Counting points in boxes : the Riesz family & friends

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Riesz interactions

- Riesz interaction, parameter s dimension d:

$$g_{s,d}(x) := \frac{1}{s}\|x\|^{-s} \text{ in } \mathbb{R}^d.$$

"Long-range" if $s \in [d-2, d)$, "short-range" if $s \in (d, +\infty)$.
Will focus on $s \geq d-2$.

- Special cases:
  - $s = d - 2$ is the Coulomb kernel (cf. Sylvia's talk):

$$-|x| \ (d=1), \quad -\log|x| \ (d=2), \quad \frac{1}{|x|} \ (d=3).$$

  - $s = 0$, $d = 1, 2$ corresponds to $-\log|x|$ (logarithmic interaction).

- Fundamental solution of fractional Laplacian $-\Delta^{\frac{d-s}{2}} g_{s,d} \propto \delta_0$.
Examples: true Laplacian for $s = d - 2$ (Coulomb cases), half-laplacian for one-dimensional log-gas.

Riesz system

- N point charges with pairwise interaction through Riesz kernel
- ∞ point charges with pairwise interaction through Riesz kernel

The second case requires a definition, especially in the "long-range case" $\mathsf{s} \leq \mathsf{d}$ for which:

$$\int_0^{+\infty} \mathsf{g}_{\mathsf{s},\mathsf{d}}(r) r^{\mathsf{d}-1} dr = +\infty.$$

- The effect of one particle at 0 is felt everywhere in the system.
- Interaction energy is not spatially additive (even up to a small error).
- What is the energy of an infinite system (even per unit volume)?

Finite Riesz system - energy

- $\mathrm{d} \geq 1$, $N \geq 1$
- $\mathrm{s} \in (\mathrm{d}-2, \mathrm{d})$ (long-range) or $\mathrm{s} \in [d, +\infty)$ (short-range / hypersingular).
- $\mathrm{X}_N := (x_1, \dots, x_N)$ positions of point particles in $\mathbb{R}^{\mathrm{d}}$
- Riesz interaction energy:

$$\frac{1}{2} \sum_{1 \leq i \neq j \leq N} \mathrm{g}_{\mathrm{s}}(x_i - x_j).$$

- $V_N : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ external potential/field/weight, fairly smooth.
- Total energy of the system in state X_N:

$$\mathrm{H}_N^V(\mathrm{X}_N) := \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \mathrm{g}(x_i - x_j) + \sum_{i=1}^{N} V_N(x_i).$$

Riesz gas - Gibbs measure

(Finite) Riesz gaz

- $\beta > 0$ inverse temperature parameter (may depend on N)
- **Canonical Gibbs measure**

$$d\mathbb{P}_{N,\beta}(\mathrm{X}_N) := \frac{1}{\mathrm{Z}_{N,\beta}} \exp\left(-\beta \mathrm{H}_N^V(\mathrm{X}_N)\right) d\mathrm{X}_N$$

- $\mathrm{Z}_{N,\beta}$ partition function:

$$\mathrm{Z}_{N,\beta} := \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \exp\left(-\beta \mathrm{H}_N(\mathrm{X}_N)\right) d\mathrm{X}_N.$$

Alternative to V: choose a density μ supported in a domain Σ and take $\prod_{i=1}^N \mu(x)_i dx_i$ as a reference measure. In physics papers: the background density $\mu =$ the uniform measure on a large box of volume N.

Defines a three-parameter family of stat. mech. systems $(\mathsf{d}, \mathsf{s}, \beta)$: called (here) the (finite-N) "Riesz gases". These form finite point processes on $\mathbb{R}^{\mathsf{d}}$ (could look at manifolds...). In some cases, the $N \to \infty$ ("thermodynamic limit") exists = infinite/limiting point process.

These are the points we would like to "count".

Motivations

- Includes Coulomb, log-gases (cf. Sylvia's talk and motivations *therein*).
- Restricted Coulomb potentials e.g. particles interacting through the "normal" $|x|^{-1}$ interaction but forced to live on a line, on a two-dimensional surface...

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- Interpolates between various interesting situations e.g. in $\mathrm{d} = 1$:
  - $\mathrm{s} = -1$, 1d Coulomb gas $g(x) = -|x|$
  - $\mathrm{s} = 0$, 1d log-gas $g(x) = -\log |x|$
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  - $s = 1$, 3d Coulomb gas restricted to the line...
- For fixed d, the parameter s determines the "long-range-ness" (behavior at infinity) and the singularity = repulsion at the origin. Coulomb cases $s = d - 2$ are *very* long-range, $s = d$ not so much, $d \ll s$ is short-range but *very* singular.
- Extra temperature parameter $\beta \to$ stat. phys. Transitions as β varies?
- Role of V? Universality?

