Universality Results in Random Lozenge Tilings

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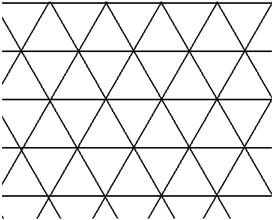
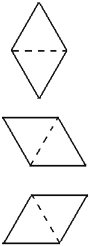
Lozenge Tilings

- Triangular lattice $\mathbb{T}$

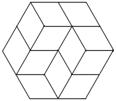


$\mathbb{T}$



- Faces are triangles
- Pairs of adjacent faces are **tiles**, also called **lozenges** or **dimers**
  - Three orientations for lozenges
- Consider tilings of subdomains of $\mathbb{T}$ using these lozenges

Tilings and Surfaces

- Tiling of a hexagon 

- Can also be interpreted as **stepped surfaces** or **height functions**

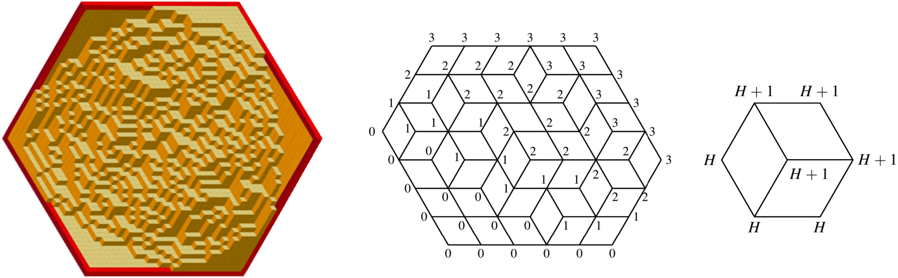


Figure from `http://math.mit.edu/~borodin/hexagon.html`.

- **Boundary height function**: Restriction of height function to ∂R
  - Boundary height function only depends on R (not on tiling)
  - Any height function with this boundary data corresponds to a tiling of R

Tilings at MSRI

Random Tilings

We consider uniformly random tilings of large domains

- Equivalently, **dimer model** on large subdomains of the hexagonal lattice

Question

How do uniformly random tilings of large domains behave?

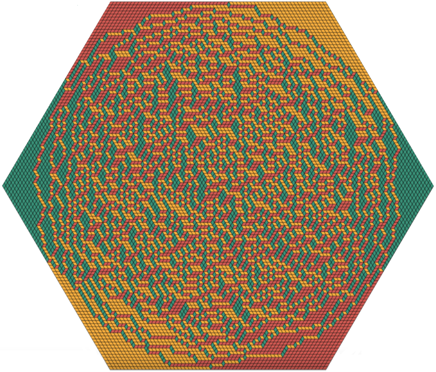
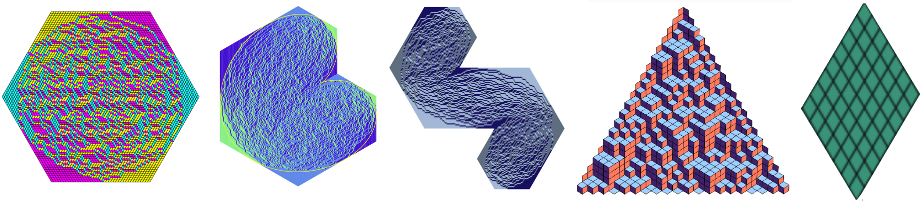


Figure from `https://storage.lpetrov.cc/img/blog/hex120_uniform.png`.

Limit Shape

Law of large numbers (Cohn–Kenyon–Propp, 2000): On general large domains, the associated height function converges to a **limit shape**

- Limit is **strongly dependent** on the geometry of the domain
- Can be very **inhomogeneous**
    - Densities of tile differ in different parts of the domain



Figures from Kenyon (2009), Kenyon–Okounkov (2005), and Keating–Sridhar (2018)

