}(\zeta)$$

has jumps close to the identity. We can then apply a small norm argument again: $\boldsymbol{R}(\zeta) = \boldsymbol{I} + \{\text{small}\}$ and

$$\boldsymbol{S}(\zeta) = \underbrace{(\boldsymbol{I} + \{\text{small}\})}_{\boldsymbol{R}(\zeta)} \cdot \underbrace{\boldsymbol{\Omega}(\zeta)}_{\substack{\text{“model”} \\ \text{problem}}}$$

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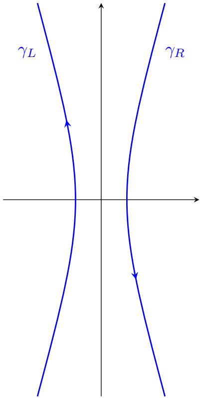
Sketch of the proof

We solve the RHP $\mathbf{\Gamma}$ with the use of a global parametrix $\boldsymbol{P}^{\infty}$ (and an explicit g-function) and two local Airy parametrices $\boldsymbol{P}^{\pm\zeta_0}$.

Step 0: the original problem $\mathbf{\Gamma}$



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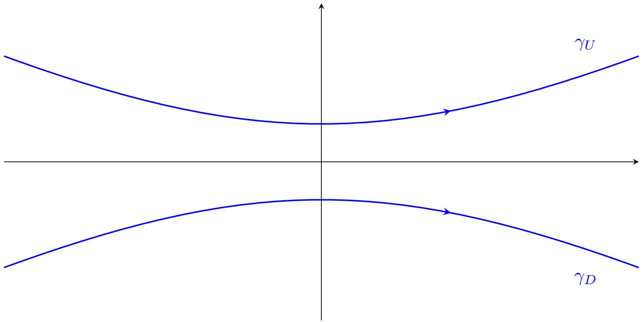
Step 1: $\mathbf{\Gamma} \mapsto \mathbf{\Psi}$, rotation



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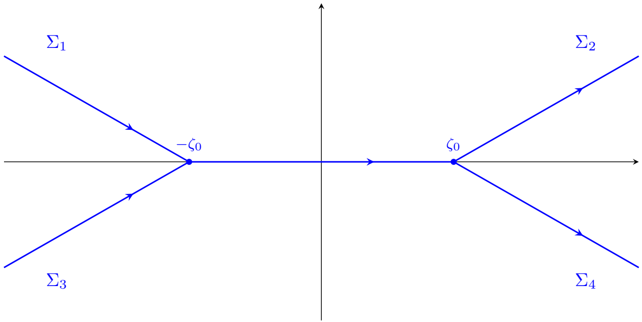
We solve the RHP $\boldsymbol{\Gamma}$ with the use of a global parametrix $\boldsymbol{P}^{\infty}$ (and an explicit g-function) and two local Airy parametrices $\boldsymbol{P}^{\pm\zeta_0}$.

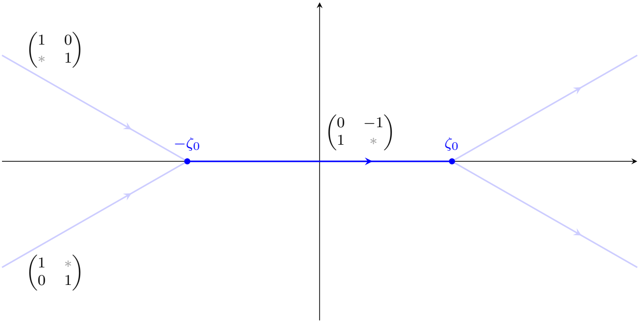
Step 2: $\boldsymbol{\Psi} \mapsto \boldsymbol{S}$, jump merging and rescaling; introduction of a g-function



Sketch of the proof

We solve the RHP $\mathbf{\Gamma}$ with the use of a global parametrix $\boldsymbol{P}^{\infty}$ (and an explicit g-function) and two local Airy parametrices $\boldsymbol{P}^{\pm\zeta_0}$.

Step 3: the magic of the g-function



A note on the g-function

Ansatz:

$$g(\zeta; s) = \sum_{j=1}^{n} c_j (\zeta^2 - \zeta_0^2)^{\frac{2j+1}{2}}$$

with $(\zeta^2 - \zeta_0^2)^{\frac{2j+1}{2}}$ analytic on $\mathbb{C} \setminus [-\zeta_0, \zeta_0]$ and such that it behaves like ζ^{2j+1} as $\zeta \to \infty$.