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- The 1d Coulomb gas $\mathsf{d} = 1, \mathsf{s} = -1, \beta > 0$ was studied by Kunz, Baxter. Admits a particularly nice structure.

Some friends

- Zeroes of Gaussian Analytic Functions (GAF). $(a_k)_{k\geq 0}$ iid standard complex Gaussian r.v.

$$f(z) = \sum_{k=0}^{+\infty} \frac{a_k}{\sqrt{k!}} z^k.$$

  Almost surely an entire function $\to$ random infinite collection of points in $\mathbb{C}$. Invariant under isometries. Some kind of algebraic structure for correlation functions, but mostly analytic techniques.
  A good friend of the Ginibre point process!
- Lattices, lattices + shift, lattices + random iid perturbations (+ shift). Potential friends of Riesz gases as $\beta \to +\infty$ (energy minimization).
- Poisson process: a reference & and friend of Riesz gases as $\beta \to 0$.

More friends?

- Zeroes of Kac polynomials: a finite point process in $\mathbb{R}^2$. The zeroes gather around the unit circle. Could be compared (to some extent) to 1d log-gases, to circular ensembles.
- Various random analytic functions in various domains.
- Hierarchical models (à la Dyson).
- Discrete particle systems (discrete β-ensembles).
- Your favorite point process?

Counting points in boxes

$\mathbf{X}$ the random point configuration: N points in a domain Λ_N or infinitely many points in $\mathbb{R}^{\mathrm{d}}$.
Take $\Lambda \subset \Lambda_N$ or $\Lambda \subset \mathbb{R}^{\mathrm{d}}$. Number of points $\#\mathbf{X} \cap \Lambda$ random.

1. N points in Λ_N. Assume volume $|\Lambda_N| = N$. Is it true that:

$$\mathbb{E}\left[\#\mathbf{X} \cap \Lambda\right] = |\Lambda| \; ??$$

   $\rightarrow$ not even clear (**at all**).

2. Infinite system in $\mathbb{R}^{\mathrm{d}}$, if stationary then constant intensity (assume $= 1$), and we have:

$$\mathbb{E}\left[\#\mathbf{X} \cap \Lambda\right] = |\Lambda| \text{ for every finite box } \Lambda.$$

Relevant quantities: $\#\mathbf{X} \cap \Lambda - |\Lambda|$ (discrepancy)
$\#\mathbf{X} \cap \Lambda - \mathbb{E}\left[\#\mathbf{X} \cap \Lambda\right]$ (charge fluctuation).

Hyperuniformity

Typical size of discrepancies / charge fluctuations?

- For a Poisson point process (intensity 1) $\mathrm{Var}(\#\mathbf{X} \cap \Lambda) = |\Lambda|$. Same (almost) for N iid points uniformly in a box of volume N. Typical charge fluctuation $= |\Lambda|^{\frac{1}{2}}$.

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$$\frac{\mathrm{Var}(\#\mathbf{X} \cap \Lambda)}{|\Lambda|} \to 0 \text{ as } |\Lambda| \to \infty.$$

  Variance in $B(0, R) \ll R^{\mathrm{d}}$, typical discrepancy $\ll R^{\mathrm{d}/2}$.

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- Best possible (random) case: "Type I" hyperuniform, variance in ball $B(0, R) \simeq R^{\mathrm{d}-1}$.
- Hyperuniformity $\leftrightarrow$ control of typical discrepancy. Related notion: equidistribution (maximal discrepancy).

(Number-)rigidity

Fix $\Lambda \subset \mathbb{R}^{d}$ bounded. Knowing $\mathbf{X} \cap \Lambda^{c}$, can I determine:

- $\#\mathbf{X} \cap \Lambda$? Yes $\to$ number-rigid. (not relevant for fixed N.)
- The center of mass in Λ? Yes $\to$ 2-rigid ("center of mass"-rigid).
- Higher moments of $\mathbf{X}$ in Λ (think $\Lambda \subset \mathbb{R}$ or $\mathcal{C}$)? $\to$ higher-rigidity.
- $\mathbf{X} \cap \Lambda$ completely ?! Yes $\to$ fully rigid.

S. Ghosh, Peres, Lebowitz
Remark: there are non-deterministic examples of high/full rigidity! (Ghosh-Krishnapur, Kiro-Nishry).