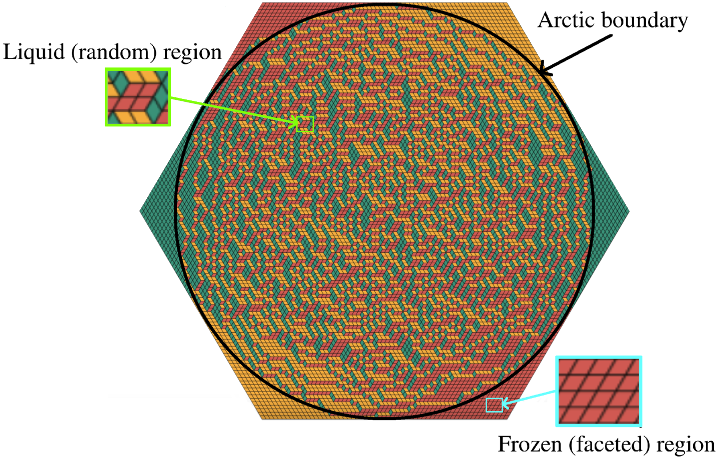
Conjecture (Kenyon–Okounkov, 2005)

The fluctuations around this limit shape converge to the Gaussian free field.

- Proven on various special domains, but open in general

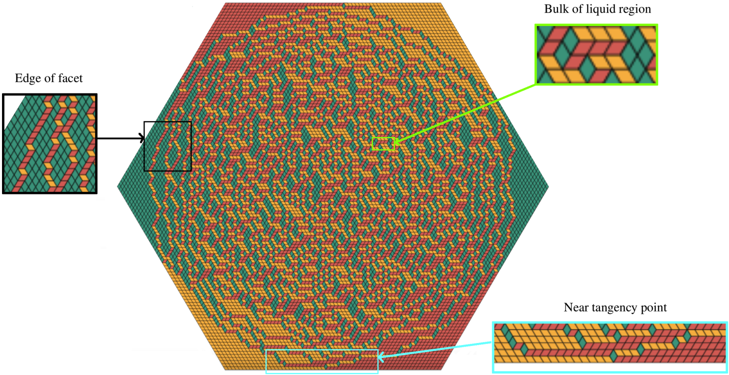
Depending on boundary, limit shape can admit frozen / liquid regions

- **Frozen regions**: Facets induced by the shape of the boundary
- **Liquid region**: Places where the surface appears rough / random



Local Features

This talk is focused on more **local features** of random tilings



- Zooming into different points of domain gives various limiting statistics

Goal: Understand these limiting statistics and their dependence on domain

1. **Compute** limiting statistics for specific domains (such as hexagons)
2. Prove these limits appear **universally** on general domains

Outline

Goal: Understand these limiting statistics and their dependence on domain

- Limiting statistics admit close connections to random matrix theory

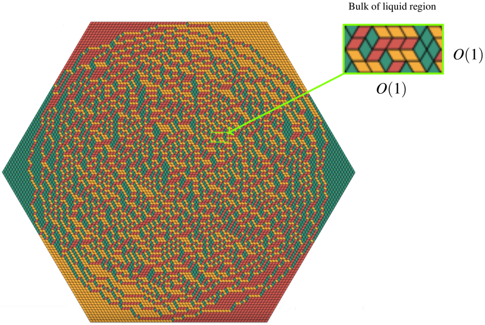
1. Bulk of liquid region
   - Cohn–Kenyon–Propp (2000): Ergodic, translation-invariant Gibbs measure (discrete sine kernel / incomplete Bessel kernel)
   - A. (2019): Appears in bulk of liquid region on general domains
2. Near tangency point
   - Johansson–Nordenstam, Okounkov–Reshetikhin (2006): Gaussian Unitary Ensemble corners process
   - A.–Gorin (2021): Appears at tangency point on general domains
3. Near edge of facet
   - Johansson (2000): Airy process / line ensemble
   - A.–Huang (2021): Appears on a wide class of polygons

Bulk of the Liquid Region

- Consider a uniformly random tiling of a domain $R \subset \mathbb{T}$.
- Fix a vertex $v \in R$ and consider an $O(1)$-neighborhood of v.
- This yields a random tiling on this $O(1)$-neighborhood
  - First taking domain size to ∞, and then taking the size of the neighborhood to ∞ gives tiling on the full plane $\mathbb{T}$

Bulk statistics question: What is the law of this random tiling?