We fix the constants c_j and the branch point $\zeta_0 > 0$ by imposing the asymptotic behaviour

$$|s|^{\frac{2n+1}{2n}} g(\zeta) = \underbrace{\Theta(|s|^{\frac{1}{2n}} \zeta)}_{\text{jump phase}} + \frac{g_1(s)}{\zeta} + \mathcal{O}(\zeta^{-2}), \qquad \text{as } \zeta \to \infty.$$

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This gives:

$$c_{n-m} = \sum_{k=0}^{m} (-1)^{m-k} 2^{2(n-m+k)-1} \tau_{n-m+k} |s|^{-\frac{m-k}{n}} \frac{\Gamma\left(n-m+k+\frac{1}{2}\right)}{k!\Gamma\left(n-m+\frac{3}{2}\right)} \zeta_0^{2k}$$

$$g_1(s) = \frac{1}{2} \sum_{k=1}^{n} (-1)^{n-k} \tau_k \binom{2k}{k-1} |s|^{\frac{2k+1}{2n}} \zeta_0^{2k+2}.$$

and $\zeta_0 = \zeta_0(s)$ defined implicitly as

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k |s|^{\frac{k-n}{n}} \zeta_0^{2k} = 1.$$

We just need to solve the equation for ζ_0, at least in the $s \to -\infty$ regime...

This gives:

$$c_{n-m} = \sum_{k=0}^{m} (-1)^{m-k} 2^{2(n-m+k)-1} \tau_{n-m+k} |s|^{-\frac{m-k}{n}} \frac{\Gamma\left(n-m+k+\frac{1}{2}\right)}{k!\Gamma\left(n-m+\frac{3}{2}\right)} \zeta_0^{2k}$$

$$g_1(s) = \frac{1}{2} \sum_{k=1}^{n} (-1)^{n-k} \tau_k \binom{2k}{k-1} |s|^{\frac{2k+1}{2n}} \zeta_0^{2k+2}.$$

and $\zeta_0 = \zeta_0(s)$ defined implicitly as

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k |s|^{\frac{k-n}{n}} \zeta_0^{2k} = 1.$$

We just need to solve the equation for ζ_0, at least in the $s \to -\infty$ regime...

$$\lambda(z) := \sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k z^{2k}, \qquad \tilde{\lambda}(z) := \sum_{k=1}^{n} (-1)^{n-k} \binom{2k}{k} \tau_k z^{k}$$

and define

$$\theta_i := \begin{cases} \binom{2n}{n}^{-\frac{1}{2n}}, & i = 0, \\ \frac{1}{2i-1} \operatorname*{res}_{z=\infty} \lambda^{\frac{2i-1}{2n}}(z), & i \geq 1. \end{cases}$$

$$\theta_i^{[2]} := \begin{cases} \theta_0, & i = 0, \\ \frac{\tau_{n-1}}{4n-2}, & i = 1, \\ \frac{1}{i-1} \operatorname*{res}_{z=\infty} \tilde{\lambda}^{\frac{i-1}{n}}(z), & i \geq 2. \end{cases}$$

Then,

$$\zeta_0(s) \sim \sum_{i=0}^{\infty} \theta_i |s|^{-\frac{i}{n}} \quad \text{and} \quad \zeta_0^2(s) \sim \sum_{i=0}^{\infty} \theta_i^{[2]} |s|^{-\frac{i}{n}}, \qquad \text{as } s \to -\infty.$$

Note: the coefficients $\{\theta_i\}_{i=0}^{\infty}$ and $\{\theta_i^{[2]}\}_{i=0}^{\infty}$ are related to *topological minimal models of type A_n* and to flat coordinates for the corresponding Frobenius manifolds.

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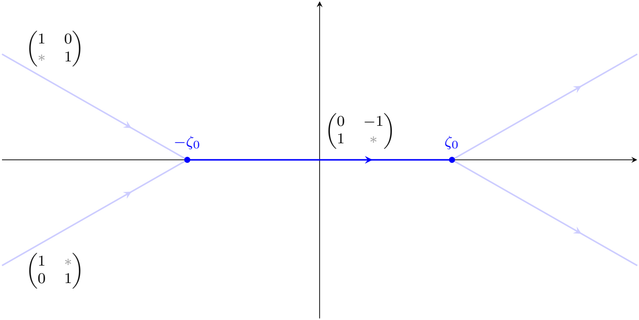
Sketch of the proof

We solve the RHP $\boldsymbol{\Gamma}$ with the use of a global parametrix $\boldsymbol{P}^{\infty}$ (and an explicit g-function) and two local Airy parametrices $\boldsymbol{P}^{\pm\zeta_0}$.

