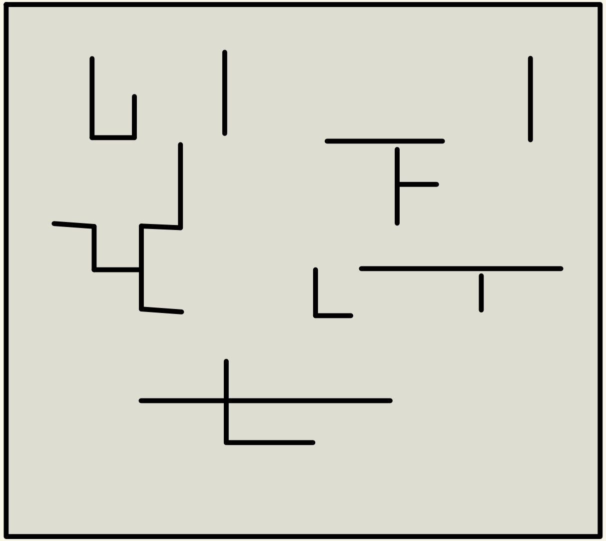
Random forests and non-linear sigma models

(and what these have to do with random matrices)



Roland Bauerschmidt (Cambridge)

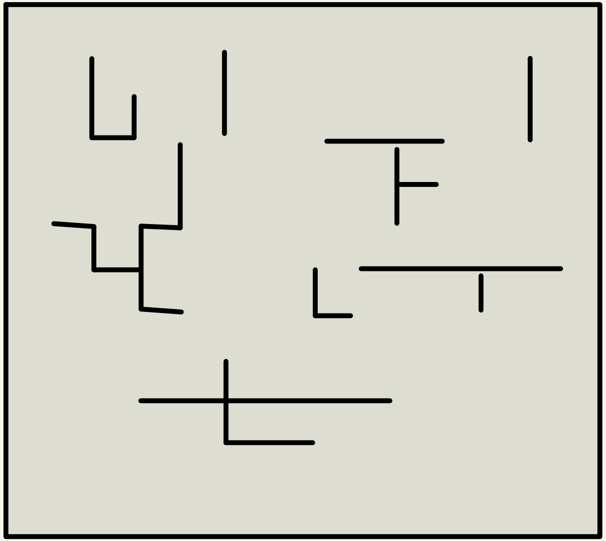
with N. Crawford, T. Helmuth, A. Swan

The arboreal gas

$G = (\Lambda, E)$ finite graph

$F = (\Lambda, E(F)) \subset G$ is a forest if it has no cycles

$$\mathbb{P}_\beta^G(F) = \frac{1}{Z_\beta} \beta^{|E(F)|} \mathbb{1}(F \text{ is a forest})$$



The arboreal gas

unrooted

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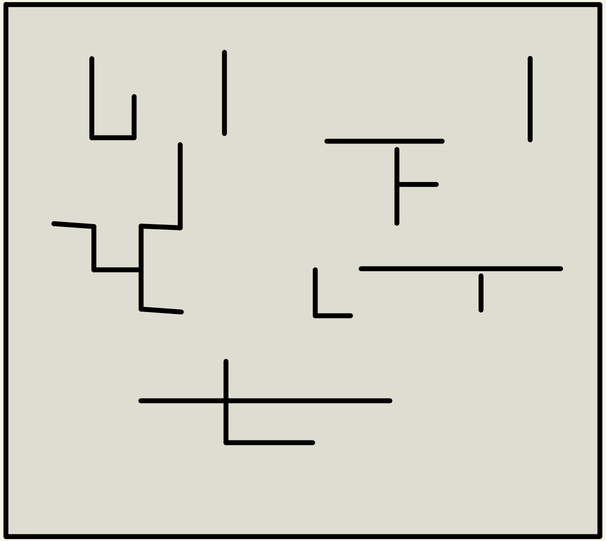