- Contains all correlation functions of nearly neighborhing tiles
- **General prediction**: Limiting measure should satisfy translation-invariance, ergodicity, and the Gibbs property

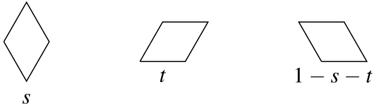


Translation Invariant Gibbs Measures

General prediction: Any candidate μ for limiting bulk statistics should satisfy three properties

- **Translation-invariance**
  - Probability measure invariant under shifts of $\mathbb{T}$
- **Ergodicity**
  - If $\mu = p\mu_1 + (1-p)\mu_2$ for $p \in (0,1)$, then $\mu_1 = \mu = \mu_2$
- **Gibbs property**
  - Conditional uniformity upon restricting to finite subdomains

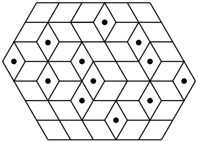
Sheffield (2003): For a given proportion $(s, t, 1-s-t)$ of tiles, exists a unique ergodic, translation-invariant Gibbs measure $\mu_{s,t}$ on tilings of $\mathbb{T}$



- We say this measure has **slope** (s, t), since its height function $H : \mathbb{T} \to \mathbb{Z}$ satisfies $\mathbb{E}\big[H(X,Y) - H(0,0)\big] = sX + tY$

Explicit Characterization of $\mu_{s,t}$

- Any tiling $\mathcal{M}$ is determined by the set $\mathcal{X} = \mathcal{X}(\mathcal{M})$ of all $(x, y) \in \mathbb{R}^2$ that are centers of **vertical lozenges** in $\mathcal{M}$



Okounkov–Reshetikhin (2001): For any $(s, t) \in (0, 1)$, we have

$$\mathbb{P}_{\mu_{s,t}}\left[\bigcap_{k=1}^{m} \{(x_k, y_k) \in \mathcal{X}(\mathcal{M})\}\right] = \det\left[\mathcal{K}_\xi(x_i, y_i; x_j, y_j)\right]_{1\le i,j\le m}, \text{ where}$$

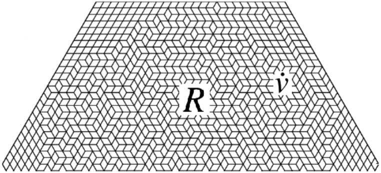
$$\mathcal{K}_\xi(x_1, y_1; x_2, y_2) = \frac{1}{2\pi\mathbf{i}}\int_{\bar{\xi}}^{\xi}(1-z)^{y_1-y_2}z^{x_2-x_1-1}dz; \qquad \xi = e^{\pi\mathbf{i}s}\frac{\sin(\pi t)}{\sin(\pi-\pi s-\pi t)}$$

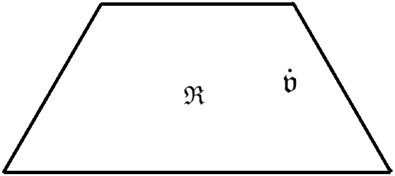
- This description as a determinantal point process, known as the **incomplete Bessel process**, is useful for computing local correlation functions

$$\mathbb{P}_{\mu_{s,t}}\left[\;\right] = s^2 - \frac{\sin^2(\pi s)}{\pi^2}$$

- Discrete analog of the sine process from random Hermitian matrix local statistics

Bulk Statistics Results





- Fix a simply connected subset $\mathfrak{R} \subset \mathbb{R}^2$ with piecewise smooth boundary
- Fix a large integer N and a tileable domain $R = R_N \subset \mathbb{T}$ with $N^{-1}R_N \approx \mathfrak{R}$
- Fix a point $\mathfrak{v} \in \mathfrak{R}$ in the liquid region and a vertex $v = v_N \in R_N$ with $N^{-1}v_N \approx \mathfrak{v}$.
- Let $\mathcal{M} = \mathcal{M}_N$ denote a uniformly random lozenge tiling of R_N

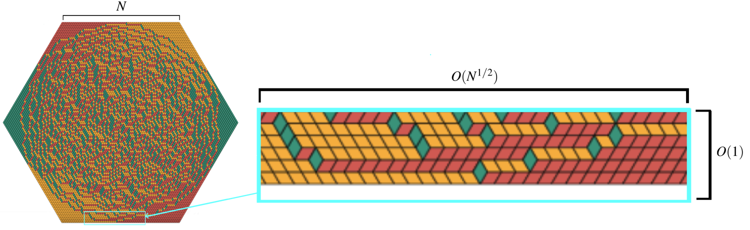
Theorem (A., 2019)

As N tends to ∞, the local statistics of $\mathcal{M}$ around v are given by $\mu_{s,t}$, where (s,t) is the gradient of the tiling limit shape at $\mathfrak{v}$.