Step 3: the magic of the g-function



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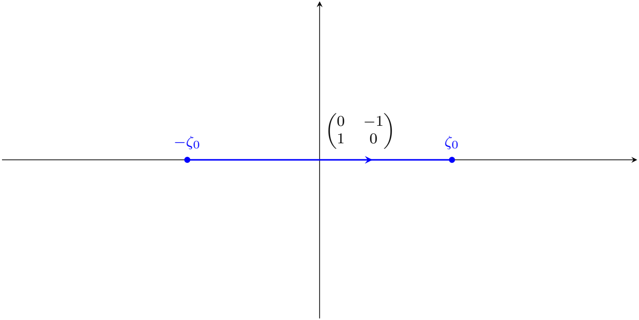

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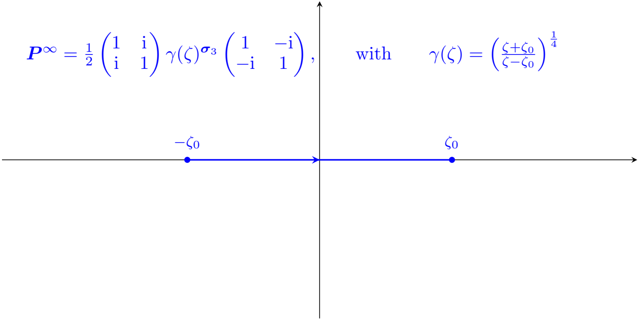
Step 4: build the model problem $\boldsymbol{\Omega}$ with parametrices $\boldsymbol{P}^{\infty}$ and $\boldsymbol{P}^{\pm\zeta_0}$

$$\boldsymbol{P}^{\infty} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \gamma(\zeta)^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}, \qquad \text{with} \qquad \gamma(\zeta) = \left(\frac{\zeta+\zeta_0}{\zeta-\zeta_0}\right)^{\frac{1}{4}}$$



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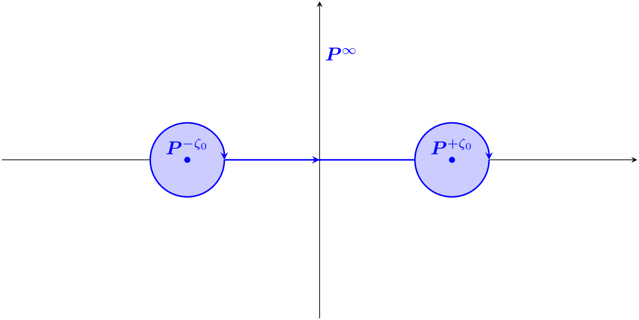

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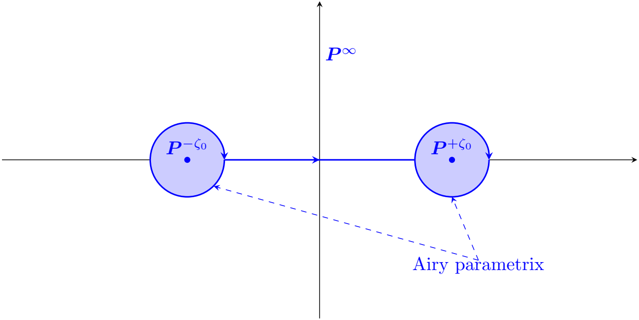

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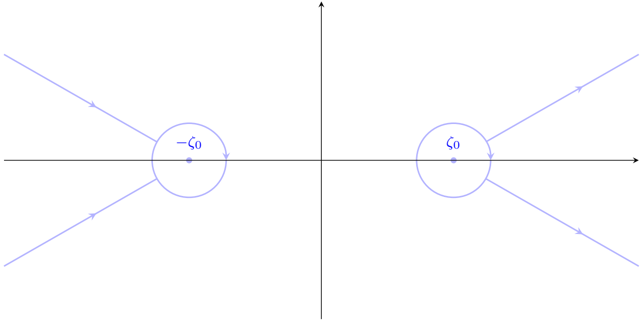
Step 5: taking care of the other small-norm jumps (the remainder $\boldsymbol{R} := \boldsymbol{S}\boldsymbol{\Omega}^{-1}$)



A note on the remainder problem

We can compute the asymptotic expansion of the remainder to arbitrary order of accuracy:

$$\boldsymbol{R}(\zeta) = \boldsymbol{I} + \boldsymbol{R}^{(1)}(\zeta)|s|^{-\frac{2n+1}{2n}} + \boldsymbol{R}^{(2)}(\zeta)|s|^{-\frac{2n+1}{n}} + \mathcal{O}\left(|s|^{-\frac{6n+3}{2n}}\right),$$

for some matrices $\boldsymbol{R}^{(1)}(\zeta), \boldsymbol{R}^{(2)}(\zeta), \ldots$ which can be computed via a recursive procedure.