The JLM law

Assume we know $\mathbb{E}[\#\mathbf{X} \cap \Lambda] = |\Lambda|$ (e.g. infinite, translation-invariant system). Consider charge fluctuations $\#\mathbf{X} \cap \Lambda - |\Lambda|$. Ask:

$$\mathbb{P}\left[\#\mathbf{X} \cap \Lambda - |\Lambda| \geq Q\right] \simeq ?$$

Deviation estimates for large excess/defects of particles?
Depending on how large Q is (regimes of deviations), the price to pay might differ.
The Jancovici-Lebowitz-Manificat law (for $\mathrm{d} = 2$).

$$\mathbb{P}\left[\#\mathbf{X} \cap B(0,R) - \pi R^2 \geq R^\alpha\right] \simeq \exp\left(-R^{\varphi(\alpha)}\right),$$

with

$$\varphi(\alpha) = \begin{cases} 2\alpha - 1 & \alpha \in (\frac{1}{2}, 1) \\ 3\alpha - 2 & \alpha \in (1, 2) \\ 2\alpha & \alpha > 2. \end{cases}$$

Other prediction for $\mathrm{d} = 3$. This is (much) stronger than "type I hyperuniform".

Ginibre

Recall: Ginibre = Coulomb gas $d = 2, s = 0$ at $\beta = 2$. For the infinite system:

- $\mathbb{E}[\#\mathbf{X} \cap \Lambda] = |\Lambda|$
- $\mathrm{Var}[\#\mathbf{X} \cap B(0, R)] \simeq R^1$. Type I hyperuniform.
- Satisfies the prediction of JLM.
- Number-rigid.

For $\beta \neq 2$, we don't know. Hyperuniformity true for hierarchical model (Chatterjee).

Sine-2 and Sine-β

Recall: GUE = log gas $\mathsf{d} = 1, \mathsf{s} = 0$ at $\beta = 2$. For the infinite system:

- $\mathbb{E}[\#\mathbf{X} \cap \Lambda] = |\Lambda|$
- $\mathrm{Var}[\#\mathbf{X} \cap B(0,R)] \simeq \log R$. Type **II** hyperuniform Costin-Lebowitz.
- Number-rigid. Bufetov, Bufetov-Nikitin-Qiu
- Deviation estimates ??

In fact for all β, it holds Kritchevski-Valkó-Virág, Killip, Najnudel-Virág

$$\mathrm{Var}[\#\mathbf{X} \cap B(0,R)] \simeq \frac{\log R}{\beta}.$$

Sine-β is number rigid $\forall \beta$ Chhaibi-Najnudel, Dereudre-Hardy-L.-Maïda.

Other examples

- 1d Coulomb gas. $\mathsf{d} = 1, \mathsf{s} = -1, \beta > 0$
  Rephrasing older results, the $1d$ Coulomb gas appears to be type I hyperuniform (bounded variance) and number-rigid (?). More rigid?

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- Lattices ($\beta \to \infty$? yes in 1d). True lattices are obviously fully rigid and no variance. Shifted lattices are type I HU (Kendall) and fully rigid. A lattice + random iid perturbation + shift remains type I HU... but not always number-rigid (Gacs-Szaz, Holroyd-Soo, Peres-Sly).

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- Poisson ($\beta \to 0$). Nothing at all.

Remark: Sine-β goes to Poisson as $\beta + 0$ (while retaining some rigidity) and to a shifted $\mathbb{Z}$ as $\beta \to +\infty$ while fluctuating a bit more.

General Riesz systems?

- For short-range systems: not HU, not rigid. Not Poisson either. What are the remaining traces of rigidity?
- Focus on long-range $s \in [\mathrm{d}-2, \mathrm{d})$. A few known facts and some guesswork:
  - $\mathrm{s} = \mathrm{d} - 2$ (Coulomb) type I HU and number-rigid?? True for $\mathrm{d} = 1$, for $\mathrm{d} = 2 \& \beta = 2$. HU is part of JLM predicition for $\mathrm{d} = 2, 3$. Number-rigidity might not be true for $\mathrm{d} = 3$.

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  - $\mathrm{s} = d$ lose HU.

Remark: we know that for $d = 1$, $s = -1$ (Coulomb) is more rigid than $s = 0$ (log-gas), despite being less repulsive.
Possible motto: long-range, rather than repulsion, is responsible for "cancellation of charge fluctuations".