- Predicted by Cohn–Kenyon–Propp (2000)
- Dependence of limiting bulk statistics on domain is isolated through (s,t)

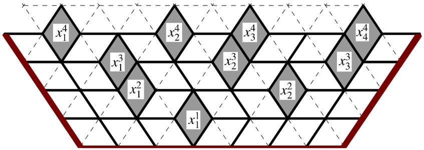
Near Tangency Points

- Fix point where arctic boundary is tangent to a (say, horizontal) side of domain
- Consider an $O(N^{1/2}) \times O(1)$ neighborhood of this point



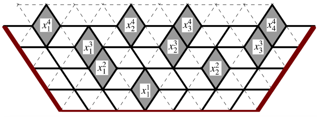
Vertical (green) tiles form an interlacing array in this neighborhood

- Denote the positions of these tiles on level k by $x_1^k < x_2^k < \cdots < x_k^k$



Tangency statistics question: What is the joint law of the $\{x_i^k\}_{i,k}$?

GUE Corners Process



- Johansson–Nordenstam (2006): On hexagon of side length N, there exist constants $\mu = \frac{1}{2}, \sigma = \sqrt{\frac{8}{3}}$ so that $\left\{\sigma N^{-1/2}\left(x_i^k - \mu N\right)\right\}_{i,k} \to \{\xi_i^k\}_{i,k}$

Here, $\{\xi_i^k\}$ is the **Gaussian Unitary Ensemble (GUE) corners process**

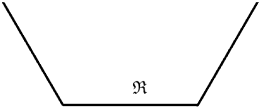
- $\mathbf{X} = [X_{ij}]$: Infinite array of independent standard complex Gaussians
- Set $\mathbf{M} = [M_{ij}] = \frac{1}{2}(X + X^*)$: Infinite Hermitian random matrix
- $\mathbf{M}^k$: The $k \times k$ matrix given by top-left $k \times k$ corner of $\mathbf{M}$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

- Set $\xi_1^k \leq \xi_2^k \leq \cdots \leq \xi_k^k$ to be the eigenvalues of $\mathbf{M}^k$

The $\{\xi_i^k\}$ interlace (as the $\{x_i^k\}$ do)

Tangency Statistics Results



- Fix $\mathfrak{R} \subset \mathbb{R}^2$ with three adjacent segments inclined 120 degrees with respect to each other
- Let $R = R_N = N \cdot \mathfrak{R}$ be a tileable domain
- Let $\mathcal{M}_N$ be a uniformly random tiling of R_N
- Denote x-coordinates of the vertical tiles around the middle (horizontal) segment by $\{x_i^k\}$

Theorem (A.–Gorin, 2021)

There exist $\mu = \mu(\mathfrak{R})$, $\sigma = \sigma(\mathfrak{R})$ *so that* $\left\{\sigma N^{-1/2}(x_i^k - \mu N)\right\}_{i,k} \to \{\xi_i^k\}_{i,k}$.

- Predicted by Johansson–Nordenstam (2006)

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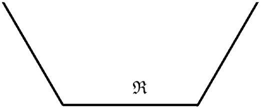

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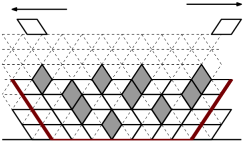
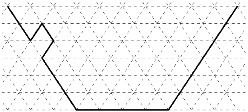
Corners process forces these three adjacent segments inclined at 120 degrees

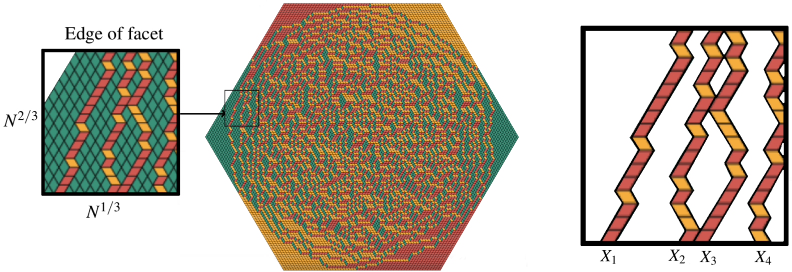
- Can fail to arise if one of the segments has microscopic dents





Near Facet Edge

- Fix a smooth point on the arctic boundary that is not a tangency location
- Consider an $O(N^{1/3}) \times O(N^{2/3})$ neighborhood of this point
  - $N^{2/3}$: Parallel (tangent) to arctic boundary
  - $N^{1/3}$: Orthogonal to arctic boundary



- The red and orange tiles form a family of paths (discrete left-right walks)
  - Denote them by $X_1, X_2, \ldots$, where $X_i = \big(X_i(t)\big)$
  - Extreme path X_1 is the interface between frozen and liquid regions

Edge statistics question: What is the joint scaling limit of these paths?