The arboreal gas

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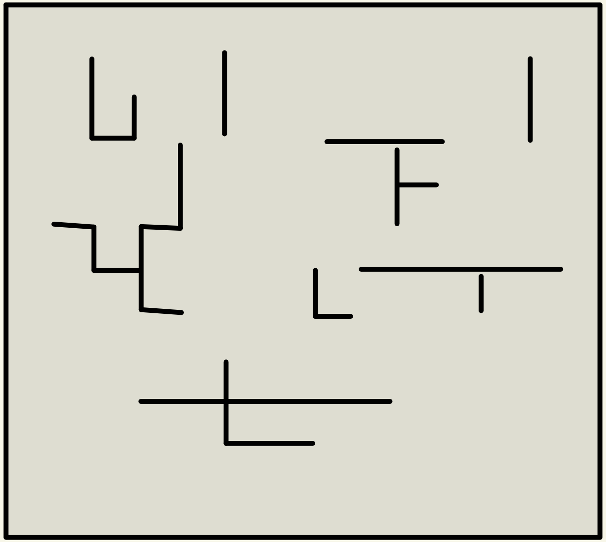
$F=(\Lambda,E(F))\subset G$ is a forest if it has no cycles

$$\mathbb{P}^G_\beta(F)=\frac{1}{Z_\beta}\beta^{|E(F)|}\,\mathbb{1}(F \text{ is a forest})$$

- Fortuin-Kasteleyn: $q\to 0$ limit of random cluster model
- Lubensky-Isaacson: model for gelation of polymers
- Caracciolo-Jacobsen-Saleur-Sokal-Sportiello: SUSY
- Luczak-Pittel, Martin-Yeo: analysis on complete graph
- Pemantle, Kahn, Grimmett-Winkler: Conj. negative dependence
- Brändén-Huh, Anari et al.: a main example of Lorentzian polyn.

When does the arboreal gas percolate?



When does the arboreal gas percolate?

<u>Prop.</u> On any graph, the arboreal gas is stochastically dominated by bond percolation with $p = \beta/(1+\beta)$.

$\longrightarrow$ Arboreal gas does not percolate on $\mathbb{Z}^d$ if β small

When does the arboreal gas percolate?

<u>Prop</u>. On any graph, the arboreal gas is stochastically dominated by bond percolation with $p = \beta/(1+\beta)$.

→ Arboreal gas does not percolate on $\mathbb{Z}^d$ if β small

<u>Thm</u> (BCHS). In $d=2$, no percolation for any $\beta > 0$.

<u>Thm</u> (BCH). In $d \geq 3$, percolation for $\beta > \beta_0$.

When does the arboreal gas percolate?

<u>Thm.</u> (BCHS). Let $d=2$. For any $\beta>0$, $\exists\, c(\beta)>0$ s.t.

$$\mathbb{P}_\beta^\Lambda(0 \leftrightarrow x) \leq O(|x|^{-c(\beta)}) \quad \text{for all } \Lambda \subset \mathbb{Z}^2.$$

<u>Thm.</u> (BCH). Let $d \geq 3$. There are $\beta_0 \in (0,\infty)$ and $\zeta(\beta) = 1 - O(1/\beta)$ s.t. for $\beta \geq \beta_0$,

$$\mathbb{P}_\beta^\Lambda(0 \leftrightarrow x) = \zeta(\beta) + \frac{c(\beta)}{\beta\,|x|^{d-2}} + O\left(\frac{1}{\beta|x|^{d-2+\kappa}} + \frac{1}{\beta L^{\kappa N}}\right)$$

if Λ is a torus $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$, $L \geq L_0(d)$.

Matrix tree theorem

delete a row and column

$$\#\text{spanning trees on } G = \det(-\Delta^{\circ})$$

$$\mathbb{P}^{UST}(e_1 \in T, \dots, e_k \in T) = \det(K(e_i, e_j))_{i,j}$$

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$\xi_x, \eta_x, x \in \Lambda$ anticommuting variables

$$e^{(\xi, -\Delta^\circ \eta)}$$

expand $e^{\sum \xi_x (-\Delta^\circ)_{xy} \eta_y}$

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$\xi_x, \eta_x, x \in \Lambda$ anticommuting variables

$$\#ST = \int \partial_\eta \partial_\xi \, e^{(\xi, -\Delta^{\circ} \eta)} \quad \text{Grassmann int.}$$

project onto top degree coefficient

expand $e^{\sum \xi_x (-\Delta^{\circ})_{xy} \eta_y}$

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Nonlinear σ-model (NLSM)