Role of temperature

- Those were β-independent statements. Unclear whether there are β-dependent properties (e.g. 3d Coulomb gases are rigid at low temperatures, not at higher ones?? in the spirit of Peres-Sly).
  There has been some interest in very low/high temperature regimes. Take $\beta = \beta(N)$.
- If $\beta(N) \to +\infty$ (low temp.) we see energy minimizers (a lattice)? Might break translation invariance, so charge fluctuations might not be centered. The regime $\beta(N) = \log N$ might already be interesting in $1d$ (since $\mathrm{Var}(\mathbf{X} \cap I) \propto \frac{\log |I|}{\beta}$), and in $2d$ (Ameur, equidistribution).
- If $\beta(N) \to 0$ (high temp.) We see Poisson in the limit, however the system might stay rigid for a while. A "rigidity scale" appears, it depends on β and on the property of interest. There is local disorder, but some order lingers at large mesoscales. Armstrong-Serfaty, Lambert, Hardy-Lambert, Akemann-Byun.

Tools - I

"The electric energy controls the fluctuations"

For some dual norm $\| \cdot \|_\star$ one gets:

$$\| \sum_{i=1}^{N} \delta_{x_i} - \mathbf{1}\Lambda_N(x) dx \|_\star \preceq \mathrm{H}_N^V(\mathrm{X}_N)$$

$+$ all sorts of controls on the energy H_N^V (large deviation principle, local laws...) $\implies$ control on the "fluctuation measure" $\sum_{i=1}^N \delta_{x_i} - \mathbf{1}\Lambda_N(x)dx$.

Problem: the dual norm may require more regularity than indicator functions possess. Need to mollify and lose precision.

In certain cases, $\mathbf{1}_\Lambda$ is an acceptable test function, e.g. for $\mathrm{d} = 1, \mathrm{s} \in (0,1)$ (Boursier). In general, need $\frac{\mathrm{d}-\mathrm{s}}{2}$ derivatives in L^2.

Tools - II

This approach gives preliminary bounds (L. - Serfaty)

$$\mathrm{Var}\left[\#\mathbf{X} \cap B(0,R)\right] = O(R^{\mathsf{d}+\mathsf{s}}).$$

For $\mathsf{d} = 1, \mathsf{s} = 0$ gives R instead of $\log R$. For $\mathsf{d} = 2, \mathsf{s} = 0$ only says "not worse than Poisson". Exponent probably off by 1.

Finer arguments

CLT-like arguments (cf. Sylvia's talk) for fluctuation of linear statistics. Test function $\to$ change of potential $\to$ local change of density $\to$ comparison of partition functions... Gives fine controls on fluctuations, but requires even higher regularity (hence even worse mollification issues when treating an indicator function).

Might grant access to e.g. intermediate regime of the JLM prediction but not the finest level.

A working-able strategy

NSV proof of the JLM law for the GAF

Proof of the small/fine regime of the JLM predicition for zeroes of the GAF. System in d = 2, infinitely many points, translation-invariant. Question:

$$\mathbb{P}\left(\mathbf{X} \cap B(0,R) - \pi R^2 \geq R^\alpha\right), \alpha \in (\frac{1}{2}, 1).$$

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   2.2 Show that the charge fluctuations are bounded on each one.

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2. Cut the boundary into R pieces of size 1. A fraction $\frac{1}{M}$ of them is "well-separated" and carries an excess of points at least $\frac{R^\alpha}{M}$.
   1. (2.1) Show that the pieces are (almost) independent.
   2. (2.2) Show that the charge fluctuations are bounded on each one.
   3. (2.3) Show that the charge fluctuations are (almost) centered on each one.

A working-able strategy

NSV proof of the JLM law for the GAF

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3. Apply Bernstein's inequality. Tail probability: $\exp\left(-\frac{R^{2\alpha-1}}{M}\right)$.

Comments

- For the first step (locate the excess near the boundary of the disk.), NSV use analytic techniques. JLM have an electrostatic justification. We can use the "fine" estimates for fluctuations of smooth functions.
- For step 2.1: well-separated pieces are almost independent? True for GAF (Gaussian functions with almost orthogonal Gaussian coefficients...), true for Riesz systems (after conditioning on the number of points in each piece).
- In fact, what is lacking is step 2.3: no reason for the charge fluctuations to be centered...
- In a remarkable way, one finds almost independence in a system of points/particles which are **far** from iid. And indeed, iid particles would never be any close to hyperuniformity and JLM prediction...

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2. Study hyperuniformity/rigidity for Riesz gases beyond “integrable” cases (Ginibre, 1d log-gas, 1d Coulomb gas).
3. Universality w.r.t. interaction? Phase portrait as s, d varies among the Riesz family, does it extend if g is only assumed to decay as $\|x\|^{-\mathsf{s}}$, with a different repulsion at the origin (e.g. Lennard-Jones potentials)? What exactly is the interplay of repulsion and long-range?

Thank you for your attention!