Airy Statistics

Baik–Kriecherbauer–McLaughlin–Miller (2007), Petrov (2012): On hexagon of side N,

$$\mathfrak{a}N^{-1/3}\big(X_i(\mathfrak{b}tN^{2/3}) - \mathfrak{c}tN^{2/3}\big) \to \big(\mathcal{A}_i(t) + t^2\big),$$

Here, $\mathcal{A}_i(t) = \big(\mathcal{A}_1(t), \mathcal{A}_2(t), \dots\big)$ is the **Airy line ensemble**

- Family $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ of continuous, random, non-intersecting curves $\mathcal{A}_1(t) > \mathcal{A}_2(t) > \cdots$ (Corwin–Hammond, 2011)
  - Prevalent in Kardar–Parisi–Zhang universality class
    - Top curve is the Airy$_2$ process (Prähofer–Spohn, 2001)
- Determinantal point process with extended Airy kernel
- Appears as **edge limit** of **Dyson Brownian motion**
  - Limit process of largest eigenvalues of large Hermitian matrix whose entries are complex Brownian motions / extremal paths in non-intersecting Brownian motions

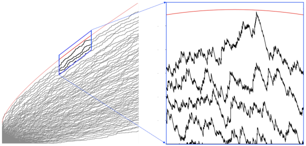
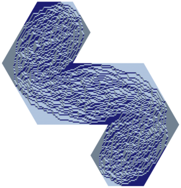


Figure by Dauvergne–Nica–Virág (2019)

Edge Statistics Results

- Fix polygon $\mathfrak{P} \subset \mathbb{R}^2$ (sides parallel to axes of $\mathbb{T}$) satisfying certain technical conditions*
- Fix regular point $\mathfrak{v} \in \mathfrak{P}$ on the arctic boundary
- Define the domain $P = P_N = N \cdot \mathfrak{P} \subset \mathbb{T}$
- Let $v = v_N \approx N \cdot \mathfrak{v} \in P$ be a vertex
- Let $\mathcal{M} = \mathcal{M}_N$ denote a uniformly random tiling of P



Consider family of discrete walks around v in $\mathcal{M}$.

Theorem (A.–Huang, 2021)

Under $(N^{1/3}, N^{2/3})$ normalization, this family of walks converges to the Airy line ensemble.

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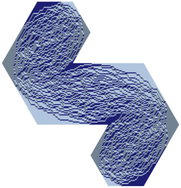

Consider family of discrete walks around v in $\mathcal{M}$.

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Under $(N^{1/3}, N^{2/3})$ normalization, this family of walks converges to the Airy line ensemble.

- *Conditions on the arctic boundary: No tacnodes, no cuspidal turning points, and no two distinct cusps on the same exact horizontal level (share the same y-coordinate)
- Polygons with generic side lengths likely satisfy this condition

On the Proofs

- Kasteleyn (1961): Random tilings are determinantal point processes
  - Correlation functions are minors of an **inverse Kasteleyn matrix**

Issue: Analyzing this matrix on arbitrary domains remains a challenge

- Known how to analyze it on specific families of "solvable" domains, including the following
  - Kenyon (1997), Cohn–Kenyon–Propp (2000): Torus
  - Okounkov–Reshetikhin (2001, 2005): q-Weighted (skew) plane partitions
  - Baik–Kreicherbauer–McLaughlin–Miller (2007), Gorin (2007): Hexagons
  - Petrov (2012): Trapezoids
  - Gorin–Petrov (2016): Nonintersecting random walks (infinite trapezoids)
- Previous limiting results often addressed one (family of) domain at a time

To prove universality results, we instead **locally couple** the tiling on a general domain to a tiling on a solvable domain

On the Proofs

Realizing this coupling qualitatively proceeds in three components

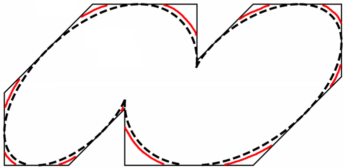
1. **Analytic**: Prove an *a priori* estimate on the tiling of a general domain
   - Example: Concentration estimate for the height function
2. **Algebraic**: Locate a specific family of solvable domains for which universal limiting statistics can be proven
   - Various examples are known (same are listed on the previous slide)
3. **Probabilistic**: Use the *a priori* estimate to locally couple the tiling on original domain to that on solvable one
   - Example: In some local neighborhood, sandwich the original tiling between two nearly equal tilings on solvable domains