Spins u_i , $i \in \Lambda$

$$\langle F \rangle_\beta = \frac{1}{Z_\beta} \int F \, e^{-\frac{\beta}{2} \sum_{ij \in E} (u_i - u_j) \cdot (u_i - u_j)} \prod_i du_i$$

Nonlinear σ-model (NLSM)

Spins u_i , $i \in \Lambda$

Haar measure

$$\langle F \rangle_\beta = \frac{1}{Z_\beta} \int F \, e^{-\frac{\beta}{2} \sum_{ij \in E} (u_i - u_j)\cdot(u_i - u_j)} \prod_i du_i$$

- Ising model: $u_i \in \{\pm 1\}$
- XY model: $u_i \in S^1$
- Heisenberg model: $u_i \in S^2$
- Hyperbolic σ-models: $u_i \in \mathbb{H}^n$

spins take values in a symmetric space

Nonlinear σ-model (NLSM)

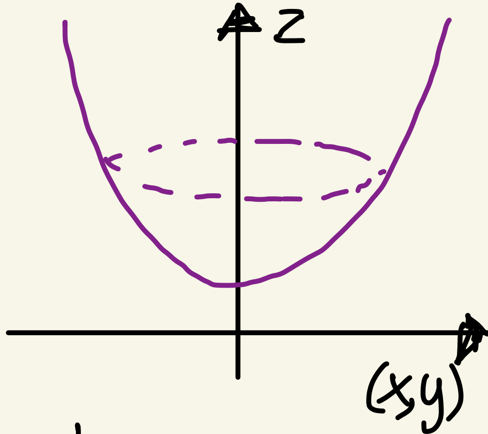
Spins u_i, $i \in \Lambda$

$z_j = u_j \cdot e$, some fixed direction e

$$\langle F \rangle_{\beta,h} = \frac{1}{Z_{\beta,h}} \int F \, e^{-\frac{\beta}{2}\sum_{ij \in E}(u_i - u_j)\cdot(u_i - u_j)} \, e^{-h\sum_{i\in\Lambda} z_i} \prod_i du_i$$

Ising model:	$u_i \in \{\pm 1\}$
XY model:	$u_i \in \mathbb{S}^1$
Heisenberg model:	$u_i \in \mathbb{S}^2$
Hyperbolic σ-models:	$u_i \in \mathbb{H}^n$

Hyperbolic plane $\mathbb{H}^2$



Vectors $u_i = (x_i, y_i, z_i)$ in $\mathbb{R}^3$

$z_i > 0$

with $u_i \cdot u_i = -1$

where $u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $SO^+(2,1)$ invariant

Concretely $z_i = \sqrt{1 + x_i^2 + y_i^2}$

$$\int_{\mathbb{H}^2} F = \underbrace{\int dx_i \, dy_i}_{\text{Lebesgue integral}} \; \underbrace{\frac{1}{z_i}}_{SO(2,1) \text{ inv. volume form}} F$$

Fermionic hyperbolic plane $\mathbb{H}^{0|2}$

Supervectors $u_i = (\underbrace{\xi_i, \eta_i}_{\text{odd}}, \underbrace{z_i}_{\text{even}})$ in $\mathbb{R}^{1|2}$

$$\left.\begin{array}{ll} \text{with} & u_i \cdot u_i = -1 \\ \text{where} & u_i \cdot u_j = \eta_i \xi_j + \eta_j \xi_i - z_i z_j \end{array}\right\} \begin{array}{c} OSp^+(1|2) \\ \text{invariant} \end{array}$$

Concretely $z_i = \sqrt{1 - 2\xi_i \eta_i} = 1 - \xi_i \eta_i$

$$\int_{\mathbb{H}^{0|2}} F = \underbrace{\int \partial_{\eta_i} \partial_{\xi_i}}_{\text{Grassmann integral}} \underbrace{\frac{1}{z_i}}_{OSp(1|2) \text{ inv. volume form}} F$$

$\mathbb{H}^{0|2}$ model $\longrightarrow$ arboreal gas

$$\langle F \rangle_{\beta,h} \propto \int_{(\mathbb{H}^{0|2})^\Lambda} e^{\frac{\beta}{2}(u,\Delta u) - h\sum_i z_i} F$$

$(u,-\Delta u) = \sum_{ij} (u_i - u_j) \cdot (u_i - u_j)$

OSp(1|2) inner prod.