In particular,

$$\boldsymbol{R}^{(1)}(\zeta) = \int_{\partial\mathcal{C}_{+\zeta_0} \cup \partial\mathcal{C}_{-\zeta_0}} \frac{\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{R}}(w)}{w - \zeta} \frac{\mathrm{d}w}{2\pi\mathrm{i}} = \frac{\boldsymbol{R}_1^{(1)}}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right), \qquad \text{as } \zeta \to \infty.$$

Back to gap probabilities

Following backwards all the transformations $\boldsymbol{\Gamma} \mapsto \boldsymbol{\Psi} \mapsto \boldsymbol{S} \mapsto \boldsymbol{R} = \boldsymbol{S}\boldsymbol{\Omega}^{-1}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \log F(s) = \Gamma_{1;11} = \ldots = 2\mathrm{i}|s|^{\frac{1}{2n}} R_{1,11} + 2|s|^{\frac{1}{2n}} g_1(s)$$

where g_1 is the residue of the g function at $\zeta = \infty$.

By explicitly calculating the terms involved (in the regime $s \to -\infty$), we obtain

$$\log F(s) = -c \log|s| - \sum_{\substack{j=0 \\ j \neq n+1}}^{2n} \frac{n^2}{(n+1-j)(2n+1-j)} \theta_j^{[2]} |s|^{\frac{2n-j+1}{n}} + \log C + o(1).$$

with $c = \frac{1}{8}$ for $n = 1$ and $c = \frac{1}{2}$ otherwise.

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Back to $q\left((-1)^{n+1}s\right)$

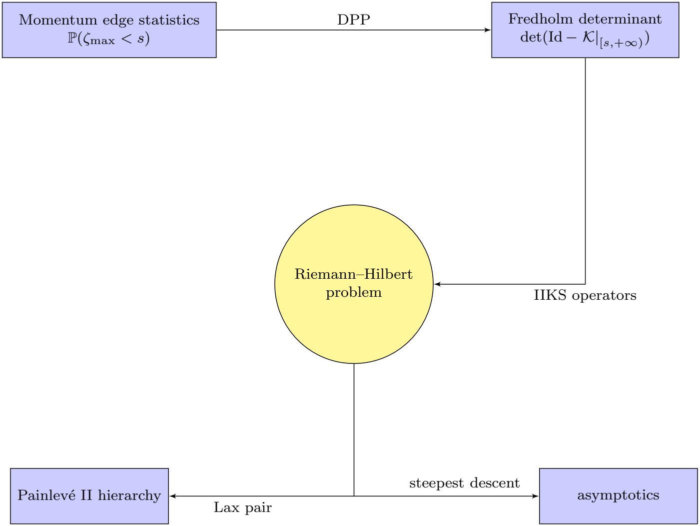
Similarly, we have

$$q\big((-1)^{n+1}s\big) = \ldots = \zeta_0(s)|s|^{\frac{1}{2n}} + 2\mathrm{i}|s|^{\frac{1}{2n}} R_{1,12}.$$

Explicitly,

$$q\big((-1)^{n+1}s\big) = \sum_{i=0}^{2n} \theta_i |s|^{\frac{1}{2n}-\frac{i}{n}} + \frac{c}{2\theta_0}|s|^{-2-\frac{1}{2n}} + \mathcal{O}\left(|s|^{-2-\frac{1}{n}}\right), \quad \text{as } s \to -\infty.$$

Conclusion



Momentum edge statistics
$\mathbb{P}(\zeta_{\max} < s)$
DPP
Fredholm determinant
$\det(\mathrm{Id} - \mathcal{K}|_{[s,+\infty)})$
IIKS operators
Riemann–Hilbert problem
Lax pair
Painlevé II hierarchy
steepest descent
asymptotics

Food for thought

Higher-order Airy kernels and their Fredholm determinant are ubiquitous...

- **universality**:
  - (Betea–Bouttier–Walsh, '20) Fredholm Determinant of higher-order Airy is connected to Schur measures (random partitions) where the edge fluctuation is of the order $\frac{1}{2n+1}$;
  - (Kimura–Zahabi, '20-'21) further work on random partition and generating functions in gauge theory.
- recent developments:
  - (Tarricone, '20) matrix-valued version of the higher-order Airy function and non-commutative PII hierarchy;
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- open problems:
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Thank you!