Implementation for the different limiting statistics proceeds very differently

- Tangency points
  - Estimate: $o(N^{1/2})$ Concentration; Solvable domain: Trapezoids
- Edge of facet
  - Estimate: $O(N^{\delta})$ Concentration; Solvable domain: Hexagons
- Bulk of liquid region
  - Estimate: Local law of large numbers; Solvable domain: Infinite trapezoid

Concentration Estimate for Edge Statistics

- Simply connected polygon $\mathfrak{P} \subset \mathbb{R}^2$ satisfying technical conditions
  - Limiting height function $\mathcal{H} : \mathfrak{P} \to \mathbb{R}$
  - **Liquid region** $\mathfrak{L} \subset \mathfrak{P}$: Region where $\mathcal{H}$ is not frozen
    - $\mathfrak{L} = \Big\{ u \in \mathfrak{P} : \big(\partial_x \mathcal{H}(u), \partial_y \mathcal{H}(u) \big) \notin \big\{ (0,0), (1,0), (0,1) \big\} \Big\}$
- Tileable domain $P = N \cdot \mathfrak{P} \subset \mathbb{T}$
  - Random tiling $\mathcal{M}$ of P with associated height function H
- **Augment** liquid region by $N^{\delta - 2/3}$ to form $\mathfrak{L}^+ = \Big\{ u \in \mathfrak{P} : \operatorname{dist}\big(u, \mathfrak{L}\big) \le N^{\delta - 2/3} \Big\} \subset \mathfrak{P}$



Theorem (A.–Huang, 2021)

The following two statements hold with probability at least $1 - N^{-1000}$.

- *For every* $u \in \mathfrak{L}^+$, *we have* $\big| H(Nu) - N\mathcal{H}(u) \big| < N^{\delta}$
- *For every* $u \in \mathfrak{P} \setminus \mathfrak{L}^+$, *we have* $H(Nu) = N\mathcal{H}(u)$

Proof Outline of Concentration Estimate

Theorem (A.–Huang, 2021)

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- *For every* $u \in \mathfrak{L}^+$, *we have* $\big|H(Nu) - N\mathcal{H}(u)\big| < N^\delta$
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Proof outline

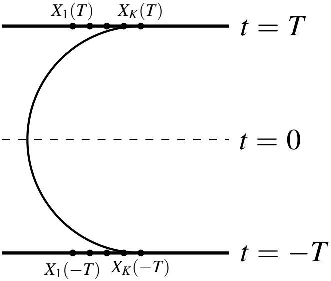
1. Huang (2021): Holds on domains whose arctic boundary has at most one cusp
   - Approximate tiling by family of discrete free random walks, with drift, conditioned to never intersect
     - Drifts can become **singular** if arctic boundary has more than one cusp
   - Analyze non-intersecting walk ensemble through discrete loop equations
2. A.–Huang (2021): Holds on more general polygons
   - Introduce Markov chain on set of tilings of $\mathfrak{P}$ that mixes in polynomial time
     - Decompose polygon into subdomains with at most one cusp
     - Repeatedly uniformly resample the tiling on each subdomain
   - Show the concentration estimate is **preserved** under these dynamics
     - Introduce barrier functions, produced from explicit perturbations of limit shape, that bound dynamics above and below

Edge Statistics

- Let $\mathcal{M}$ be a random tiling of P, and denote its extreme paths by $X_1(t), X_2(t), \ldots$
- Fix a nonsingular point $\mathfrak{v} = (0,0) \in \mathfrak{A}$ on the arctic curve of $\mathfrak{A}$ of $\mathfrak{P}$
- Locally around $\mathfrak{v}$, the curve $\mathfrak{A}$ is parabolic
  - Exist $\mathfrak{l}, \mathfrak{q}$ such that $x = \mathfrak{l}y + \mathfrak{q}y^2 + O\big(|y|^3\big)$ for $(x,y) \in \mathfrak{A}$
- Fix a small $\delta > 0$; set $K = N^{2\delta}$; and let $T = N^{2/3+20\delta}$