(Λ finite lattice)

<u>Thm.</u> (CJSSS, BCHS) Nonlinear Matrix Tree Thm.

$$\mathbb{P}_{\beta,0}(0 \longleftrightarrow x) = -\langle u_0 \cdot u_x \rangle_{\beta,0}$$

$$\mathbb{P}_{\beta,0}(e_1 \in F, \ldots, e_k \in F) = \left(\frac{-\beta}{2}\right)^k \left\langle \prod_{i=1}^k (\nabla_{e_i} u) \cdot (\nabla_{e_i} u) \right\rangle_{\beta,0}$$

$\vdots$

$\mathbb{H}^{0|2}$ model $\longrightarrow$ arboreal gas

$$\mathbb{P}_{\beta,0}(0 \longleftrightarrow x) = -\langle u_0 \cdot u_x \rangle_{\beta,0}$$

$\Rightarrow$ percolation $\leftrightarrow$ spontaneously broken symmetry

"spins point in same direction over large dist."

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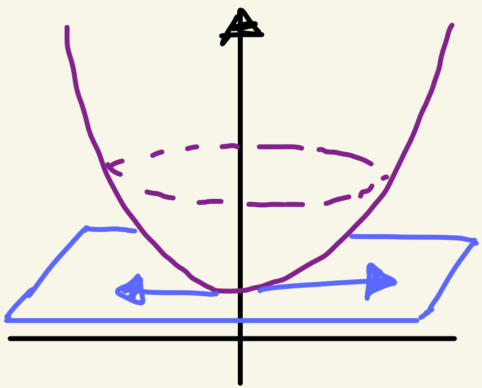

Continuous symmetry

$\rightarrow$ expect diffusive corrections
(free field)

Ward identity: $\dfrac{\langle z_i \rangle_{\beta,h}}{h} = \sum_j \langle \xi_i \eta_j \rangle_{\beta,h}$

'Magic formula' $\longrightarrow$ No percolation in $d=2$

Mermin–Wagner theorem: "Continuous symmetries cannot be spontaneously broken in 2d."

→ Does not directly apply to superspaces, but...

<u>Thm</u> (BCHS). For the arboreal gas,

'Magic formula' (c.f. Sabot–Tarres, Diaconis)

$$\mathbb{P}_\beta(0 \longleftrightarrow x) \propto \int e^{t_x} \, e^{-\sum_{i\sim j} \beta \cosh(t_i - t_j)} \left(\det(-\Delta^{o}_{\beta(t)})\right)^{3/2} \prod_{i\neq 0} e^{-3t_i} \, dt_i .$$

→ Versions of Mermin–Wagner can be adapted to RHS.

Review of Anderson transition

$\Lambda_L = \mathbb{Z}^d / L\mathbb{Z}^d$, $d \geq 2$

$H_L = -\Delta + \lambda V$, V i.i.d. (or other finite range RM)

DOS: $\rho_{E,\eta} = \mathbb{E}\,\mathrm{Im}(H-E-i\eta)^{-1}(0,0)$

2-pt funct: $\tau_{E,\eta}(j) = \mathbb{E}|(H-E-i\eta)^{-1}(0,j)|^2$, $j \in \Lambda$

Ward identity: $\frac{1}{\eta}\rho_{E,\eta} = \sum_j \tau_{E,\eta}(j)$

Non-linear σ-model: $\frac{1}{\eta}\langle z_0\rangle_{\beta,\eta} = \sum_j \langle x_0 x_j\rangle_{\beta,\eta}$

Review of Anderson transition

$\Lambda_L = \mathbb{Z}^d / L\mathbb{Z}^d$, $d \geq 2$

$H_L = -\Delta + \lambda V$, V i.i.d. Gaussian

DOS: $\rho_{E,h} = \mathbb{E}\, \mathrm{Im}(H-E-ih)^{-1}(0,0)$

2-pt funct: $\tau_{E,h}(x) = \mathbb{E}|(H-E-ih)^{-1}(0,x)|^2$

Localisation: $\tau_{E,h}(x) \lesssim \frac{1}{h} e^{-|x|/\ell}$ (FS: $\lambda \gg 1$)