Concentration estimate: With high probability, X_K is close to a parabola

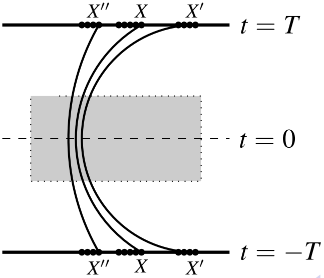
- Exists $\mathfrak{c} \in \mathbb{R}$ so that $\big|X_K(s) - \mathfrak{c}K^{2/3} - \mathfrak{l}s - \mathfrak{q}s^2\big| = O(N^{1/3-\delta})$ for $s \in [-T, T]$
- For each $i \in [1, K]$ and $t \in \{-T, T\}$, we have $\big|X_i(t) - X_K(t)\big| \le N^{1/3+3\delta}$



Comparison to Hexagons

Couple $X = (X_1, X_2, \ldots, X_K)$ with the extreme paths $X' = (X'_1, X'_2, \ldots, X'_K)$ associated with a hexagon P', so that they differ by $o(N^{1/3})$

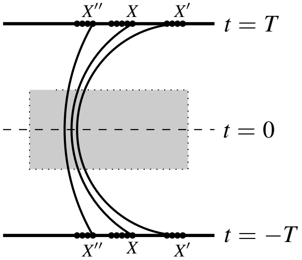
- Bound X between the extreme paths X, X'' associated with two hexagons P', P''
- Edge statistics X', X'' of the hexagonal domains P', P'' converge to Airy line ensemble
  - Baik–Kriecherbauer–McLaughlin–Miller (2007): One-point Tracy–Widom limit
  - Petrov (2012), Duse–Metcalfe (2017): Extended Airy kernel correlation function limit
  - Dauvergne–Nica–Virág (2019): Airy line ensemble
- Parameters of the hexagons will be close, implying X' and X'' are close
- Implies X converges to Airy line ensemble



Comparison to Hexagons

Recall $x = \mathfrak{l}y + \mathfrak{q}y^2 + O\big(|y|^3\big)$ for $(x, y) \in \mathfrak{A}$

- Fix $\mathfrak{q}' = \mathfrak{q} - N^{-10\delta}$ and $\mathfrak{q}'' = \mathfrak{q} + N^{-10\delta}$, so that $\mathfrak{q}' < \mathfrak{q} < \mathfrak{q}''$ and $\mathfrak{q}' \approx \mathfrak{q}''$
- Find two hexagons $\mathfrak{P}'$ and $\mathfrak{P}''$ with arctic boundaries $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively, and two points $\mathfrak{v}' = (0,0) \in \mathfrak{A}'$ and $\mathfrak{v}'' = (0,0) \in \mathfrak{A}''$ so the following holds
  - For $(x', y') \in \mathfrak{A}'$, we have $x' = \mathfrak{l}y' + \mathfrak{q}'y'^2 + O\big(|y'|^3\big)$
  - For $(x'', y'') \in \mathfrak{A}''$, we have $x'' = \mathfrak{l}y'' + \mathfrak{q}''y''^2 + O\big(|y''|^3\big)$
- Set $P' = N \cdot \mathfrak{P}'$ and $P'' = N \cdot \mathfrak{P}''$; let associated extreme paths be X' and X''
- Concentration estimate also applies to P' and P''
  - Gives $X_i''(t) < X_i(t) < X_i''(t)$ if $t \in \{-T, T\}$ and $X_K(s) < X_K(s) < X_K(s)$ if $s \in [-T, T]$
- Can couple between (X'', X, X') so that $X_i''(s) \leq X_i(s) \leq X_i'(s)$ for $s \in [-T, T]$



Summary

- Random lozenge tilings are very sensitive to boundary conditions
  - Local behaviors are different depending on where one looks in domain
    - Bulk of liquid region: Ergodic, translation invariant Gibbs measure (incomplete Bessel process)
    - Tangency location: GUE corners process
    - Edge of facet: Airy process / line ensemble
  - Previous results showed these statistics held on specific domains
- Recent results: Established universality pheneomena
  - These limiting statistics universally appear for random tilings on fairly general domains
- Proofs are based on a combination of algebra / analysis / probability
  - Analytic: Obtain coarse estimates for tiling on general domains
  - Algebraic: Obtain refined asymptotics for tiling on specific domains
  - Probabilistic: Couple tiling on general domain to one on specific domain