Delocalisation: $\tau_{E,h}(0) \lesssim 1$ as $L \to \infty$ then $h \downarrow 0$

\+ Diffusion: $\tau_{E,h}(x) \sim D(E)\, |x|^{-(d-2)}$ ((Conj.: $\lambda \ll 1$, $\rho_E > 0$)

Many questions

Universality. Is the order statistics of the component sizes on $\mathbb{Z}^d / L\mathbb{Z}^d$, $d \geq 3$, $\beta > \beta_0$, the same as on the complete graph?

Expected similarly to universality of Wigner-Dyson.

Distributions known explicitly for complete graph:

$$\left(T^{(1)} - \alpha N\right) N^{-2/3} \longrightarrow \text{limiting distribution}$$

$$T^{(k)} N^{-2/3} \longrightarrow \text{limiting distribution}$$

k-th largest component

Many questions

Negative correlation conjecture. (Pemantle, Grimmett-Winkler, Kahn)

$$\mathbb{P}_\beta(e_1 \in F, e_2 \in F) \le \mathbb{P}_\beta(e_1 \in F)\,\mathbb{P}_\beta(e_2 \in F)$$

Known only in limit $\beta \to \infty$ (UST).

Brändén-Huh proved

$$\mathbb{P}_\beta(e_1 \in F, e_2 \in F) \le 2\,\mathbb{P}_\beta(e_1 \in F)\,\mathbb{P}_\beta(e_2 \in F)$$

and the partition function with edge dependent weights is log concave (Lorentzian polynomial).

Thank you!

Magic formula for ERRW and VRJP

ERRW: Edge e has weight $\alpha + N_e(t)$

# of crossings of edge e

parameter

α small: strong reinforcement

α large: weak reinforcement

Magic formula for ERRW and VRJP

ERRW: Edge e has weight $\alpha + N_e(t)$

random conductances

Diaconis-Freedman
& Coppersmith-Diaconis: $\mathbb{P}_0^{ERRW(\alpha)} = \int \mathbb{P}_0^{SRW(c)} \, d\mu_\alpha(c)$

walk starting at vertex 0

mixing measure given by 'magic formula'

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VRJP: Vertex reinforced version

Sabot - Tarrès: $\mathbb{P}_0^{VRJP(\beta)} = \int \mathbb{P}_0^{SRW(c(t))} \, d\nu_\beta(t)$

Thm (Sabot–Tarrès). Mixing measure for VRJP:

$$d\nu_\beta(t) \propto e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^0(-\Delta_{\beta(t)})\right)^{1/2}\prod_{i\neq 0} e^{-t_i}\,dt_i$$

$$(\Delta_{\beta(t)}f)_i = \sum_{j\sim i}\beta e^{t_i+t_j}(f_j-f_i)$$

Thm (BCHS). For the arboreal gas,

$$\mathbb{P}_\beta(0\leftrightarrow x) \propto \int e^{t_x}\, e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)} \left(\det{}^0(-\Delta_{\beta(t)})\right)^{3/2}\prod_{i\neq 0} e^{-3t_i}\,dt_i.$$

Three sources for the magic.

Hyperbolic σ-model

$$\propto e^{-\frac{\beta}{2}\sum_{ij\in E}(u_i-u_j)\cdot(u_i-u_j)} \prod du_i$$

$$= e^{-\beta\sum_{ij\in E}\left(\cosh(t_i-t_j)+\frac{1}{2}e^{t_i+t_j}(s_i-s_j)^2\right)} \prod_i e^{-(n-1)t_i} dt_i\, ds_i$$

Gaussian in s!

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Gaussian in s!

Thus t-marginal is

$$e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^0(-\Delta_{\beta(t)})\right)^{-(n-1)/2}\prod_{i\neq 0}e^{-(n-1)t_i}dt_i$$

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$n=0$

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$n=-2$

What is $\mathbb{H}^n$ if $n\leq 0$?

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What is $\mathbb{H}^n$ if $n\leq 0$? $\mathbb{H}^{p|2q}$ has dim. $n=p-2q$.